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A NEW EXTENSION OF THE M_b-METRIC SPACES

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Abstract. In this paper, we present a new notion which is called an extended M_b-metric space as a generalization of an M_b-metric space. We investigate some basic and topological properties of this new space. Furthermore, an extended M_b-metric space is a new generalization of an M-metric space and partial metric space. So it is important to study fixed-point theorems for non-M-metric (or non-partial metric) functions on an extended M_b-metric space. Also we generalize some known results in literature.

1. Introduction and Preliminaries

An M-metric space was introduced by Asadi in [2], which is an extension of partial metric spaces, for more on $M-$metric spaces see [21]. $b-$metric spaces was introduced as a generalization of metric spaces see [22], [23], [24], [25], [26], [27]. Some relationships between a partial metric and an M-metric were investigated in [1]. So, first we remind the reader of the definition of a partial metric space and an M-metric space along with some other notationtions.

Definition 1.1. [9] [15] *A partial metric on a nonempty set X is a function $p : X^2 \to [0,\infty)$ such that for all $x, y, z \in X$*

($p1$) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$,
($p2$) $p(x,x) \le p(x,y)$,
($p3$) $p(x,y) = p(y,x)$,
($p4$) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Notation 1.2. [2]

1. $m_{x,y} := min\{m(x,x), m(y,y)\}$
2. $M_{x,y} := max\{m(x,x), m(y,y)\}$

Definition 1.3. [2] *Let X be a nonempty set. If the function $m : X^2 \to [0,\infty)$ satisfies the following conditions*

(1) $m(x,x) = m(y,y) = m(x,y)$ *if and only if* $x = y$,
(2) $m_{x,y} \le m(x,y)$,
(3) $m(x,y) = m(y,x)$,

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(4) $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$,
for all $x, y, z \in X$, then the pair (X, m) is called an M-metric space.

Recently, Mlaiki et al. [10], developed the concept of an M_b-metric space which extends an M-metric space and some fixed point theorems are established which was also a generalization of $b-$metric spaces see [18], [19], and [20]. Now, we remind the reader of some definitions and notationtions of M_b-metric spaces.

Notation 1.4. [10]
1. $m_{bx,y} := min\{m_b(x,x), m_b(y,y)\}$
2. $M_{bx,y} := max\{m_b(x,x), m_b(y,y)\}$

Definition 1.5. [10] *An M_b-metric on a nonempty set X is a function $m_b : X^2 \to [0,\infty)$ that satisfies the following conditions*
(1) $m_b(x,x) = m_b(y,y) = m_b(x,y)$ *if and only if* $x = y$,
(2) $m_{bx,y} \leq m_b(x,y)$,
(3) $m_b(x,y) = m_b(y,x)$,
(4) *There exists a real number $s \geq 1$ such that for all $x, y, z \in X$ we have*

$$(m_b(x,y) - m_{bx,y}) \leq s[(m_b(x,z) - m_{bx,z}) + (m_b(z,y) - m_{bz,y})] - m_b(z,z),$$

for all $x, y, z \in X$. Then the pair (X, m_b) is called an M_b-metric space and the number s is called the coefficient of the M_b-metric space (X, m_b).

We note that the condition (4) given in Definition 1.5 is equivalent to the following condition:
$(4)'$ There exists a real number $s \geq 1$ such that for all $x, y, z \in X$ we have

$$(m_b(x,y) - m_{bx,y}) \leq s[(m_b(x,z) - m_{bx,z}) + (m_b(z,y) - m_{bz,y})],$$

for all $x, y, z \in X$.

Indeed, if we take $x = y$ in the condition (4) then we get

$$m_b(x,x) - m_{bx,x} = m_b(x,x) - \min\{m_b(x,x), m_b(x,x)\} = 0$$

and so we have

$$0 \leq s[(m_b(x,x) - m_{bx,x}) + (m_b(x,x) - m_{bx,x})] - m_b(x,x) \leq -m_b(x,x),$$

for $z = x$. Therefore we get $m_b(x,x) = 0$ for all $x \in X$ since $m_b(x,x) \in [0,\infty)$.

Motivated by the above studies, in this paper we introduce the notion of an extended M_b-metric space and prove some fixed-point results on this new space. In Section 2, we investigate some basic properties of this space and determine the relationships between an extended M_b-metric space and some known metric spaces. In Section 3, we give some topological notions on an extended M_b-metric space. In Section 4, we prove some fixed-point results on an extended M_b-metric space using the techniques of the classical fixed-point theorems such as the Banach's contraction principle, Kannan's fixed-point results etc.

2. Extended M_b-Metric Spaces

In this section, we introduce the concept of an extended M_b-metric space, which is a generalization of an M_b-metric space. We give basic properties of this new space and its relation with some known metric spaces.

First, we give the following notationtion.

Notation 2.1.
(1) $m_{\theta x,y} := min\{m_\theta(x,x), m_\theta(y,y)\}$
(2) $M_{\theta x,y} := max\{m_\theta(x,x), m_\theta(y,y)\}$

Definition 2.2. *Let $\theta : X^2 \to [1,\infty)$ be a function. An extended M_b-metric on a nonempty set X is a function $m_\theta : X^2 \to [0,\infty)$ satisfying the following conditions*
(1) $m_\theta(x,x) = m_\theta(y,y) = m_\theta(x,y)$ *if and only if* $x = y$,
(2) $m_{\theta x,y} \leq m_\theta(x,y)$,
(3) $m_\theta(x,y) = m_\theta(y,x)$,
(4) $(m_\theta(x,y) - m_{\theta x,y}) \leq \theta(x,y)[(m_\theta(x,z) - m_{\theta x,z}) + (m_\theta(z,y) - m_{\theta z,y})]$,
for all $x, y, z \in X$. Then the pair (X, m_θ) is called an extended M_b-metric space.

We note that if $\theta(x,y) = s$ for $s \geq 1$, then we get the definition of an M_b-metric space.

