A New Example of Deficiency One Groups

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Abstract. The main purpose of this paper is to present a new example of deficiency one groups by considering the split extension of a finite cyclic group by a free abelian group having rank two.

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INTRODUCTION AND PRELIMINARIES

Let G be a finitely presented group, and let $\mathscr{P} = \langle \mathbf{x};\mathbf{r}\rangle$ be a finite presentation for G. If we regard the above $\mathscr{P}$ as a 2-complex with single 0-cell whose 1-cells are in bijective correspondence with the elements of $\mathbf{x}$, and whose 2-cells are attached by the boundary path determined by the spelling of the corresponding element of $\mathbf{r}$ in the standard way, then G is just the fundamental group of $\mathscr{P}$. Therefore the *deficiency* of $\mathscr{P}$ is defined by $def(\mathscr{P}) = -|\mathbf{x}| + |\mathbf{r}|$. Let $\delta(G) = -rk_{\mathbf{Z}}(H_1(G)) + d(H_2(G))$, where $rk_{\mathbf{Z}}(\,.\,)$ denotes the $\mathbf{Z}$-rank of the torsion-free part and $d(\,.\,)$ means the minimal number of generators. Then it is a well known fact that for the presentation $\mathscr{P}$, the inequality $def(\mathscr{P}) \geq \delta(G)$ always holds. Thus we define the deficiency $def(G)$ of a finitely presented group G is the maximum deficiency over all such presentations $\mathscr{P}$. Moreover we say G is *efficient* if $def(G) = \delta(G)$, and $\mathscr{P}$ such that $def(\mathscr{P}) = \delta(G)$ is then called an *efficient presentation*.

One of the most effective way to show efficiency for the group G is to use *spherical pictures* ([2, 11]) over $\mathscr{P}$. These geometric configurations are the representative elements of the second homotopy group $\pi_2(\mathscr{P})$ of $\mathscr{P}$ which is a left $\mathbf{Z}G$-module. There are certain operations on spherical pictures. Suppose $\mathbf{Y}$ is a collection of spherical pictures over $\mathbf{P}$. Allowing these operations lead to the notion of *equivalence (rel* $\mathbf{Y}$*) of spherical pictures*. Then it has been proved that the elements $\langle \mathbf{P}\rangle$, where $\mathbf{P}$ is in the set $\mathbf{Y}$, generate $\pi_2(\mathscr{P})$ as a module if and only if every spherical picture is equivalent (rel $\mathbf{Y}$) to the empty picture. Therefore one can easily say that if the elements $\langle \mathbf{P}\rangle$ generate $\pi_2(\mathscr{P})$ then $\mathbf{Y}$ *generates* $\pi_2(\mathscr{P})$. For any picture $\mathbf{P}$ over $\mathscr{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbf{P}$, denoted by $exp_R(\mathbf{P})$, is the number of discs of $\mathbf{P}$ labeled by R minus the number of discs labeled by R^{-1}. We remark that if any two pictures $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent then for all $R \in \mathbf{r}$ their exponent sums are equivalent. Let n be a non-negative integer. Then $\mathscr{P}$ is said to be *n-Cockcroft* if $exp_R(\mathbf{P}) \equiv 0 \,(\mathrm{mod}\ n)$ (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbf{P}$ over $\mathscr{P}$. Then a group G is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation. To verify that the n-Cockcroft property holds, it is enough to check for pictures $\mathbf{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a set of generating pictures. The case $n = 0$ is just called Cockcroft. For a connection between Cockcroft property and efficiency, we should give the following result which is essentially due to Epstein [7] that can also be found in [9]. So let us consider a presentation $\mathscr{P} = \langle \mathbf{x};\mathbf{r}\rangle$ for the group G.

Theorem 1 *$\mathscr{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

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As a consequence of this above theorem, it is easy to see that if $\mathscr{P}$ is Cockcroft then it is efficient. These two facts will be used in the proof of main result of this paper.

