REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 55, No. 2, 2014, Pages 81–100
Published online: October 14, 2014

ON SLANT CURVES IN TRANS-SASAKIAN MANIFOLDS

ŞABAN GÜVENÇ AND CIHAN ÖZGÜR

Abstract. We find the characterizations of the curvatures of slant curves in trans-Sasakian manifolds with C-parallel and C-proper mean curvature vector field in the tangent and normal bundles.

1. Introduction

Let γ be a curve in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. In [14], Lee, Suh and Lee introduced the notions of C-parallel and C-proper curves in the tangent and normal bundles. A curve γ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined to be *C-parallel* if $\nabla_T H = \lambda\xi$, *C-proper* if $\Delta H = \lambda\xi$, *C-parallel in the normal bundle* if $\nabla^{\perp}_T H = \lambda\xi$, *$C$-proper in the normal bundle* if $\Delta^{\perp} H = \lambda\xi$, where T is the unit tangent vector field of γ, H is the mean curvature vector field, Δ is the Laplacian, λ is a non-zero differentiable function along the curve γ, $\nabla^{\perp}$ and $\Delta^{\perp}$ denote the normal connection and Laplacian in the normal bundle, respectively [14]. For a submanifold M of an arbitrary Riemannian manifold $\widetilde{M}$, if $\Delta H = \lambda H$, then M is a *submanifold with proper mean curvature vector field H* [7]. If $\Delta^{\perp} H = \lambda H$, then M is a *submanifold with proper mean curvature vector field H in the normal bundle* [1].

Let M be an almost contact metric manifold and $\gamma(s)$ a Frenet curve in M parametrized by the arc-length parameter s. The contact angle $\alpha(s)$ is a function defined by $\cos[\alpha(s)] = g(T(s), \xi)$. A curve γ is called a *slant curve* [8] if its contact angle is a constant. Slant curves with contact angle $\frac{\pi}{2}$ are traditionally called *Legendre curves* [4].

In [18], Srivastava studied Legendre curves in trans-Sasakian 3-manifolds. In [11], Inoguchi and Lee studied almost contact curves in normal almost contact 3-manifolds. In [12], the same authors studied slant curves in normal almost contact metric 3-manifolds. In [14], Lee, Suh and Lee studied slant curves in Sasakian 3-manifolds. They find the curvature characterizations of C-parallel and C-proper curves in the tangent and normal bundles. In the present study, our aim is to generalize results of [14] to a curve in a trans-Sasakian manifold.

2010 *Mathematics Subject Classification.* 53C25 (53C40 53A05).

Key words and phrases. trans-Sasakian manifold, slant curve, C-parallel mean curvature vector field, C-proper mean curvature vector field.

2. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold M is said to be an *almost contact metric manifold* [4], if there exist on M a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X,\xi),$$

for any vector fields X, Y on M. Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \varphi Y)$ is called the *fundamental* 2*-form* of M [4].

The almost contact metric structure of M is said to be *normal* if

$$[\varphi, \varphi](X,Y) = -2d\eta(X,Y)\xi,$$

for any vector fields X, Y on M, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ. A normal contact metric manifold is called a *Sasakian manifold* [4]. It is easy to see that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

An almost contact metric manifold M is called a *trans-Sasakian manifold* [17] if there exist two functions α and β on M such that

$$(\nabla_X \varphi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{2.1}$$

for any vector fields X, Y on M. From (2.1), it is easily obtained that

$$\nabla_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi]. \tag{2.2}$$

If $\beta = 0$ (resp. $\alpha = 0$), then M is said to be an *α-Sasakian manifold* (resp. *β-Kenmotsu manifold*). Sasakian manifolds (resp. Kenmotsu manifolds [13]) appear as examples of α-Sasakian manifolds (resp. β-Kenmotsu manifolds), with $\alpha = 1$ (resp. $\beta = 1$). For $\alpha = \beta = 0$, we get *cosymplectic manifolds* [15]. From (2.2), for a cosymplectic manifold we obtain

$$\nabla_X \xi = 0.$$

Hence ξ is a Killing vector field for a cosymplectic manifold [3].

Proposition 2.1. [16] *A trans-Sasakian manifold of dimension greater than or equal to* 5 *is either α-Sasakian, β-Kenmotsu or cosymplectic.*

From now on, we state "(α, β)-trans-Sasakian manifold", when the dimension of the manifold is 3 and $\alpha \neq 0, \beta \neq 0$.

The contact distribution of an almost contact metric manifold M with an almost contact metric structure (φ, ξ, η, g) is defined by

$$\{X \in TM : \eta(X) = 0\}$$

and an integral curve of the contact distribution is called a *Legendre curve* [4].