Example 2.3. *Let $X = C([a,d],\mathbb{R})$ be the set of all continuous real valued functions on $[a,b]$. We define the functions $m_\theta : X^2 \to [0,\infty)$ and $\theta : X^2 \to [1,\infty)$ by*
$$m_\theta(x(t), y(t)) = \sup_{t\in[a,b]} |x(t) - y(t)|^2,$$
and
$$\theta(x(t), y(t)) = |x(t)| + |y(t)| + 2.$$
Then (X, m_θ) is an extended M_b-metric space with the function θ.

Now we give the following proposition.

Proposition 2.4. *Let (X, m_θ) be an extended M_b-metric space and $x, y, z \in X$. Then we have*
(1) $M_{\theta x,y} + m_{\theta x,y} = m_\theta(x,x) + m_\theta(y,y) \geq 0$,
(2) $M_{\theta x,y} - m_{\theta x,y} = |m_\theta(x,x) - m_\theta(y,y)| \geq 0$,
(3) $M_{\theta x,y} - m_{\theta x,y} \leq \theta(x,y)\left[(M_{\theta x,z} - m_{\theta x,z}) + (M_{\theta z,y} - m_{\theta z,y})\right]$.

Proof. (1) Let $m_\theta(x,x) \geq m_\theta(y,y)$. Then we get $M_{\theta x,y} = m_\theta(x,x)$ and $m_{\theta x,y} = m_\theta(y,y)$ and so
$$M_{\theta x,y} + m_{\theta x,y} = m_\theta(x,x) + m_\theta(y,y) \geq 0.$$
On the other hand, if $m_\theta(x,x) \leq m_\theta(y,y)$, then the condition (1) follows by similar arguments used above.

(2) By the similar argument used in the proof of the condition (1), we can see the desired result.

(3) Let $m_\theta(x,x) > m_\theta(y,y)$. Then we get $M_{\theta x,y} = m_\theta(x,x)$ and $m_{\theta x,y} = m_\theta(y,y)$. Also, assume that
$$m_\theta(y,y) < m_\theta(z,z) < m_\theta(x,x).$$
Therefore, we obtain
$$\begin{aligned} m_\theta(x,x) - m_\theta(y,y) &\leq \theta(x,y)\left[(m_\theta(x,x) - m_\theta(z,z)) + (m_\theta(z,z) - m_\theta(y,y))\right] \\ &= \theta(x,y)\left[m_\theta(x,x) - m_\theta(y,y)\right]. \end{aligned}$$
Since $\theta(x,y) \geq 1$, the condition (3) is satisfied in this case. For other cases, it can be easily checked that the condition (3) is satisfied. □

Also, the notion of an extended b-metric was introduced as a generalization of a b-metric space in [7]. Now we recall the following definitions and an example related to an extended b-metric space.

Definition 2.5. [7] *Let X be a nonempty set and $\theta : X^2 \to [1,\infty)$ be a function. An extended b-metric is a function $d_\theta : X^2 \to [0,\infty)$ satisfying the following conditions*
($d_\theta 1$) $d_\theta(x,y) = 0$ *if and only if* $x = y$,
($d_\theta 2$) $d_\theta(x,y) = d_\theta(y,x)$,
($d_\theta 3$) $d_\theta(x,z) \le \theta(x,z)\,[d_\theta(x,y) + d_\theta(y,z)]$,
for all $x,y,z \in X$. Then the pair (X,d_θ) is called an extended b-metric space.

If $\theta(x,y) = s$ for $s \ge 1$ then it is obtained the definition of a b-metric space given in [3].

Definition 2.6. [7] *Let (X,d_θ) be an extended b-metric space. Then we have*
(1) *A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x) < \varepsilon$ for all $n \ge n_0$. It is denoted by*

$$\lim_{n\to\infty} x_n = x.$$

(2) *A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x_m) < \varepsilon$ for all $n,m \ge n_0$.*
(3) *X is complete if every Cauchy sequence in X is convergent.*

Notice that a b-metric function is not always continuous and so an extended b-metric function is not always continuous as seen in the following example.

Example 2.7. [6] *Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X^2 \to [0,\infty)$ be a function defined as*

$$d(x,y) = \begin{cases} 0 & \text{if} \quad m = n \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if} \quad m,n \text{ are even or } mn = \infty \\ 5 & \text{if} \quad m,n \text{ are odd and } m \ne n \\ 2 & \text{otherwise} \end{cases}.$$

Then (X,d) be a b-metric space with $s = 3$ but it is not continuous.

Remark. *Every extended M_b-metric is not continuous.*

In the following proposition, we see the relationship between an extended b-metric and an extended M_b-metric.

Proposition 2.8. *Let (X,m_θ) be an extended M_b−metric space and $m^b_\theta : X^2 \to [0,\infty)$ be a function defined as*

$$m^b_\theta(x,y) = m_\theta(x,y) - 2m_{\theta x,y} + M_{\theta x,y},$$

for all $x,y \in X$. Then m^b_θ is an extended b-metric and the pair (X,m^b_θ) is an extended b-metric space.

Proof. ($d_\theta 1$) Using the conditions (1) and (2) given in Definition 2.2, we have

$$\begin{aligned} m^b_\theta(x,y) &= 0 \Leftrightarrow m_\theta(x,y) - 2m_{\theta x,y} + M_{\theta x,y} = 0 \\ &\Leftrightarrow m_\theta(x,y) = 2m_{\theta x,y} - M_{\theta x,y} \end{aligned}$$

and

$$\begin{aligned} m_{\theta x,y} &\le m_\theta(x,y) = 2m_{\theta x,y} - M_{\theta x,y} \Leftrightarrow M_{\theta x,y} \le m_\theta(x,y) \Leftrightarrow M_{\theta x,y} = m_\theta(x,y) \\ &\Leftrightarrow m_\theta(x,x) = m_\theta(y,y) = m_\theta(x,y) \Leftrightarrow x = y. \end{aligned}$$