Let A be a finite cyclic group of order t and D be the group F_2 (the free abelian group having rank 2), with respective presentations $\mathscr{P}_A = \langle a\,;\,a^t\rangle$ and $\mathscr{P}_D = \langle s,c\,;\,sc=cs\rangle$. It is a well known fact that if we want to obtain a semidirect product $G = D \times_\theta A$, then we need to define a regular homomorphsim θ from A to automorphism group of D. Now if we regard the elements $[c^m d^n]_D$ of D as 1×2 matrices $[m\,n]$, then we can represent automorphisms of D by 2×2 matrices with integer entries. In other words we can represent automorphisms $\theta_{[a]}$ of D by the matrix

$$\mathscr{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

For simplicity, let us label $\mathscr{M}$ as the form $\begin{bmatrix} U_1 & V_1 \\ W_1 & Z_1 \end{bmatrix}$, and then let us multiply it by itself. Now by relabelling the matrix $\mathscr{M}^2$ as $\begin{bmatrix} U_2 & V_2 \\ W_2 & Z_2 \end{bmatrix}$ and iterating this procedure, we finally have

$$\mathscr{M}^t = \begin{bmatrix} U_{t-1}\alpha_{11}+V_{t-1}\alpha_{21} & U_{t-1}\alpha_{12}+V_{t-1}\alpha_{22} \\ W_{t-1}\alpha_{11}+Z_{t-1}\alpha_{21} & W_{t-1}\alpha_{12}+Z_{t-1}\alpha_{22} \end{bmatrix},$$

say

$$\begin{bmatrix} U_t & V_t \\ W_t & Z_t \end{bmatrix}.$$

In fact this t^{th} power of $\mathscr{M}$ will be needed for the following lemma.

In general, if we have any two groups G_1 and G_2 that generated by the sets $\mathbf{x}$ and $\mathbf{y}$, respectively, then for each $x \in \mathbf{x}$ and $y \in \mathbf{y}$ and for a given homomorphism θ, we are allowed to choose a word $y\theta_x$ on $\mathbf{y}$ with $[y\theta_x]_{G_2} = [y]_{G_2}\theta_{[x]_{G_1}}$ (see, for instance, [6]). In our case, we will restrict ourselves only to the choice

$$s\theta_a = s^{\alpha_{11}}c^{\alpha_{12}} \quad \textit{and} \quad c\theta_a = s^{\alpha_{21}}c^{\alpha_{22}}.$$

Hence, for the function $\theta : A \to Aut(D)$ to be a well-defined homomorphism, we must require $\theta_{[a^t]} = \theta_{[1]}$ or equivalently that $\mathscr{M}^t$ is equal to identity matrix. So we have the following lemma that will be played an important role to have a semidirect product.

Lemma 2 *The function $\theta : A \to Aut(D)$ defined by $[a] \mapsto \theta_{[a]}$ is a well-defined group homomorphism if and only if*

$$U_t = 1\,,\ V_t = 0\,,\ W_t = 0 \quad \textit{and} \quad Z_t = 1.$$

Proof This follows immediately from the equality of $\mathscr{M}^t = I_{2\times 2}$.

By this lemma, we definitely have a homomorphism and so, have a semidirect product $G = D \times_\theta A$ (of the cyclic group of order t by the free abelian group rank 2) with a presentation

$$\mathscr{P}_G = \left\langle a,s,c\,;\,a^t,\ [s,c],\ T_{sa},\ T_{ca}\right\rangle \tag{1}$$

(see [8]), where

$$T_{sa} : sa = as^{\alpha_{11}}c^{\alpha_{12}}, \quad T_{ca} : ca = as^{\alpha_{21}}c^{\alpha_{22}},$$

respectively.

Therefore the main result of this paper is the following:

Theorem 3 *Let p be a prime or 0. Then $\mathscr{P}_G$, as in (1), is p-Cockcroft if and only if the following conditions hold:*

(i) $det\mathscr{M} \equiv 1\,(mod\ p)$,

(ii) $\sum_{i=1}^{t-1} U_i \equiv 1\,(mod\ p)$, $\quad \sum_{i=1}^{t-1} V_i \equiv 0\,(mod\ p)$,

$\sum_{i=1}^{t-1} W_i \equiv 0\,(mod\ p)$, $\quad \sum_{i=1}^{t-1} Z_i \equiv 1\,(mod\ p)$,

(iii) $exp_S(\mathbf{B}_{y,a^t}) \equiv 0\,(mod\ p)$, *for* $y \in \{s,c\}$.

