Commun.Fac.Sci.Univ.Ank.Series A1
Volume 56, Number 1, Pages 1–6 (2007)
ISSN 1303–5991

THE DISCRIMINANT OF SECOND FUNDAMENTAL FORM

BENGU KILIC

Abstract. In this study we consider the discriminant of the second fundamental form. As application we also give necessary condition for Vranceanu surface in $\mathbb{E}^4$ to have vanishing discriminant.

1. Introduction

Let M be an n-dimensional Riemannian manifolds. For the vector fields X, Y, Z on M the curvature tensor R of M is defined by

$$R(X,Y)Z = \bigtriangledown_X(\bigtriangledown_Y Z) - \bigtriangledown_Y(\bigtriangledown_X Z) - \bigtriangledown_{[X,Y]}Z \tag{1.1}$$

where $\bigtriangledown$ is the Levi-Civita connection of M, and $[\ ,\ ]$ is Lie parantheses operator.

Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$, the real number

$$K(\sigma) = \frac{g(\ R(X,Y)X,Y\ )}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1.2}$$

is called *the Sectional Curvature* of σ at point p, where X, Y is any basis of σ [1].

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. For all local formulas and computations, we may assume f as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \widetilde{M}$. The tangent space T_pM is identified with a subspace $f_*(T_pM)$ of $T_p\widetilde{M}$ where f_* is the differential map of f . Letters X, Y and Z (resp. ζ, μ and ξ) vector fields tangent (resp. normal) to M. We denote the tangent bundle of M (resp. $\widetilde{M}$) by TM (resp. $T\widetilde{M}$) , unit tangent bundle of M by UM and the normal bundle by $T^{\perp}M$. Let $\widetilde{\bigtriangledown}$ and $\bigtriangledown$ be the Levi-Civita connections of $\widetilde{M}$ and M , resp. Then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.3}$$

Received by the editors March 21, 2006; Rev.: Feb. 23, 2007; Accepted: Feb. 26, 2007.
2000 *Mathematics Subject Classification.* 53C40,
Key words and phrases. Second fundamental form, discriminant.

where h denotes the second fundamental form. If the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.4}$$

where A denotes the shape operator and D the normal connection. Clearly $h(X,Y) = h(Y,X)$ and A is related to h as $\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle$, where $\langle\ ,\rangle$ denotes the Riemannian metrics of M and $\widetilde{M}$.

For the second fundamental form, we define their covariant derivatives by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\bigtriangledown_X Y, Z) - h(Y, \bigtriangledown_X Z) \tag{1.5}$$

where X, Y, Z tangent vector fields over M and $\overline{\bigtriangledown}$ is the van der Waerden Bortolotti connection [1].The equation of Codazzi implie, that $\overline{\bigtriangledown} h$ is symmetric hence

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) = (\bar{\nabla}_Z h)(X,Y) \tag{1.6}$$

If $\bar{\nabla} h = 0$ then the second fundamental form of M is called *parallel* [7] (i.e. M is *1-parallel*) [4].

2. Discriminant of The Second Fundamental Form

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. The main invariant of the second fundamental form h is its discriminant Δ, (see [2]) the real valued function on the planes (through 0) in T_xM such that if the linearly independent tangent vectors X, Y span σ, then

$$\Delta_{XY} = \Delta(\sigma) = \frac{\langle h(X,X), h(Y,Y)\rangle - \|h(X,Y)\|^2}{\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2}. \tag{2.1}$$

For an isometric immersion $f : M \to \widetilde{M}$, the Gauss equation asserts that

$$K(\sigma) = \Delta(\sigma) + \widetilde{K}(df(\sigma)) \tag{2.2}$$

where K and $\widetilde{K}$ are the sectional curvatures of M and $\widetilde{M}$, and σ is any plane tangent to M [6].

If the vectors in T_xM are orthonormal then, the formula (2.1) reduces to

$$\Delta_{XY} = \langle h(X,X), h(Y,Y)\rangle - \|h(X,Y)\|^2 \tag{2.3}$$

Definition 2.1. We say that h is λ-*isotropic* provided that $\|h(X,X)\| = \lambda$ for all unit vectors X in T_xM . Clearly, an isometric immersion is isotropic provided that all its normal curvature vectors have the same length [5].

Lemma 2.2. [5]*Suppose that h is λ- isotropic on T_xM and let X, Y be orthonormal vectors in T_xM. Then*

$$\langle h(X,X), h(Y,Y)\rangle + 2\|h(X,Y)\|^2 = \lambda^2. \tag{2.4}$$

The assertation (2.1) in the preceding lemma yields the following result.

