

On the Power Subgroups of the Extended Modular Group $\bar{\Gamma}$

Recep Şahin, Sebahattin İkikardeş, Özden Koruoğlu

Abstract

In this paper we describe the group structure of power subgroups $\bar{\Gamma}^m$ of the extended modular group $\bar{\Gamma}$ and the quotients to them. Then we give some relations between the power subgroups $\bar{\Gamma}^m$, the commutator subgroups $\bar{\Gamma}'$ and $\bar{\Gamma}''$ and also the information of interest about free normal subgroups of the extended modular group $\bar{\Gamma}$.

Key Words: Extended Modular Group, Power Subgroup, Commutator Subgroup, Free Subgroup

1. Introduction

The modular group Γ is the discrete subgroup of $PSL(2, \mathbb{Z})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + 1.$$

Let $S = T \cdot U$, that is,

$$S(z) = -\frac{1}{z+1}.$$

Then modular group Γ has a presentation

$$\Gamma = \langle T, S \mid T^2 = S^3 = I \rangle \cong C_2 * C_3.$$

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By adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ , the extended modular group $\bar{\Gamma}$ has been defined in [1]. The extended modular group $\bar{\Gamma}$ has a presentation

$$\bar{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = I, RT = TR, RS = S^{-1}R \rangle$$

or

$$\bar{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_3. \quad (1)$$

The modular group Γ is a subgroup of index 2 in $\bar{\Gamma}$.

Let us define $\bar{\Gamma}^m$ to the subgroup generated by the m^{th} powers of all elements of $\bar{\Gamma}$, for some positive integer m . $\bar{\Gamma}^m$ is called the m^{th} -power subgroup of $\bar{\Gamma}$. As fully invariant subgroups, they are normal in $\bar{\Gamma}$.

From the definition one can easily deduce that

$$\bar{\Gamma}^{mk} < \bar{\Gamma}^m$$

and that

$$\bar{\Gamma}^{mk} < (\bar{\Gamma}^m)^k.$$

Also, it is easy to deduce that

$$\bar{\Gamma}^m \cdot \bar{\Gamma}^k = \bar{\Gamma}^{(m,k)},$$

where (m, k) denotes the greatest common divisor of m and k .

The power subgroups of the modular group Γ was studied by [4]. In [4], M. Newman showed that

$$\begin{aligned} \Gamma^2 &= \langle S \rangle * \langle TST \rangle, \\ \Gamma^3 &= \langle T \rangle * \langle STS^2 \rangle * \langle S^2TS \rangle, \\ \Gamma' &= \Gamma^2 \cap \Gamma^3, \Gamma' = \langle TSTS^2 \rangle * \langle TS^2TS \rangle \text{ and } \Gamma'' \subset \Gamma^6 \subset \Gamma'. \end{aligned} \quad (2)$$

Also, M. Newman proved that the groups Γ^{6m} are free groups and the index $|\Gamma : \Gamma^{6m}| = \infty$ for $m \geq 72$ and $|\Gamma : \Gamma^{6m}|$ when $2 \leq m \leq 71$ is unknown. Γ^6 is a free group of rank 37.

The commutator subgroup of $\bar{\Gamma}$ is denoted by $\bar{\Gamma}'$ and defined by

$$\langle [g, h] \mid g, h \in \bar{\Gamma} \rangle,$$

where $[g, h] = ghg^{-1}h^{-1}$. $\bar{\Gamma}'$ is a normal subgroup of $\bar{\Gamma}$, and therefore we can form the quotient group $\bar{\Gamma}/\bar{\Gamma}'$.

The commutator subgroup $\bar{\Gamma}'$ of the extended modular group $\bar{\Gamma}$ was investigated in [1], and it was shown that

$$\begin{aligned} |\bar{\Gamma} : \bar{\Gamma}'| &= 4, \\ \bar{\Gamma}' &= \langle S \rangle * \langle TST \rangle, \\ |\bar{\Gamma} : \bar{\Gamma}''| &= 36, \end{aligned} \tag{3}$$

so that $\bar{\Gamma}''$ is a free group with basis $[S, TST], [S, TS^2T], [S^2, TST], [S^2, TS^2T]$.

The purpose of this paper is to determine the structure of the power subgroups $\bar{\Gamma}^m$ of the extended modular group $\bar{\Gamma}$ and to give some relations between them, the commutator subgroups $\bar{\Gamma}'$ and $\bar{\Gamma}''$ and also to investigate free normal subgroups of the extended modular group $\bar{\Gamma}$. In our discussion we use Reidemeister-Schreier method, (for more detail about this method, see [2]).