Example 4 *By Lemma* 2*, a group G having one of the presentation*

$i)$ $\mathscr{P}_1 = \langle a,s,c\ ;\ a^2,\ [s,c],\ sa = as^k c^{1-k},\ ca = as^{1+k}c^{-k}\rangle$,

$ii)$ $\mathscr{P}_2 = \langle a,s,c\ ;\ a^2,\ [s,c],\ sa = as^{-1},\ ca = as^k c\rangle$, *where* $k = 2n \in \mathbf{Z}$,

$iii)$ $\mathscr{P}_3 = \langle a,s,c\ ;\ a^3,\ [s,c],\ sa = asc,\ ca = as^{-3}c^{-2}\rangle$,

$iv)$ $\mathscr{P}_4 = \langle a,s,c\ ;\ a^3,\ [s,c],\ sa = ac,\ ca = as^{-1}c^{-1}\rangle$,

defines a semidirect product. Also each of $\mathscr{P}_1$*,* $\mathscr{P}_2$*,* $\mathscr{P}_3$ *and* $\mathscr{P}_4$ *has deficiency* 1.

In the remaning part of this paper, by introducing the generating pictures for the presentation $\mathscr{P}_G$ in (1), we will prove Theorem 3.

DEFICIENCY OF $\mathscr{P}_G$

In this section, by [1], we will first obtain a generating set (i.e. the generating pictures) of $\pi_2(\mathscr{P}_G)$, where $\mathscr{P}_G$ as in (1). After that, by considering this set, we will prove the main result which was stated the result for $\mathscr{P}_G$ to be p-Cockcroft (and so, by Theorem 1, to be efficient) for some prime p or 0. Then, by picking one of the presentation given in Example 4, we will show that it is efficient (more precisely, it is a deficiency one presentation) for the group G.

The generating set of $\pi_2(\mathscr{P}_G)$

Let us consider the group $G = D \times_\theta A$ with the presentation $\mathscr{P}_G$ in (1), where A and D are presented by $\mathscr{P}_A = \langle a\ ;\ a^t\rangle$ and $\mathscr{P}_D = \langle s, c\ ;\ sc = cs\rangle$, respectively. Recall that T_{sa} and T_{ca} denote the relators $sa = a(s\theta_a)$ and $ca = a(c\theta_a)$, respectively, where

$$s\theta_a = s^{\alpha_{11}}c^{\alpha_{12}} \text{ and } c\theta_a = s^{\alpha_{21}}c^{\alpha_{22}}.$$

For the relator a^t $(t \in \mathbf{Z}^+)$ and for any $y \in \{s,c\}$, we denote the word $(\cdots((y\theta_a)\theta_a)\theta_a\cdots)\theta_a)$ by $y\theta_{a^t}$, and this can be represented by a picture, say $\mathbf{A}_{a^t,y}$, as drawn in Figure 1 in [4].

Moreover, if $W = s^{\varepsilon_1}c^{\varepsilon_2}s^{\varepsilon_3}c^{\varepsilon_4}\cdots s^{\varepsilon_{m-1}}c^{\varepsilon_m}$ is a word on the set $\{s,c\}$, then for the generator a, we denote the word $(s^{\varepsilon_1}\theta_a)(c^{\varepsilon_2}\theta_a)\cdots(s^{\varepsilon_{m-1}}\theta_a)(s^{\varepsilon_m}\theta_a)$ by $W\theta_a$.

Let X_A and X_D be a generating set of $\pi_2(\mathscr{P}_A)$ and $\pi_2(\mathscr{P}_D)$, respectively. By [2], each of X_A and X_D contains a single generating picture $\mathbf{P}_A$ and $\mathbf{P}_D$, respectively as drawn in Figure 2 in [4].

For simplicity, let us denote the commutator relator $[s,c]$ by $\mathscr{R}$.

Since $[\mathscr{R}\theta_a]_{\mathscr{P}_D} = [1\theta_a]_{\mathscr{P}_D}$, there is a non-spherical picture, say $\mathbf{B}_{s,c}$, over $\mathscr{P}_D$ with the boundary label

$$\mathscr{R}\theta_a = s^{\alpha_{11}}c^{\alpha_{12}}s^{\alpha_{21}}c^{\alpha_{22}}(s^{\alpha_{21}}c^{\alpha_{22}}s^{\alpha_{11}}c^{\alpha_{12}})^{-1}.$$

We note that, by the dependence on the choice of homomorphism θ_a (i.e. choice of matrix $\mathscr{M}$), there are various $\mathbf{B}_{s,c}$ pictures which can be drawn.