Lemma 2.3. [5] *If h is λ- isotropic then for orthonormal vectors X, Y in T_xM*
i) $\Delta_{XY} + 3\left\|h(X,Y)\right\|^2 = \lambda^2$.
ii) $2\Delta_{XY} + \lambda^2 = 3\left\langle h(X,X), h(Y,Y)\right\rangle$.

In the case of $\widetilde{M} = \mathbb{E}^{\,n+d}$ the sectional curvature $K(\sigma)$ of M reduces to

$$K(\sigma) = \Delta_{XY}. \tag{2.5}$$

Remark 2.4. Let K be a Gaussian curvature of the surface $M \subseteq \mathbb{E}^m$. Then $K = \Delta_{XY}$. If $\Delta_{XY} = 0$ then M is said to be *flat.*

Proposition 1. [7] *Let $f : M^2 \rightarrow \mathbb{E}^{2+d}$ be isometric immersion. If the second fundamental form of M^2 is 1-parallel (i.e. $\overline{\bigtriangledown} h = 0$) then $f(M^2)$ is one of the following surfaces*

i)$\mathbb{E}^2$
ii) $S^2 \subset \mathbb{E}^3$
iii) $IR^1 \times S^1 \subset \mathbb{E}^3$
iv) $S^1(a) \times S^1(b) \subset \mathbb{E}^4$
v) $V^2 \subset \mathbb{E}^5$.

Proposition 2. *Let M be a ruled surface of the form*

$$x(u,v) = \beta(u) + v\delta(u).$$

If $\Delta xy = 0$ (i.e M is flat) then M is one of the following surfaces;

i) a cone of the form $x(u,v) = p + v\delta(u)$ or,
ii) a cylinder of the form $x(u,v) = \beta(u) + vq$ or,
iii) a tangent developable surface of the form $x(u,v) = \beta(u) + v\beta'(u)$, $(v > 0)$.

Proof. (see [6]).
For more details for the following Examples see [3].

Example 2.5. For the following surfaces $K = \Delta_{XY} = 0$;
1) The torus T^2 embedded in $\mathbb{E}^4$ by

$$T^2 = \{(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) : \theta, \varphi \in IR\}$$

2) The helicel cylinder H^2 embedded in $\mathbb{E}^4$ by

$$H^2 = \{(u, c\cos v, c\sin v, dv) : u, v \in IR\}$$

3)The cylinder C embedded in $\mathbb{E}^3$ by

$$C = \{(a\cos s, a\sin s, t) : s, t \in IR\}.$$

Example 2.6. For the following surfaces $K = \Delta_{XY} \neq 0$;

1) The sphere S^2 embedded in $\mathbb{E}^3$ by

$$\begin{aligned} S^2 &= \{(a\cos s\cos t, a\cos s\sin t, a\sin s) : s,t \in IR\}, \\ \Delta_{XY} &= \frac{1}{a^2}. \end{aligned}$$

2)The helicoid H embedded in $\mathbb{E}^3$ by

$$\begin{aligned} H &= \{(s\cos t, s\sin t, at) : s,t \in IR\} \\ \Delta_{XY} &= -\frac{a^2}{(s^2+a^2)^2}. \end{aligned}$$

Proposition 3. *The Veronese surface parametrized by*

$$V^2 = \{\frac{1}{\sqrt{3}}yz, \frac{1}{\sqrt{3}}zx, \frac{1}{\sqrt{3}}xy, \frac{1}{2\sqrt{3}}(x^2-y^2), \frac{1}{6}(x^2+y^2-2z^2)\}$$

is spherical.

Proof. The parametric representation of V^2 defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x,y,z) and $(-x,-y,-z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane imbedded in $S^4(1)$ is called *the Veronese surface* [1] which is minimal in $S^4(1) \subset \mathbb{E}^5$.

A submanifolds (or immersion) is called *non-spherical* in the fact that it does not lie in a sphere.

Theorem 2.7. *Let $f : M^n \to \mathbb{E}^{n+d}$ be non-spherical isometric immersion. If M is 1-parallel then $\Delta_{XY} = 0$.*

Proof. Since $f(M)$ is not spherical therefore by Proposition 3 the possible non-spherical 1-parallel surfaces are cylinder $IR^1 \times S^1 \subset \mathbb{E}^3$and torus $S^1(a)\times S^1(b) \subset \mathbb{E}^4$. On the other hand , both of them have vanishing sectional curvature.