($d_\theta 2$) From the condition (3) given in Definition 2.2, it can be easily seen

$$m^b_\theta(x,y) = m^b_\theta(y,x).$$

($d_\theta 3$) Using the condition (4) given in Definition 2.2 and the inequality (3) given in Proposition 2.4, we obtain

$$\begin{aligned} m_\theta^b(x,y) &= m_\theta(x,y) - 2m_{\theta x,y} + M_{\theta x,y} = (m_\theta(x,y) - m_{\theta x,y}) + (M_{\theta x,y} - m_{\theta x,y}) \\ &\leq \theta(x,y)[(m_\theta(x,z) - m_{\theta x,z}) + (m_\theta(z,y) - m_{\theta z,y})] + (M_{\theta x,y} - m_{\theta x,y}) \\ &\leq \theta(x,y)[(m_\theta(x,z) - m_{\theta x,z}) + (m_\theta(z,y) - m_{\theta z,y})] \\ &\quad + \theta(x,y)\left[(M_{\theta x,z} - m_{\theta x,z}) + (M_{\theta z,y} - m_{\theta z,y})\right] \\ &= \theta(x,y)\left[m_\theta^b(x,z) + m_\theta^b(z,y)\right]. \end{aligned}$$

Consequently, m_θ^b is an extended b-metric and the pair (X, m_θ^b) is an extended b-metric space. □

Proposition 2.9. *Let (X, m_θ) be an extended M_b-metric space and $x, y \in X$. Then we have*

$$m_\theta(x,y) - M_{\theta x,y} \leq m_\theta^b(x,y) \leq m_\theta(x,y) + M_{\theta x,y}.$$

Proof. By Proposition 2.8, the proof follows easily. □

In the following propositions, we see the relationship between an extended M_b-metric space and an M_b-metric space (resp. a partial metric space).

Proposition 2.10. *Let (X, m_θ) be an extended M_b-metric space and $\theta : X^2 \to [1, \infty)$ be a function defined as*

$$\theta(x,y) = 1,$$

for all $x, y \in X$. Then m_θ is an M-metric.

Proof. By the conditions (1), (2) and (3) given in Definition 2.2, we can easily seen that the condition (1), (2) and (3) given in Definition 1.3. From the condition (4) given in Definition 2.2, we get

$$\begin{aligned} (m_\theta(x,y) - m_{\theta x,y}) &\leq \theta(x,y)[(m_\theta(x,z) - m_{\theta x,z}) + (m_\theta(z,y) - m_{\theta z,y})] \\ &= (m_\theta(x,z) - m_{\theta x,z}) + (m_\theta(z,y) - m_{\theta z,y}). \end{aligned}$$

Consequently, an extended M_b-metric m_θ is an M-metric. □

Proposition 2.11. *Let (X, p) be a partial metric space. Then the partial metric p is an extended M_b-metric.*

Proof. (1) It can be easily proved by the condition ($p1$).

(2) Using the condition ($p2$), we have

$$p(x,x) \leq p(x,y)$$

and

$$p_{x,y} = \min\{p(x,x), p(y,y)\} \leq p(x,x) \leq p(x,y),$$

for all $x, y \in X$.

(3) It follows easily from the condition ($p3$).

(4) We get the following cases:

1.	$p(x,x) = p(y,y) = p(z,z)$,
2.	$p(x,x) < p(y,y) < p(z,z)$,
3.	$p(x,x) = p(y,y) < p(z,z)$,
4.	$p(x,x) = p(y,y) > p(z,z)$,
5.	$p(x,x) < p(y,y) = p(z,z)$,
6.	$p(x,x) > p(y,y) = p(z,z)$.

Under the above cases, the condition (4) given in Definition 2.2 is satisfied. For example, if we consider the case 2, we obtain

$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z) \leq p(x,z) + p(z,y) - p(y,y)$$

and so

$$\begin{aligned} p(x,y) - p_{x,y} &\leq p(x,y) - p(x,x) \leq p(x,z) - p(x,x) + p(z,y) - p(y,y) \\ &\leq \theta(x,y)\left[p(x,z) - p(x,x) + p(z,y) - p(y,y)\right] \\ &\leq \theta(x,y)\left[(p(x,z) - p_{x,z}) + (p(z,y) - p_{z,y})\right], \end{aligned}$$

for all $x, y, z \in X$.

Consequently, the partial metric p is an extended M_b-metric. □

Example 2.12. *Let $X = \{1,2,3\}$ and the function $\theta : X^2 \to [1,\infty)$ be defined by*

$$\theta(x,y) = xy,$$

for all $x, y \in X$. Let us define the function $m_\theta : X^2 \to [0,\infty)$ as

$$\begin{aligned} m_\theta(1,1) &= m_\theta(2,2) = m_\theta(3,3) = 1, \\ m_\theta(1,2) &= m_\theta(2,1) = 6, \\ m_\theta(1,3) &= m_\theta(3,1) = 4, \\ m_\theta(2,3) &= m_\theta(3,2) = 2, \end{aligned}$$

for all $x, y \in X$. Then m_θ is an extended M_b-metric, but neither it is an M-metric nor a partial metric. Indeed, for $x = 1$, $y = 2$, $z = 3$, we have

$$m_\theta(1,2) - m_{\theta 1,2} = 5 \leq \left[(m_\theta(1,3) - m_{\theta 1,3}) + (m_\theta(3,2) - m_{\theta 3,2})\right] = 4$$

and

$$m_\theta(1,2) = 6 \leq m_\theta(1,3) + m_\theta(3,2) - m_\theta(3,3) = 5,$$

which is a contradiction. Therefore, the condition (4) given in Definition 1.3 and the condition (p4) are not satisfied, respectively.

3. Topological Structure of Extended M_b-Metric Spaces

In this section, we give some topological notions on an extended M_b-metric space.

Definition 3.1. *Let (X, m_θ) be an extended M_b-metric space. Then*

(1) *A sequence $\{x_n\}$ in X converges to a point x if and only if*

$$\lim_{n\to\infty} (m_\theta(x_n, x) - m_{\theta x_n, x}) = 0.$$

(2) *A sequence $\{x_n\}$ in X is said to be m_θ-Cauchy sequence if*

$$\lim_{n,m\to\infty} (m_\theta(x_n, x_m) - m_{\theta x_n, x_m})$$

and

$$\lim_{n\to\infty} (M_{\theta x_n, x_m} - m_{\theta x_n, x_m})$$

exist and finite.