Let us consider the relator a^t and the set of generators $\{s,c\}$ for the presentation $\mathscr{P}_D$. Then we get non-spherical pictures $\mathbf{A}_{a^t,y}$, for each $y \in \{s,c\}$. It is clear that $\mathbf{A}_{a^t,y}$ pictures consist of only T_{ya} $(y \in \{s,c\})$ discs.

In addition to above non-spherical pictures, since $[y\theta_{a^t}]_{\mathscr{P}_D} = [y\theta_1]_{\mathscr{P}_D}$, for each $y \in \{s,c\}$, there is a non-spherical picture, $\mathbf{B}_{y,a^t}$ say, over $\mathscr{P}_D$ with boundary label $y\theta_{a^t}$.

Our aim is now to contruct spherical pictures by using these above non-spherical pictures:

Let us consider the single $\mathbf{B}_{s,c}$ picture. If we process the boundary of $\mathbf{B}_{s,c}$ by a single a-arc, then for each fixed $y \in \{s,c\}$, we get one positive and one negative T_{ya}-discs. Therefore, for the same T_{ya}-dics, we have two discs with opposite sign and so these give us that we have one $\mathscr{R}$-disc. Hence we have a new non-spherical picture containing the single $\mathbf{B}_{s,c}$ picture, two different types of T_{ya}-discs (such that each of has one positive and one negative disc) and one $\mathscr{R}$-disc. The boundary label of this new picture is $a^{-1}a$. Clearly to obtain a spherical picture, say $\mathbf{P}_{sc}$, from this last non-spherical picture, we must combine a and a^{-1} by an arc (see Figure 3-(a) in [4]). Thus let X_{sc} be the set $\{\mathbf{P}_{sc}\}$.

Now let us consider one of the non-spherical picture $\mathbf{A}_{a^t,y}$ with the boundary label

$$ya^ty^{-1}(y\theta_a)^{-1}a^{-t}.$$

To obtain a spherical picture from this non-spherical picture, we first need to fix two a^t-discs which one of them is positive and the other is negative. After that we can combine y and y^{-1} by an arc. So we finally need to fix the subpicture $(\mathbf{B}_{y,a^t})^{-1}$ for the part of the boundary $(y\theta_{a^t})^{-1}$. Thus, for each $y \in \{s,c\}$, we have a spherical picture, say $\mathbf{P}_{ya}$, as in Figure 3-(b) in [4]. Therefore let $X_{sca} = \{\mathbf{P}_{sa}, \mathbf{P}_{ca}\}$.

Although the monoid version of the following proposition can be found in [13], the group version can be either proved directly by the result in [1] or seen at the first author's thesis in the same reference.

Proposition 5 *Suppose $G = D \times_\theta A$ is a semidirect product with associated presentation $\mathscr{P}_G$, as in* (1). *Then a generating set of the second homotopy module $\pi_2(\mathscr{P}_G)$ is*

$$X_A \cup X_D \cup X_{sc} \cup X_{sca}.$$

We should note that, by applying completely the same progress, the above proposition could be constructed for the semidirect product of any two groups G_1 and G_2 with associated presentations $\mathscr{P}_{G_1} = \langle \mathbf{x};\mathbf{r}\rangle$ and $\mathscr{P}_{G_2} = \langle \mathbf{y};\mathbf{s}\rangle$, respectively.

The proof of Theorem 3

By concerning the generating pictures defined in Proposition 5, we will count the exponent sums in these pictures to deduce the p-Cockcroft property and so efficiency. In other words, in the proof, we will basically count the number of discs in each of spherical pictures $\mathbf{P}_A$, $\mathbf{P}_D$, $\mathbf{P}_{sc}$ and $\mathbf{P}_{ya}$, where $y \in \{s,c\}$. It is quite clear that $\mathbf{P}_A$ and $\mathbf{P}_D$ are Cockcroft, and so p-Cockcroft.