2. The Power Subgroups of the Extended Modular Group

We consider the presentation of the extended modular group $\bar{\Gamma}$ given in (1):

$$\bar{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I \rangle.$$

We find a presentation for the quotient $\bar{\Gamma}/\bar{\Gamma}^m$ by adding the relation $X^m = I$ to the presentation of $\bar{\Gamma}$. The order of $\bar{\Gamma}/\bar{\Gamma}^m$ gives us the index. We have

$$\begin{aligned} \bar{\Gamma}/\bar{\Gamma}^m &\cong \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I, \\ &T^m = S^m = R^m = (TR)^m = (RS)^m = I \rangle. \end{aligned} \tag{4}$$

Thus we use Reidemeister-Schreier process to find the presentation of the power subgroups $\bar{\Gamma}^m$. First we have the following theorem.

Theorem 2.1 *i) The normal subgroup $\bar{\Gamma}^2$ is isomorphic to the free product of two finite*

cyclic groups of order 3. Also

$$\begin{aligned} |\bar{\Gamma} : \bar{\Gamma}^2| &= 4, \\ \bar{\Gamma}^2 &= \langle S \rangle * \langle TST \rangle, \\ \bar{\Gamma} &= \bar{\Gamma}^2 \cup T \bar{\Gamma}^2 \cup R \bar{\Gamma}^2 \cup TR \bar{\Gamma}^2. \end{aligned}$$

The elements of $\bar{\Gamma}^2$ are characterised by the property that the sum of the exponents of T is even.

ii) The normal subgroup $\bar{\Gamma}^3$ is isomorphic to the extended modular group $\bar{\Gamma}$, i.e.

$$\bar{\Gamma}^3 = \bar{\Gamma}.$$

Proof. i) By (4), we have

$$\begin{aligned} \bar{\Gamma} / \bar{\Gamma}^2 &\cong \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I, \\ &T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = I \rangle. \end{aligned}$$

Since

$$S^3 = S^2 = I,$$

we obtain $S = T^2 = R^2 = I$. Therefore

$$\bar{\Gamma} / \bar{\Gamma}^2 \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong D_2$$

and

$$|\bar{\Gamma} : \bar{\Gamma}^2| = 4.$$

Now we choose $\{I, T, R, TR\}$ as a Schreier transversal for $\bar{\Gamma}^2$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{aligned} I.T.(T)^{-1} &= I, & I.S.(I)^{-1} &= S, & I.R.(R)^{-1} &= I, \\ T.T.(I)^{-1} &= I, & T.S.(T)^{-1} &= TST, & T.R.(TR)^{-1} &= I, \\ R.T.(TR)^{-1} &= RTRT, & R.S.(R)^{-1} &= RSR, & R.R.(I)^{-1} &= I, \\ TR.T.(R)^{-1} &= TRTR, & TR.S.(TR)^{-1} &= TRSRT, & TR.R.(T)^{-1} &= I. \end{aligned}$$

Since $RTRT = I$, $TRTR = I$, $RSR = S^{-1}$, $TRSRT = TS^{-1}T = (TST)^{-1}$, the generators are S and TST . Thus we have

$$\bar{\Gamma}^2 = \langle S, TST \mid S^3 = (TST)^3 = I \rangle \cong C_3 * C_3,$$

and

$$\bar{\Gamma}^2 = \bar{\Gamma}^2 \cup T \bar{\Gamma}^2 \cup R \bar{\Gamma}^2 \cup TR \bar{\Gamma}^2.$$

ii) By (4), we have

$$\begin{aligned} \bar{\Gamma} / \bar{\Gamma}^3 \cong \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I, \\ T^3 = S^3 = R^3 = (TR)^3 = (RS)^3 = I \rangle. \end{aligned}$$

Therefore we find $S = T = R = I$ from the relations

$$R^2 = R^3 = I, S^3 = (SR)^2 = I, T^2 = T^3 = I.$$

Thus we have

$$|\bar{\Gamma} : \bar{\Gamma}^3| = 1;$$

that is,

$$\bar{\Gamma}^3 = \bar{\Gamma}.$$

□

The following results are easy to see:

Theorem 2.2 i) $\bar{\Gamma}^2 = \Gamma^2 = \bar{\Gamma}' = \bar{\Gamma}^2 \cap \bar{\Gamma}^3$

ii) $(\bar{\Gamma}')^3 \subset \bar{\Gamma}''$.

Now we have

Theorem 2.3 Let m be a positive integer. The normal subgroups $\bar{\Gamma}^m$ satisfy the following:

i) $\bar{\Gamma}^m = \bar{\Gamma}$ if $2 \nmid m$,

ii) $\bar{\Gamma}^m = \bar{\Gamma}^2$ if $2 \mid m$ but $6 \nmid m$.