3. Slant curves with C-parallel mean curvature vector field

Let (M,g) be an m-dimensional Riemannian manifold and $\gamma : I \to M$ a curve parametrized by arc length. Then γ is called a Frenet curve of osculating order r, $1 \leq r \leq m$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\cdots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{3.1}$$

where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I.

A *geodesic* is a Frenet curve of osculating order 1; a *circle* is a Frenet curve of osculating order 2 such that κ_1 is a non-zero positive constant; a *helix of order* r, $r \geq 3$, is a Frenet curve of osculating order r such that $\kappa_1, \ldots, \kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is called simply a *helix*.

Now let (M,g) be a Riemannian manifold and $\gamma : I \to M$ a Frenet curve of osculating order r. By the use of (3.1), it can be easily seen that

$$\begin{aligned} \nabla_T \nabla_T T &= -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \\ \nabla_T \nabla_T \nabla_T T &= -3\kappa_1 \kappa_1' E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2\right) E_2 \\ &\quad + \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4, \\ \nabla_T^\perp \nabla_T^\perp T &= \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \\ \nabla_T^\perp \nabla_T^\perp \nabla_T^\perp T &= \left(\kappa_1'' - \kappa_1 \kappa_2^2\right) E_2 + \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4. \end{aligned}$$

So we have (see [1])

$$\nabla_T H = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \tag{3.2}$$

$$\begin{aligned} \Delta H &= -\nabla_T \nabla_T \nabla_T T \\ &= 3\kappa_1 \kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 \\ &\quad - (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 - \kappa_1 \kappa_2 \kappa_3 E_4, \end{aligned} \tag{3.3}$$

$$\nabla_T^\perp H = \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \tag{3.4}$$

$$\begin{aligned} \Delta^\perp H &= -\nabla_T^\perp \nabla_T^\perp \nabla_T^\perp T \\ &= \left(\kappa_1 \kappa_2^2 - \kappa_1''\right) E_2 - \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 \\ &\quad - \kappa_1 \kappa_2 \kappa_3 E_4. \end{aligned} \tag{3.5}$$

By the use of equations (3.2), (3.3), (3.4) and (3.5), we can directly state the following proposition:

Proposition 3.1. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Frenet curve in a trans-Sasakian manifold M. Then*

i) γ has C-parallel mean curvature vector field if and only if

$$-\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \xi; \quad \textit{or} \tag{3.6}$$

ii) γ has C-proper mean curvature vector field if and only if

$$3\kappa_1\kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1''\right) E_2 - (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 - \kappa_1\kappa_2\kappa_3 E_4 = \lambda\xi; \quad or \tag{3.7}$$

iii) γ has C-parallel mean curvature vector field in the normal bundle if and only if

$$\kappa_1' E_2 + \kappa_1\kappa_2 E_3 = \lambda\xi; \quad or \tag{3.8}$$

iv) γ has C-proper mean curvature vector field in the normal bundle if and only if

$$\left(\kappa_1\kappa_2^2 - \kappa_1''\right) E_2 - \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right) E_3 - \kappa_1\kappa_2\kappa_3 E_4 = \lambda\xi, \tag{3.9}$$

where λ is a non-zero differentiable function along the curve γ.

Now, let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order r with contact angle α_0 in an n-dimensional trans-Sasakian manifold. By the use of (2.1), (2.2) and (3.1), we obtain

$$\eta(T) = \cos\alpha_0, \tag{3.10}$$

$$\kappa_1\eta(E_2) = -\beta\sin^2\alpha_0, \tag{3.11}$$

$$\nabla_T\xi = -\alpha\varphi T + \beta[T - \cos\alpha_0\xi], \tag{3.12}$$

$$\nabla_T\varphi T = \alpha[\xi - \cos\alpha_0 T] - \beta\cos\alpha_0\varphi T + \kappa_1\varphi E_2. \tag{3.13}$$

So we have the following theorem:

Theorem 3.1. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order r in a trans-Sasakian manifold. If γ has C-parallel or C-proper mean curvature vector field in the normal bundle, then it is a Legendre curve.*

Proof. By the use of (3.8), (3.9) and (3.10), the proof is clear. □

We consider the following cases:

Case I. The osculating order $r = 2$.

For this case, we have the following results:

Theorem 3.2. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 2 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-parallel mean curvature vector field if and only if it satisfies*

$$\kappa_1 = \frac{\mp\cot\alpha_0}{c - s}, \tag{3.14}$$

$$\lambda = \frac{-\cot\alpha_0\csc\alpha_0}{(c - s)^2}, \tag{3.15}$$

where c is an arbitrary constant and s is the arc-length parameter of γ. In this case, M becomes an (α, β)-trans-Sasakian or a β-Kenmotsu manifold with

$$\beta = \frac{\cot\alpha_0\csc\alpha_0}{c - s}.$$

Proof. Let γ have C-parallel mean curvature vector field. From (3.6), we have

$$-\kappa_1^2 E_1 + \kappa_1' E_2 = \lambda\xi. \tag{3.16}$$

If $\alpha_0 = \frac{\pi}{2}$, we find $\kappa_1 = 0$, which is a contradiction. Thus, $\alpha_0 \neq \frac{\pi}{2}$.