Now let us consider the picture $\mathbf{P}_{sc}$ as drawn in Figure 3-(a) in [4]. It contains a single negative $\mathscr{R}$-disc, a single $\mathbf{B}_{s,c}$ picture and balanced (one positive and one negative) number of T_{sa} and T_{ca}-discs. We first note that the boundary of $\mathbf{B}_{s,c}$ is equal to the $\mathscr{R}\theta_a$, more clearly,

$$s^{\alpha_{11}}c^{\alpha_{12}}s^{\alpha_{21}}c^{\alpha_{22}}(s^{\alpha_{21}}c^{\alpha_{22}}s^{\alpha_{11}}c^{\alpha_{12}})^{-1}.$$

That means, inside $\mathbf{B}_{s,c}$, we have $\alpha_{11}\alpha_{22}$-times positive and $\alpha_{12}\alpha_{21}$-times negative $\mathscr{R}$-discs, i.e.

$$exp_{\mathscr{R}}(\mathbf{B}_{s,c}) = det\mathscr{M} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}.$$

So to balanced the single negative $\mathscr{R}$-disc in $\mathbf{P}_{sc}$, we must have $det\mathscr{M} \equiv 1\,(mod\ p)$, as required. This gives the condition (i).

For a fixed $y \in \{s,c\}$, let us consider a picture $\mathbf{P}_{ya}$ (see Figure 3-(b) in [4]). It contains one positive and one negative a^t-discs and two subpictures $\mathbf{A}_{a^t,y}$ and $\mathbf{B}_{y,a^t}$, where $y \in \{s,c\}$. Clearly $exp_{a^t}(\mathbf{P}_{ya}) = 1-1 = 0$, and so there is nothing to do. Now let us consider the matrices $\mathscr{M}$, $\mathscr{M}^2$, $\cdots$, $\mathscr{M}^{t-1}$ to use in the calculation of exponent sums in the subpicture $\mathbf{A}_{a^t,y}$. We know that the each of the subpicture $\mathbf{A}_{a^t,y}$ consists of only T_{ya}-discs ($y \in \{s,c\}$). By using the morphism $\theta_{[a]}$ of D defined by $[s] \mapsto [s^{\alpha_{11}}c^{\alpha_{12}}]$ and $[c] \mapsto [s^{\alpha_{21}}c^{\alpha_{22}}]$, a simple calculation shows that the sum of first row and first column elements for all $\mathscr{M}^j$ $(1 \le j \le t-1)$ matrices gives the exponent sum of T_{sa}-discs in $\mathbf{A}_{a^t,s}$, the sum of first row and second column elements gives the exponent sum of T_{ca}-discs in $\mathbf{A}_{a^t,c}$, etc. In other words

$$\begin{aligned}
U_1 + U_2 + \cdots + U_{t-1} &= exp_{T_{sa}}(\mathbf{A}_{a^t,s}),\\
V_1 + V_2 + \cdots + V_{t-1} &= exp_{T_{ca}}(\mathbf{A}_{a^t,s}),\\
W_1 + W_2 + \cdots + W_{t-1} &= exp_{T_{sa}}(\mathbf{A}_{a^t,c}),\\
Z_1 + Z_2 + \cdots + Z_{t-1} &= exp_{T_{ca}}(\mathbf{A}_{a^t,c}).
\end{aligned}$$

Therefore to p-Cockcroft property be hold, we must have

$$\sum_{i=1}^{t-1} U_i \equiv 1\,(mod\ p), \qquad \sum_{i=1}^{t-1} V_i \equiv 0\,(mod\ p),$$
$$\sum_{i=1}^{t-1} W_i \equiv 0\,(mod\ p), \qquad \sum_{i=1}^{t-1} Z_i \equiv 1\,(mod\ p),$$

as required. This gives the condition (ii).

In picture $\mathbf{P}_{ya}$, we also have a subpicture $\mathbf{B}_{y,a^t}$ having boundary label $y\theta_{a^t}$. (We note that the boundary word $y\theta_{a^t}$ is actually a piece of the boundary label $a^{-t}da^td^{-1}(y\theta_{a^t})^{-1}$ of the subpicture $\mathbf{A}_{a^t,y}$). In fact the word $y\theta_{a^t}$ contains a finite number of only "s" and "c" letters, and so the subpicture $\mathbf{B}_{y,a^t}$ contains only commutator $\mathscr{R}$-discs. Therefore the exponent sum of $\mathscr{R}$-discs in $\mathbf{B}_{y,a^t}$ must congruent to zero by modulo p, as required.

Conversely suppose that these three conditions (i), (ii) and (iii) hold. Then, by using the generating set of $\pi_2(\mathscr{P}_G)$, it is easy to see that the presentation $\mathscr{P}_G$ is p-Cockcroft for a prime p or 0.

Hence the result.