Proof. i) If $2 \nmid m$ then by (4), we find $S = T = R = I$ from the relations

$$R^2 = R^m = I, S^3 = S^m = (SR)^2 = (SR)^m = I = I, T^2 = T^m = I.$$

Thus $\bar{\Gamma}/\bar{\Gamma}^m$ is trivial and hence $\bar{\Gamma}^m = \bar{\Gamma}$.

ii) If $2 \mid m$ but $6 \nmid m$ then $(m, 3) = 1$. By (4), we obtain $S = T^2 = R^2 = I$ from the relations

$$R^2 = R^m = I, S^3 = S^m = I, T^2 = T^m = I$$

as $2 \mid m$ but $6 \nmid m$. These show that

$$\bar{\Gamma}/\bar{\Gamma}^m \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong D_2$$

and

$$|\bar{\Gamma} : \bar{\Gamma}^m| = 4.$$

Since $\bar{\Gamma}^2$ is the only normal subgroup of index 4 we have $\bar{\Gamma}^m = \bar{\Gamma}^2$. □

Therefore the only case left is that when m is divisible by 6. In this case, the above techniques do not say much about $\bar{\Gamma}^m$. To do this we use the second commutator subgroup $\bar{\Gamma}''$ of $\bar{\Gamma}$.

Theorem 2.4 *Let m be a positive integer. The groups $\bar{\Gamma}^{6m}$ are the subgroups of the second commutator subgroup $\bar{\Gamma}''$.*

Proof. i) Since $\bar{\Gamma}^6 \subset (\bar{\Gamma}^2)^3 \subset \bar{\Gamma}^2$ and $\bar{\Gamma}' = \bar{\Gamma}^2$ implies that $\bar{\Gamma}^6 \subset (\bar{\Gamma}')^3 \subset \bar{\Gamma}'$ and $\bar{\Gamma}^{6m} \subset \bar{\Gamma}^6 \subset \bar{\Gamma}''$. Since $\bar{\Gamma}'$ does not contain any reflection, $\bar{\Gamma}^{6m}$ does not contain any reflection. Also we know that $\bar{\Gamma}^{6m} \subset \bar{\Gamma}^{6m}$. Thus we get

$$\bar{\Gamma}^{6m} = \bar{\Gamma}^{6m} \subset \bar{\Gamma}''.$$

□

Then because $\bar{\Gamma}''$ is a free group and $\bar{\Gamma}^{6m} \subset \bar{\Gamma}''$, we have by Schreier's theorem the following theorem

Theorem 2.5 *The groups $\bar{\Gamma}^{6m}$ are free groups.*

Therefore

$$\begin{aligned} |\bar{\Gamma} : \bar{\Gamma}^{6m}| &= |\bar{\Gamma} : \Gamma^{6m}| \\ &= |\bar{\Gamma} : \Gamma| \cdot |\Gamma : \Gamma^{6m}| \\ &= 2 |\Gamma : \Gamma^{6m}| \end{aligned}$$

since $|\bar{\Gamma} : \Gamma| = 2$. In [4], the index $|\Gamma : \Gamma^6|$ was computed as 216. Therefore

$$|\bar{\Gamma} : \bar{\Gamma}^6| = 432.$$

Also, the index $|\bar{\Gamma} : \bar{\Gamma}^{6m}|$ is unknown since $|\Gamma : \Gamma^{6m}|$, $2 \leq m \leq 71$, is unknown.

Corollary 2.6 $\bar{\Gamma}^6$ is a free group of rank 37.

3. Free Normal Subgroups of the Extended Modular Group

As $\bar{\Gamma}$ is isomorphic to the free product of dihedral groups D_2 and D_3 with amalgamation \mathbb{Z}_2 , it has two kinds of normal subgroups : Free ones and free products of some infinite cyclic groups, some cyclic groups of order 2 and order 3, some dihedral groups D_2 and D_3 with some dihedral groups D_2 and D_3 with amalgamation \mathbb{Z}_2 . Therefore the study of free normal subgroups and their group theoretical structures will be important to us. Here we discuss them for extended modular group $\bar{\Gamma}$. This has been done for modular group by Newman in [3]. His results can be generalized to the extended modular group.

Before giving the main theorem we need the following lemmas.