(3) *An extended M_b-metric space is said to be m_θ-complete if every m_θ-Cauchy sequence $\{x_n\}$ converges to a point x such that*

$$\lim_{n\to\infty} (m_\theta(x_n, x) - m_{\theta x_n, x}) = 0$$

and

$$\lim_{n\to\infty} (M_{\theta x_n, x} - m_{\theta x_n, x}) = 0.$$

Remark. *If we consider Example 2.3, then it is not difficult to see that,* (X, m_θ) *is a complete extended* M_b*-metric space.*

Lemma 3.2. *Let* (X, m_θ) *be an extended* M_b*-metric space. Then we get*

(1) $\{x_n\}$ *is an* m_θ*-Cauchy sequence in* (X, m_θ) *if and only if* $\{x_n\}$ *is a Cauchy sequence in* (X, m_θ^b).

(2) (X, m_θ) *is complete if and only if* (X, m_θ^b) *is complete.*

Proof. Using Proposition 2.8, the proof follows easily. □

Lemma 3.3. *Let* (X, m_θ) *be an extended* M_b*-metric space. If the sequence* $\{x_n\}$ *in* X *converges to two points* x *and* y *with* $x \neq y$*, then we have* $m_\theta(x, y) - m_{\theta x,y} = 0$.

Proof. Let $\{x_n\}$ converges to two points x and y with $x \neq y$. Then we get

$$\lim_{n\to\infty}(m_\theta(x_n, x) - m_{\theta x_n,x}) = 0$$

and

$$\lim_{n\to\infty}(m_\theta(x_n, y) - m_{\theta x_n,y}) = 0.$$

Using the conditions (3) and (4) given in Definition 2.2, we obtain

$$\begin{aligned} m_\theta(x, y) - m_{\theta x,y} &\leq \theta(x, y)[(m_\theta(x, x_n) - m_{\theta x,x_n}) + (m_\theta(x_n, y) - m_{\theta x_n,y})] - m_\theta(x_n, x_n) \\ &\leq \theta(x, y)[(m_\theta(x, x_n) - m_{\theta x,x_n}) + (m_\theta(x_n, y) - m_{\theta x_n,y})] \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}[m_\theta(x, y) - m_{\theta x,y}] &\leq \lim_{n\to\infty}\theta(x, y)[\lim_{n\to\infty}(m_\theta(x, x_n) - m_{\theta x,x_n}) \\ &\quad + \lim_{n\to\infty}(m_\theta(x_n, y) - m_{\theta x_n,y})]. \end{aligned}$$

Therefore, we get $m_\theta(x, y) - m_{\theta x,y} = 0$ by the condition (2) given in Definition 2.2. □

As seen in the proof of Lemma 3.3, the limit of a sequence is not to be unique. Then we give the following lemma.

Lemma 3.4. *Let* (X, m_θ) *be an extended* M_b*-metric space. If* m_θ *is a continuous function then every convergent sequence has a unique limit.*

We use the following lemma in the next section.

Lemma 3.5. *Let* (X, m_θ) *be an extended* M_b*-metric space such that* m_θ *is continuous and* T *be a self mapping on* X*. If there exists* $k \in [0, 1)$ *such that*

$$m_\theta(Tx, Ty) \leq k m_\theta(x, y) \text{ for all } x, y \in X, \quad (\bigstar)$$

then the sequence $\{x_n\}_{n\geq 0}$ *is defined by* $x_{n+1} = Tx_n$*. If* $x_n \to u$ *as* $n \to \infty$*, then* $Tx_n \to Tu$ *as* $n \to \infty$,

Proof. First, note that if $m_\theta(Tx_n, Tu) = 0$, then $m_{\theta Tx_n,Tu} = 0$ and that is due to the fact that $m_{\theta Tx_n,Tu} \leq m_\theta(Tx_n, Tu)$, which implies that

$$m_\theta(Tx_n, Tu) - m_{\theta Tx_n,Tu} \to 0 \text{ as } n \to \infty \text{ and that is } Tx_n \to Tu \text{ as } n \to \infty.$$

So, we may assume that $m_\theta(Tx_n, Tu) > 0$, since by $(\bigstar)$ we have $m_\theta(Tx_n, Tu) < m_\theta(x_n, u)$, then we have the following two cases:

Case 1: If $m_\theta(u, u) \leq m_\theta(x_n, x_n)$, then it is easy to see that $m_\theta(x_n, x_n) \to 0$, which implies that $m_\theta(u, u) = 0$ and since $m_\theta(Tu, Tu) < m_\theta(u, u) = 0$ we deduce that $m_\theta(Tu, Tu) = m_\theta(u, u) = 0$, and $m_\theta(x_n, u) \to 0$, on the other words we have

$m_\theta(Tx_n, Tu) \le m_\theta(x_n, u) \to 0$. Hence, $m_\theta(Tx_n, Tu) - m_{\theta Tx_n,Tu} \to 0$ and thus $Tx_n \to Tu$.

Case 2: If $m_\theta(u, u) \ge m_\theta(x_n, x_n)$, and once again it is easy to see that $m(x_n, x_n) \to 0$, which implies that $m_{\theta x_n,u} \to 0$. Hence, $m_\theta(x_n, u) \to 0$ and since $m_\theta(Tx_n, Tu) < m_\theta(x_n, u) \to 0$ then we have $m_\theta(Tx_n, Tu) - m_{\theta Tx_n,Tu} \to 0$ and thus $Tx_n \to Tu$ as desired. □

Finally, we define the following topological concepts.

Definition 3.6. *Let (X, m_θ) be an extended M_b-metric space. For $\varepsilon > 0$ and $x \in X$, the open ball $B(x, \varepsilon)$ and the closed ball $B[x, \varepsilon]$ are defined as follows:*

$$B(x, \varepsilon) = \{y \in X \mid m_\theta(x, y) - m_{\theta x,y} < \varepsilon\}$$

and

$$B[x, \varepsilon] = \{y \in X \mid m_\theta(x, y) - m_{\theta x,y} \le \varepsilon\},$$

respectively.