After completed this above proof, we can easily say that $\mathscr{P}_G$ is efficient (by Theorem 1). Since number of relators is precisely one more than number of generators, $\mathscr{P}_G$ is actually a *deficiency one presentation*.

Let us consider the presentation $\mathscr{P}_1$ in Example 4. Clearly it presents a semidirect product since the square of matrix $\begin{bmatrix} k & 1-k \\ 1+k & -k \end{bmatrix}$ is equal to the identity (by Lemma 2). Assume $k=1$ in $\mathscr{P}_1$. By considering Figures 1, 2 and 3 in [4], one can easily draw the generating pictures for $\pi_2(\mathscr{P}_1)$ while $k=1$. In this case, the subpicture $\mathbf{B}_{s,c}$ contains only a single positive $\mathscr{R}$-disc that balanced one negative $\mathscr{R}$-disc in $\mathbf{P}_{sc}$. Thus all discs in the spherical picture $\mathbf{P}_{sc}$ are balanced. Also, for the picture $\mathbf{P}_{sa}$, there is no subpicture $\mathbf{B}_{s,a^2}$. In $\mathbf{P}_{sa}$, we actually have one positive and one negative a^2-discs, and again one positive and one negative T_{sa}-discs. So, as in $\mathbf{P}_{sc}$, all discs in $\mathbf{P}_{sa}$ are balanced as well. Finally, for the subpicture $\mathbf{A}_{a^2,c}$ of $\mathbf{P}_{ca}$, we have one positive and one negative T_{ca}-discs, and two positive T_{sa}-discs. In other words, $exp_{T_{ca}}(\mathbf{A}_{a^2,c}) = 1-1 = 0$ and $exp_{T_{sa}}(\mathbf{A}_{a^2,c}) = 2$. Additionally, in the subpicture $\mathbf{A}_{a^2,c}$ of $\mathbf{P}_{ca}$, we have two positive $\mathscr{R}$-discs. Therefore the presentation

$$\mathscr{P}_1^1 = \langle a,s,c \,;\, a^2,\ [s,c],\ sa = as,\ ca = as^2c^{-1} \rangle$$

is 2-Cockcroft and so efficient (by Theorem 1). More precisely, $\mathscr{P}_1^1$ is a deficiency 1 presentation.

In fact, the deficiencies of other presentations $\mathscr{P}_2$, $\mathscr{P}_3$ and $\mathscr{P}_4$ in Example 4 can be seen quite similar as in $\mathscr{P}_1$ case. In detailed, while $\mathscr{P}_2$ is 2-Cockcroft, $\mathscr{P}_3$ and $\mathscr{P}_4$ are Cockcroft and so p-Cockcroft for any prime p.

Note that

1) In *[10]*, Lustig developed a test to investigate the *minimality* of a group presentation. In fact this test has been widely used while the presentation is inefficient. By this test, one can easily says that if if a group has an efficient presentation while this presentation is minimal, then this group is inefficient. In other words, there is no way to prove that this group (presented by this minimal but inefficient presentation) is efficient. Lustig test basically works on the Fox ideals obtained from the generating pictures of the second homotopy modules. In our case, by concerning presentation $\mathscr{P}_G$ in *(1)* and using this Lustig test, we could not get a minimal but inefficient presentation example. (For instance, in the presentation $\mathscr{P}_2$ given in Example *4*, if we take $k \neq 2n$ for any integer n, then $\mathscr{P}_2$ becomes an inefficient presentation. But Lustig test does not give an answer whether it is minimal while $k \neq 2n$). Therefore obtaining minimality while having inefficiency and constructing relationship (if any) between some other algebraic properties and inefficiency can be studied for a future project.
2) The monoid version of the p-Cockcroft property and minimality while having inefficiency of the semidirect product have been defined and examined in detail in *[4]* and *[5]*, respectively. In fact it is not hard to find deficiency one monoid presentations.
3) It is known that a semidirect product $A \times B$ is *residually finite* $\mathscr{R}F$ (i.e. the intersection of all its subgroups of finite index is trivial) if both A and B are $\mathscr{R}F$ and A is finitely generated. It is also well known that there is a relationship between the properties $\mathscr{R}F$ and Largeness of groups. After that, one can ask whether our group G with presentation *(1)* is large or not. In deficiency one presentations, there are significant studies on to have large property *(*see, for instance, *([3, Theorem 3.6], [12])*.

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