Lemma 3.1 *Let N be a non-trivial normal subgroup of finite index in $\bar{\Gamma}$. Then N is free if and only if it contains no elements of finite order.*

Proof. By (1), $\bar{\Gamma}$ is isomorphic to a free product of $D_2 = C_2 \times C_2$ and $D_3 = C_2 \times C_3$ each amalgamated over \mathbb{Z}_2 . A subgroup of finite index in $\bar{\Gamma}$ is isomorphic to a free product of the groups F , C_r , and $D_{m_1} *_{\mathbb{Z}_2} D_{m_2}$, where r and each m_i divide 2 or 3. Thus if N is a subgroup of finite index in $\bar{\Gamma}$, it follows that

$$N = F * \prod_* C_r * \prod_* (D_{m_1} *_{\mathbb{Z}_2} D_{m_2}), \tag{5}$$

where F is either free or $\{I\}$ and each C_r is conjugate to $\{T\}$ or to $\{S\}$ or to $\{R\}$ and each D_{m_i} is conjugate to $\{T, R\}$ or to $\{S, R\}$. As N contains no elements of finite order the free product $\prod_* C_r * \prod_* (D_{m_1} *_{\mathbb{Z}_2} D_{m_2})$ is vacuous; and also as N is non-trivial, N must be free.

Conversely, if N is free, then by definition, it contains no elements of finite order. \square

Lemma 3.2 *The only normal subgroups of finite index in $\bar{\Gamma}$ containing elements of finite order are*

$$\bar{\Gamma}, \Gamma, \Gamma^2 \text{ and } \Gamma^3.$$

Proof. Let N be a normal subgroup of finite index in $\bar{\Gamma}$ containing an element of finite order. Then N contains an element of order 2 or an element of order 3 or two elements of order 2 or two elements of order 2 and 3 or three elements so that two elements of order 2 and an element of order 3. An element of order 2 in $\bar{\Gamma}$ is conjugate to T or to R and an element of order 3 in $\bar{\Gamma}$ is conjugate to a power of S . Therefore if a normal subgroup N contains an element of finite order, then it contains T or R or S . Therefore there are seven cases:

(i) N contains T, R and S . Then $N = \bar{\Gamma}$.

(ii) N contains T but not R and S . Then $N \neq \bar{\Gamma}, \Gamma$ and $\Gamma^3 \subset N$, as N is normal. Since $|\bar{\Gamma} : \Gamma^3| = 6$ we have $N = \Gamma^3$.

(iii) N contains T, R but not S . Then $N \neq \bar{\Gamma}$ and $\Gamma^3 \subset N$, the fact that N is normal and by (ii). Since $|\bar{\Gamma} : \Gamma^3| = 6$, we have $N = \bar{\Gamma}$ or Γ or Γ^3 . But this is not possible since $S \in \bar{\Gamma}, S \in \Gamma$ and $R \notin \Gamma^3$.

(iv) N contains T and S , but not R . Then $N \neq \bar{\Gamma}$ and $\Gamma \subset N$, by (1) and the fact that N is normal. Since $|\bar{\Gamma} : \Gamma| = 2$ it follows that $N = \Gamma$.

(v) N contains S but not T and R . Then $N \neq \bar{\Gamma}$ and $\Gamma^2 \subset N$, by (2) and the fact that N is normal. Since $|\bar{\Gamma} : \Gamma^2| = 4$, it follows that $N = \Gamma^2$.

(vi) N contains S, R but not T . Then $N \neq \bar{\Gamma}$ and $\Gamma^2 \subset N$, as N is normal and by (v). Since $|\bar{\Gamma} : \Gamma^2| = 4$, we have $N = \bar{\Gamma}$ or Γ or Γ^2 . But this is not possible since $T \in \bar{\Gamma}, T \in \Gamma$ and $R \notin \Gamma^2$.

(vii) N contains R but not T and S . This is not possible by (iii) and by (vi). \square

Theorem 3.3 *Let N be a non-trivial normal subgroup of finite index in $\bar{\Gamma}$ different from $\bar{\Gamma}, \Gamma, \Gamma^2, \Gamma^3$. Then N is a free group.*

Proof. It can be easily seen as an immediate consequence of the lemmas. □

Theorem 3.4 *Let N be a normal subgroup of finite index in $\bar{\Gamma}$ different from $\bar{\Gamma}, \Gamma, \Gamma^2, \Gamma^3$ such that $|\bar{\Gamma} : N| = \mu < \infty$. Then μ is divisible by 12.*

Proof. The quotient group contains subgroups of orders 2, 4 and 6, so its order is divisible by 12. □

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Recep ŞAHİN, Sebahattin İKİKARDEŞ,
Özden KORUOĞLU
Department of Mathematics,
Faculty of Arts and Sciences,
Balıkesir University,
10100 Balıkesir-TURKEY
e-mail : rsahin@balikesir.edu.tr

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