Definition 3.7. *Let (X, m_θ) be an extended M_b-metric space and $A \subset X$. If there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$ for all $x \in A$, then A is called an open subset of X.*

Definition 3.8. *Let (X, m_θ) and (Y, m^*_θ) be two extended M_b-metric spaces and $T : X \to Y$ be a function. Then T is continuous at $x \in X$ if and only if $\{Tx_n\}$ converges to a point Tx whenever $\{x_n\}$ converges to a point x.*

4. Fixed-Point Theorems on Extended M_b-Metric Spaces

In this section, we prove some fixed-point theorems on a complete extended M_b-metric space. Using the technique of the Banach's contraction principle [4], we obtain the following theorem.

Theorem 4.1. *Let (X, m_θ) be a complete extended M_b-metric space such that m_θ is continuous and T be a self mapping on X satisfy the following condition:*

$$m_\theta(Tx, Ty) \le k m_\theta(x, y), \quad (\blacklozenge)$$

for all $x, y \in X$ where $0 \le k < 1$ be such that $\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0) < \frac{1}{k}$ for every $x_0 \in X$. Then T has a unique fixed point say u. Also we have $\lim_{n\to\infty} T^n y = u$ for every $y \in X$. Moreover, we get $m_\theta(u, u) = 0$.

Proof. Since X is a nonempty set, consider $x_0 \in X$ and define the sequence $\{x_n\}$ as follow:

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_n = T^n x_0, \cdots$$

By using $(\blacklozenge)$ we obtain

$$m_\theta(x_n, x_{n+1}) \le k m_\theta(x_{n-1}, x_n) \le \cdots \le k^n m_\theta(x_0, x_1).$$

Now, consider two natural numbers $n < m$. Thus, by the triangle inequality of the extended M_b-metric space we deduce

$$\begin{aligned} m_\theta(x_n, x_m) - m_{\theta x_n,x_m} &\le \theta(x_n, x_m)(k)^n m_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)(k)^{n+1} m_\theta(x_0, x_1) \\ &+ \cdots + \theta(x_n, x_m) \cdots \theta(x_{m-1}, x_m)(k)^{m-1} m_\theta(x_0, x_1) \\ &\le m_\theta(x_0, x_1)[\theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)(k)^n \\ &+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m)(k)^{n+1} \\ &+ \cdots + \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)(k)^{m-1}]. \end{aligned}$$

It is not difficult to see that

$$\lim_{n,m\to\infty}\theta(x_n,x_m)(k)<1.$$

Hence, by the Ratio test the series $\sum_{n=1}^{\infty}(k)^n\prod_{i=1}^{n}\theta(x_i,x_m)$ converges. Let

$$B=\sum_{n=1}^{\infty}(k)^n\prod_{i=1}^{n}\theta(x_i,x_m) \text{ and } B_n=\sum_{j=1}^{n}(k)^j\prod_{i=1}^{j}\theta(x_i,x_m).$$

Thus, for $m>n$ we deduce that

$$m_\theta(x_n,x_m)-m_{\theta x_n,x_m}\le m_\theta(x_0,x_1)[B_{m-1}-B_n].$$

Taking the limit as $n,m\to\infty$, we conclude that

$$\lim_{n,m\to\infty}(m_\theta(x_n,x_m)-m_{\theta x_n,x_m})=0.$$

On the other hand, without loss of generality we may assume that

$$M_{\theta x_n,x_m}=m_\theta(x_n,x_n).$$

Hence, we obtain

$$\begin{aligned}M_{\theta x_n,x_m}-m_{\theta x_n,x_m}&\le M_{\theta x_n,x_m}\\&\le m_\theta(x_n,x_n)\\&\le km_\theta(x_{n-1},x_{n-1})\\&\le\cdots\\&\le k^n m_\theta(x_0,x_0).\end{aligned}$$

Taking the limit of the above inequality as $n\to\infty$ we deduce that

$$\lim_{n\to\infty}(M_{\theta x_n,x_m}-m_{\theta x_n,x_m})=0.$$

Therefore, $\{x_n\}$ is an m_θ-Cauchy sequence. Since X is m_θ-complete, hence $\{x_n\}$ converges to some $u\in X$.

Now, we show that $Tu=u$. By Lemma 3.5, we have for any natural number n

$$\begin{aligned}\lim_{n\to\infty}m_b(x_n,u)-m_{bx_n,u}&=0\\&=\lim_{n\to\infty}m_b(x_{n+1},u)-m_{bx_{n+1},u}\\&=\lim_{n\to\infty}m_b(Tx_n,u)-m_{bTx_n,u}\\&=m_b(Tu,u)-m_{bTu,u}.\end{aligned}$$

Hence, we find

$$m_b(Tu,u)=m_{bu,Tu}.$$

Note that, since $m_\theta(Tx,Ty)\le km_\theta(x,y)$ for all $x,y\in X$ then we have

$$M_{\theta x_n,Tx_n}=m_\theta(x_n,x_n)\le km_\theta(x_{n-1},x_{n-1})\le\cdots\le k^n m_\theta(x_0,x_0).$$

Taking the limit of the above inequality as $n\to\infty$ we conclude that $M_{\theta u,Tu}=0$, and that leads us to conclude the following:

$$m_\theta(Tu,u)=m_{\theta u,Tu}\le M_{\theta u,Tu}=0$$

and that implies that $Tu = u$. To show the uniqueness of the fixed point u, first we show that if u is a fixed point, then $m_\theta(u, u) = 0$, assume that u is a fixed point of T, hence

$$\begin{aligned} m_\theta(u, u) &= m_\theta(Tu, Tu) \\ &\leq k m_\theta(u, u) \\ &< m_\theta(u, u) \;\; \text{since } k \in [0, 1), \end{aligned}$$

thus $m_\theta(u, u) = 0$. Now,assume that T has two fixed points $u \neq v \in X$, that is, $Tu = u$ and $Tv = v$. Thus,

$$m_\theta(u, v) = m_\theta(Tu, Tv) \leq k m_\theta(u, v) < m_\theta(u, v),$$

which implies that $m_\theta(u, v) = 0$, and hence $u = v$ as desired. Therefore, T has a unique fixed point $u \in X$ such that $m_\theta(u, u) = 0$ as desired. □

In the following theorem, we extend the classical Kannan's fixed-point result [8] using appropriate condition defined on a complete extended M_b-metric space.