Definition 2.8. The Vranceanu surface is defined by the parametrized

$$x(s,t) = \{u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t\}. \tag{2.6}$$

Theorem 2.9. *Let the Vranceanu surface is given by the parametrized (2.6) . The Vranceanu surface has vanishing Gaussian curvature ($K = \Delta_{XY} = 0$) if and only if $(u')^2 - uu'' = 0$ (i.e. $u = Ce^{ks}$ for the real constant $0 \neq C$ and k).*

Proof. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as given by the following

$$\begin{aligned} e_1 &= (-\cos s \sin t, \cos s \cos t, -\sin s \sin t, \sin s \cos t) \\ e_2 &= \frac{1}{A}(B \cos t, B \sin t, C \cos t, C \sin t) \\ e_3 &= \frac{1}{A}(-C \cos t, -C \sin t, B \cos t, B \sin t) \\ e_4 &= (-\sin s \sin t, \sin s \cos t, \cos s \sin t, -\cos s \cos t) \end{aligned} \tag{2.7}$$

where we put $A = \sqrt{u^2 + (u')^2}, B = u' \cos s - u \sin s, C = u' \sin s + u \cos s$. Then we have

$$e_1 = \frac{1}{u}\frac{\partial}{\partial t},\ e_2 = \frac{1}{A}\frac{\partial}{\partial s}. \tag{2.8}$$

Then the structure equations of $\mathbb{E}^m$ are obtained as follows:

$$\widetilde{\nabla}_{e_i} e_j = w_j^k(e_i) e_k + h_{ij}^\alpha e_\alpha,\ 1 \leq i, j, k \leq 2 \tag{2.9}$$

$$\begin{aligned} \widetilde{\nabla}_{e_i} e_\alpha &= -h_{ij}^\alpha e_j + w_\alpha^\beta(e_i) e_\beta,\ 3 \leq \alpha, \beta \leq 4 \\ D_{e_\alpha} e_\beta &= w_\alpha^\beta(e_i) e_\beta \end{aligned} \tag{2.10}$$

where D is the normal connection and h_{ij}^α the coefficients of the second fundamental form h. Using (2.7), (2.8), (2.9) and (2.10) we can get that the coefficients of the second fundamental form h and the connection form w_B^A are as following:

$$\begin{aligned} h_{11}^3 &= \frac{1}{\sqrt{u^2 + (u')^2}} = \alpha,\ h_{12}^3 = h_{21}^3 = 0 \\ h_{22}^3 &= \frac{2(u')^2 - uu'' + u^2}{(u^2 + (u')^2)^{3/2}} = \beta \\ h_{11}^4 &= h_{22}^4 = 0,\ h_{12}^4 = h_{21}^4 = -\frac{1}{\sqrt{u^2 + (u')^2}}. \end{aligned}$$

The Gauss curvature is given by

$$\begin{aligned} K &= \det(h_{ij}^3) + \det(h_{ij}^4),\ 1 \leq i, j \leq 2 \\ &= \frac{(u')^2 - uu''}{(u^2 + (u')^2)^2}. \end{aligned} \tag{2.11}$$

Thus this completes the proof of the theorem.

ÖZET: Bu çalışmada, ikinci temel formun diskriminantı gözönünde bulunduruldu. $\mathbb{E}^4$ de Vranceanu yüzeyinin sıfır diskriminantlı olması için gerekli koşul verildi.

References

[1] B.Y.Chen, *Geometry of Submanifolds,* (1973).
[2] S.Fueki, *Helices and Isometric Immersions*, Tsukuba J.Math.Vol 22, No 2 (1998),427-445.
[3] B.Kilic, *Sonlu Tipte Egriler ve Yüzeyler,* PhD Thesis, Hacettepe Üniversitesi, (2002).
[4] U. Lumiste, *Submanifolds with parallel fundamental form,* Handbook of Differential Geometry , (1999),Vol 1, Chapter 7, 86 p.
[5] B. O,Neill, *Isotropic and Kahler Immersions*, Canadian J. Math. 17(1965), 907-915.
[6] B. O,Neill, *Elementary Differential Geometry,*(1966).
[7] R.Walden, *Untermannigfaltigkeiten mit Paralleler Zweiter Fundamentalform in Euklidischen Räumen und Sphären*, Manuscripta Math.10,91-102 (1973).

Current address: Balikesir University,Faculty of Art and Sciences, Department of Mathematics, Balikesir, TURKEY.
E-mail address: benguk@balikesir.edu.tr
URL: http://math.science.ankara.edu.tr