Theorem 4.2. *Let (X, m_θ) be a complete extended M_b-metric space such that m_θ is continuous and T be a continuous self mapping on X satisfy the following condition*:

$$m_\theta(Tx, Ty) \leq \lambda[m_\theta(x, Tx) + m_\theta(y, Ty)], \quad (\blacktriangle)$$

for all $x, y \in X$ where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point u such that $m_\theta(u, u) = 0$.

Proof. Let $x_0 \in X$ be an arbitrary point. Consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ and $m_{\theta_n} = m_\theta(x_n, x_{n+1})$. Note that if there exists a natural number n such that $x_{n+1} = x_n$, then x_n is a fixed point of T and we are done. Assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. By $(\blacktriangle)$, we obtain for any $n \geq 0$,

$$\begin{aligned} m_{\theta_n} &= m_\theta(x_n, x_{n+1}) = m_\theta(Tx_{n-1}, Tx_n) \\ &\leq \lambda[m_\theta(x_{n-1}, Tx_{n-1}) + m_\theta(x_n, Tx_n)] \\ &= \lambda[m_\theta(x_{n-1}, x_n) + m_\theta(x_n, x_{n+1})] \\ &= \lambda[m_{\theta_{n-1}} + m_{\theta_n}]. \end{aligned}$$

Hence, $m_{\theta_n} \leq \lambda m_{\theta_{n-1}} + \lambda m_{\theta_n}$, which implies $m_{\theta_n} \leq \mu m_{\theta_{n-1}}$, where $\mu = \frac{\lambda}{1-\lambda} < 1$ as $\lambda \in \left[0, \frac{1}{2}\right)$. By repeating this process, we obtain

$$m_{\theta_n} \leq \mu^n m_{\theta_0}.$$

Thus, $\lim_{n\to\infty} m_{\theta_n} = 0$. By $(\blacktriangle)$, for all natural numbers n, m we have

$$\begin{aligned} m_\theta(x_n, x_m) &= m_\theta(T^n x_0, T^m x_0) = m_\theta(Tx_{n-1}, Tx_{m-1}) \\ &\leq \lambda[m_\theta(x_{n-1}, Tx_{n-1}) + m_\theta(x_{m-1}, Tx_{m-1})] \\ &= \lambda[m_\theta(x_{n-1}, x_n) + m_\theta(x_{m-1}, x_m)] \\ &= \lambda[m_{\theta_{n-1}} + m_{\theta_{m-1}}]. \end{aligned}$$

As $\lim_{n\to\infty} m_{\theta_n} = 0$, for every $\varepsilon > 0$ we can find a natural number n_0 such that $m_{\theta_n} < \frac{\varepsilon}{2}$ and $m_{\theta_m} < \frac{\varepsilon}{2}$ for all $m, n > n_0$. Therefore, it follows that

$$m_\theta(x_n, x_m) \leq \lambda[m_{\theta_{n-1}} + m_{\theta_{m-1}}] < \lambda\left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $n, m > n_0$ which implies that

$$m_\theta(x_n, x_m) - m_{\theta x_n, x_m} < \varepsilon,$$

for all $n, m > n_0$. Now, for all natural numbers n, m we have

$$\begin{aligned} M_{\theta x_n, x_m} &= m_\theta(Tx_{n-1}, Tx_{n-1}) \\ &\leq \lambda[m_\theta(x_{n-1}, Tx_{n-1}) + m_\theta(x_{n-1}, Tx_{n-1})] \\ &= \lambda[m_\theta(x_{n-1}, x_n) + m_\theta(x_{n-1}, x_n)] \\ &= \lambda[m_{\theta_{n-1}} + m_{\theta_{n-1}}] \\ &= 2\lambda m_{\theta_{n-1}}. \end{aligned}$$

As $\lim_{n\to\infty} m_{\theta_{n-1}} = 0$, for every $\varepsilon > 0$ we can find a natural number n_0 such that $m_{\theta_n} < \frac{\varepsilon}{2}$ and for all $m, n > n_0$. Therefore, it follows that

$$M_{\theta x_n, x_m} \leq \lambda[m_{\theta_{n-1}} + m_{\theta_{n-1}}] < \lambda\left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $n, m > n_0$ which implies that

$$M_{\theta x_n, x_m} - m_{\theta x_n, x_m} < \varepsilon,$$

for all $n, m > n_0$. Thus, $\{x_n\}$ is an m_θ-Cauchy sequence in X. Since X is complete there exists $u \in X$ such that

$$\lim_{n\to\infty} m_\theta(x_n, u) - m_{\theta x_n, u} = 0.$$

Now, we show that u is a fixed point of T in X. For any natural number n and by the continuity of T, we have

$$\begin{aligned} \lim_{n\to\infty} m_\theta(x_n, u) - m_{\theta x_n, u} = 0& \\ &= \lim_{n\to\infty} m_\theta(x_{n+1}, u) - m_{\theta x_{n+1}, u} \\ &= \lim_{n\to\infty} m_\theta(Tx_n, u) - m_{\theta Tx_n, u} \\ &= m_\theta(Tu, u) - m_{\theta Tu, u}, \end{aligned}$$

which implies that $m_\theta(Tu, u) - m_{\theta u, Tu} = 0$, hence $m_\theta(Tu, u) = m_{\theta u, Tu}$. Using the fact that $\lim_{n\to\infty}(M_{\theta x_n, u} - m_{\theta x_n, u}) = 0$ it not difficult to deduce that $Tu = u$. Thus, u is a fixed point of T. Now, we show that if u is a fixed point, then $m_\theta(u, u) = 0$, assume that u is a fixed point of T, hence

$$\begin{aligned} m_\theta(u, u) &= m_\theta(Tu, Tu) \\ &\leq \lambda[m_\theta(u, Tu) + m_\theta(u, Tu)] \\ &= 2\lambda m_\theta(u, Tu) \\ &= 2\lambda m_\theta(u, u) \\ &< m_\theta(u, u) \quad \text{since } \lambda \in \left[0, \frac{1}{2}\right), \end{aligned}$$

that is $m_\theta(u, u) = 0$. To prove uniqueness, assume that T has two fixed points say $u, v \in X$, hence we get

$$m_\theta(u, v) = m_\theta(Tu, Tv) \leq \lambda[m_\theta(u, Tu) + m_\theta(v, Tv)] = \lambda[m_\theta(u, u) + m_\theta(v, v)] = 0,$$

which implies that $m_\theta(u, v) = 0$, and hence $u = v$ as required. □

In the following theorem, we generalize the classical Chatterjea's fixed-point result [5] using appropriate condition defined on a complete extended M_b-metric space.

Theorem 4.3. *Let (X, m_θ) be a complete extended M_b-metric space such that m_θ is continuous and let T be a continuous self mapping on X satisfy the following condition*:

$$m_\theta(Tx, Ty) \leq \lambda[m_\theta(x, Ty) + m_\theta(y, Tx)],$$

for all $x, y \in X$ where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point u such that $m_\theta(u, u) = 0$.

Proof. By the similar arguments used in the proof of Theorem 4.2, the proof follows easily. □

Finally, we prove the following fixed-point result.

Theorem 4.4. *Let (X, m_θ) be a complete extended M_b-metric space such that m_θ is continuous and T be a continuous self mapping on X satisfying the following condition*:

$$m_\theta(Tx, Ty) \leq \lambda \max\{m_\theta(x, y), m_\theta(x, Tx), m_\theta(y, Ty)\}, \qquad (\clubsuit)$$

for all $x, y \in X$ where $\lambda \in \left[0, \frac{1}{2}\right)$ and there exists $x_0 \in X$ such that for all $i \geq 0$ we have $m_\theta(x_0, T^i x_0) \leq k$, for some real number k. Then T has a unique fixed point $u \in X$ and $m_\theta(u, u) = 0$.

Proof. Let $x_0 \in X$ be the point that satisfies the hypothesis of the theorem and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$ (i.e. $x_n = T^n x_0$). Let $m_{\theta_n} = m_\theta(x_n, x_{n+1})$. Note that if there exists a natural number n such that $x_n = x_{n+1}$, then x_n is a fixed of T and hence we are done. So, we may assume that $m_{\theta_n} > 0$ for all $n \geq 0$. By $(\clubsuit)$, we obtain

$$\begin{aligned} m_{\theta_n} = m_\theta(x_n, x_{n+1}) &= m_\theta(Tx_{n-1}, Tx_n) \\ &\leq \lambda \max\{m_\theta(x_{n-1}, x_n), m_\theta(x_{n-1}, Tx_{n-1}), m_\theta(x_n, Tx_n)\} \\ &= \lambda \max\{m_\theta(x_{n-1}, x_n), m_\theta(x_{n-1}, x_n), m_\theta(x_n, x_{n+1})\} \\ &= \lambda \max\{m_\theta(x_{n-1}, x_n), m_\theta(x_n, x_{n+1})\}. \end{aligned}$$

If $\max\{m_\theta(x_{n-1}, x_n), m_\theta(x_n, x_{n+1})\} = m_\theta(x_n, x_{n+1})$, then by using the above inequality we deduce that

$$m_\theta(x_n, x_{n+1}) \leq \lambda m_\theta(x_n, x_{n+1}) < m_\theta(x_n, x_{n+1}),$$

which leads us to a contradiction. Hence, we must have

$$\max\{m_\theta(x_{n-1}, x_n), m_\theta(x_n, x_{n+1})\} = m_\theta(x_{n-1}, x_n),$$

by using the above inequality we obtain

$$m_\theta(x_n, x_{n+1}) \leq \lambda m_\theta(x_{n-1}, x_n),$$

where $\lambda \in [0, \frac{1}{2})$. By repeating this process we obtain

$$m_{\theta_n} = m_\theta(x_n, x_{n+1}) \leq \lambda^n m_\theta(x_0, x_1),$$

for all $n \geq 0$. Thus, $\lim_{n\to\infty} m_{\theta_n} = 0$. For any two natural numbers $m > n$, we obtain

$$\begin{aligned} m_\theta(x_n, x_m) &= m_\theta(T^n x_0, T^m x_0) \\ &= m_\theta(x_{n-1}, x_{m-1}) \\ &\leq \lambda \max\{m_\theta(x_{n-1}, x_{m-1}), m_\theta(x_{n-1}, Tx_{n-1}), m_\theta(x_{m-1}, Tx_{m-1})\} \\ &= \lambda \max\{m_\theta(x_{n-1}, x_{m-1}), m_\theta(x_{n-1}, x_n), m_\theta(x_{m-1}, x_m)\}. \end{aligned}$$

If $\max\{m_\theta(x_{n-1}, x_{m-1}), m_\theta(x_{n-1}, x_n), m_\theta(x_{m-1}, x_m)\} = m_\theta(x_{n-1}, x_n) = m_{\theta_{n-1}}$, then

$$m_\theta(x_n, x_m) \leq \lambda m_{\theta_{n-1}} < m_{\theta_{n-1}},$$

which leads us to a conclude that

$$\lim_{n,m\to\infty} m_\theta(x_n, x_m) - m_{\theta x_n, x_m} = 0.$$

Similarly, if $\max\{m_\theta(x_{n-1}, x_{m-1}), m_\theta(x_{n-1}, x_n), m_\theta(x_{m-1}, x_m)\} = m_\theta(x_{m-1}, x_m) = m_{\theta_{m-1}}$, then

$$\lim_{n,m\to\infty} m_\theta(x_n, x_m) - m_{\theta x_n, x_m} = 0.$$

Hence, we may assume that $\max\{m_\theta(x_{n-1}, x_{m-1}), m_\theta(x_{n-1}, x_n), m_\theta(x_{m-1}, x_m)\} = m_\theta(x_{n-1}, x_{m-1})$. Thus, from the above inequality we deduce that

$$m_\theta(x_n, x_m) \leq \lambda m_\theta(x_{n-1}, x_{m-1}),$$

for all $n \geq 0$. By repeating this process, we get

$$m_\theta(x_n, x_m) \leq \lambda^n m_\theta(x_0, x_{m-n}),$$

for all $n \geq 0$. Hence, we obtain

$$\begin{aligned} m_\theta(x_n, x_m) - m_{\theta x_n, x_m} &\leq \lambda^n m_\theta(x_0, x_{m-n}) \\ &\leq \lambda^n k. \end{aligned}$$

As $\lambda \in [0, \frac{1}{2})$, it follows from the above inequality that

$$\lim_{n,m\to\infty} m_\theta(x_n, x_m) - m_{\theta x_n, x_m} = 0.$$

Similarly, one can show that

$$\lim_{n,m\to\infty} M_{\theta x_n, x_m} - m_{\theta x_n, x_m} = 0.$$

Thus, $\{x_n\}$ is an m_θ-Cauchy sequence in X. Since X is complete there exists $u \in X$ such that

$$\lim_{n\to\infty} m_\theta(x_n, u) - m_{\theta x_n, u} = 0.$$

Now, we show that u is a fixed point of T in X. For any natural number n, by continuity of T we have,

$$\begin{aligned} \lim_{n\to\infty} m_\theta(x_n, u) - m_{\theta x_n, u} &= 0 \\ &= \lim_{n\to\infty} m_\theta(x_{n+1}, u) - m_{\theta x_{n+1}, u} \\ &= \lim_{n\to\infty} m_\theta(Tx_n, u) - m_{\theta Tx_n, u} \\ &= m_\theta(Tu, u) - m_{\theta Tu, u}. \end{aligned}$$

This is $m_\theta(Tu,u) = m_{\theta Tu,u}$ Using the fact that $\lim_{n\to\infty}(M_{\theta x_n,u} - m_{\theta x_n,u}) = 0$ it not difficult to deduce that

$$M_{\theta Tu,u} = m_{\theta Tu,u}.$$

Thus, $Tu = u$ as desired. To show that if u is a fixed point, then $m_\theta(u,u) = 0$. Consider the following

$$\begin{aligned} m_\theta(u,u) &= m_\theta(Tu,Tu) \\ &\le \lambda max\{m_\theta(u,u), m_\theta(u,Tu), m_\theta(u,Tu)\} \\ &= \lambda m_\theta(u,u) \\ &< m_\theta(u,u) \end{aligned}$$

which leads to a contradiction. Thus, $m_\theta(u,u) = 0$ as required. To prove uniqueness, assume that T has two fixed points in X say u and v, hence

$$\begin{aligned} m_\theta(u,v) &= m_\theta(Tu,Tv) \\ &\le \lambda max\{m_\theta(u,v), m_\theta(u,Tu), m_\theta(v,Tv)\} \\ &= \lambda max\{m_\theta(u,v), 0, 0\} \\ &= \lambda m_\theta(u,v) \\ &< m_\theta(u,v), \end{aligned}$$

which implies that $m_\theta(u,v) = 0$, and thus $u = v$. □

5. Conclusion and Future Work

We have introduced a new generalized metric space which is called an extended M_b-metric space. We obtain some fixed-point theorems as the generalizations of some known fixed-point results. More recently, a new direction of extension called fixed-circle problem has been studied on various metric spaces (see [11], [12], [13], [14], [16] and [17] for more details). Now we define the concepts of a circle and of a fixed circle on an extended M_b-metric space (X, m_θ) as follows:

For $r > 0$ and $x_0 \in X$, the circle $C^{m_\theta}_{x_0,r}$ with the center x_0 and the radius r is defined by

$$C^{m_\theta}_{x_0,r} = \{x \in X \mid m_\theta(x,y) - m_{\theta x,y} = r\}.$$

Let (X, m_θ) be an extended M_b-metric space, $C^{m_\theta}_{x_0,r}$ be a circle and $T : X \to X$ be a self-mapping. If $Tx = x$ for every $x \in C^{m_\theta}_{x_0,r}$ then the circle $C^{m_\theta}_{x_0,r}$ is called as the fixed circle of T.

Let us consider the following example:

Let $A_1 = \left\{z \mid z = x + iy,\ x^2 + y^2 = 9\right\}$, $A_2 = \left\{z \mid z = x + iy,\ x^2 + y^2 = 1\right\} \subseteq \mathbb{C}$ where $\mathbb{C}$ is the set of all complex numbers and $X = A_1 \cup A_2$. If we define the functions $\theta : X^2 \to [1,\infty)$ and $m_\theta : X^2 \to [0,\infty)$ as

$$\theta(z_1, z_2) = |z_1|\,|z_2|$$

and

$$m_\theta(z_1, z_2) = |z_1 - z_2|\,,$$

for all $z_1, z_2 \in X$, respectively, then (X, m_θ) is an extended M_b-metric space. Let us consider the circle $C^{m_\theta}_{0,3} = \{z \in X \mid m_\theta(z,0) - m_{\theta z,0} = 3\} = \{z \in X \mid |z| = 3\}$ and two self-mappings $T_{1,2} : X \to X$ defined by

$$T_1 z = \begin{cases} \frac{9}{\bar{z}} & \text{if} \quad z \in A_1 \\ \alpha & \text{if} \quad z \in A_2 \end{cases}$$

and

$$T_2z = \begin{cases} \frac{9}{\bar{z}} & \text{if} \quad z \in A_1 \\ \frac{1}{z} & \text{if} \quad z \in A_2 \end{cases},$$

for all $z \in X$ where $\overline{z}$ is the complex conjugate of the complex number z and α is a constant with $|\alpha| = 1$. Some straightforward computations show that the circle $C_{0,3}^{m_\theta}$ is the fixed circle of T_1 while it is not fixed by T_2. Then it is natural to consider the following question:

What are the existence and uniqueness conditions for a fixed circle of a self-mapping on an extended M_b-metric space?

For a future work, it can be investigated some fixed-circle theorems and their applications.

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