Let $\beta \neq 0$. Hence M is an (α, β)-trans-Sasakian or a β-Kenmotsu manifold. Since $\eta(E_2) = \pm\sin\alpha_0$, (3.11) gives us

$$\kappa_1 = \mp\beta\sin\alpha_0. \tag{3.17}$$

By the use of (3.10), (3.11) and (3.16), we get

$$\lambda = \frac{-\kappa_1^2}{\cos\alpha_0}, \tag{3.18}$$

$$\kappa_1' = \kappa_1\beta\sin\alpha_0\tan\alpha_0. \tag{3.19}$$

Differentiating (3.17) and using (3.19), we have

$$\beta' = \beta^2\sin\alpha_0\tan\alpha_0,$$

which gives us

$$\beta = \frac{\cot\alpha_0\csc\alpha_0}{c-s}, \tag{3.20}$$

where c is an arbitrary constant. Using (3.20) in (3.18) and (3.19), we obtain (3.14) and (3.15).

Now, let $\beta = 0$. Hence M is an α-Sasakian or cosymplectic manifold. In this case, we have $\eta(E_2) = 0$. Thus (3.16) gives us κ_1 =constant. So we get

$$-\kappa_1^2 E_1 = \lambda\xi.$$

Thus $\xi = \pm E_1$. From (3.1) and (3.12), we have

$$\nabla_T\xi = -\alpha\varphi T = 0 = \pm\kappa_1 E_2. \tag{3.21}$$

Since γ is non-geodesic, (3.21) causes a contradiction.

Conversely, if the above conditions are satisfied, one can easily show that γ has C-parallel mean curvature vector field. □

Using the proof of Theorem 3.2, we have the following corollary:

Corollary 3.1. *There does not exist any non-geodesic slant curve of order* 2 *with C-parallel mean curvature vector field in an α-Sasakian or a cosymplectic manifold.*

In the normal bundle, we can state the following theorem:

Theorem 3.3. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order* 2 *with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if it is a Legendre curve with*

$$\kappa_1 = \mp\beta, \quad \xi = \pm E_2, \quad \lambda = \pm\beta'. \tag{3.22}$$

In this case, $\alpha = 0$ and β is not a constant along the curve γ.

Proof. Let γ have C-parallel mean curvature vector field in the normal bundle. From (3.8) and Theorem 3.1, we have

$$\kappa_1' E_2 = \lambda\xi. \tag{3.23}$$

So we have

$$\lambda = \pm\kappa_1',$$
$$\xi = \pm E_2. \tag{3.24}$$

Differentiating (3.24), we find

$$-\alpha\varphi E_1 + \beta E_1 = \mp\kappa_1 E_1. \tag{3.25}$$

(3.25) gives us (3.22) and $\alpha = 0$ along the curve. □

Case II. The osculating order $r = 3$.

For slant curves of order 3, we have the following theorem:

Theorem 3.4. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 3 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-parallel mean curvature vector field if and only if*

i) it is a curve with

$$\kappa_1 = c.e^{\sin\alpha_0 \tan\alpha_0 \int \beta(s)ds}, \tag{3.26}$$

$$\kappa_2 = |\tan\alpha_0| \sqrt{\kappa_1^2 - \beta^2 \sin^2\alpha_0}, \tag{3.27}$$

$$\xi = \cos\alpha_0 E_1 - \frac{\beta\sin^2\alpha_0}{\kappa_1} E_2 - \frac{\kappa_2\cos\alpha_0}{\kappa_1} E_3 \tag{3.28}$$

and

$$\lambda = \frac{-\kappa_1^2}{\cos\alpha_0}, \tag{3.29}$$

where $\kappa_1^2 > \beta^2\sin^2\alpha_0$, $\alpha_0 \neq \frac{\pi}{2}$, c is an arbitrary constant, s is the arc-length parameter of γ, (in this case, M becomes an (α,β)-trans-Sasakian or a β-Kenmotsu manifold); or

ii) it is a helix with

$$\lambda = \frac{-\kappa_1^2}{\cos\alpha_0}, \quad \alpha_0 \neq \frac{\pi}{2},$$
$$\kappa_2 = -\kappa_1\tan\alpha_0$$

and

$$\xi = \cos\alpha_0 E_1 + \sin\alpha_0 E_3.$$

(*In this case, $\alpha \neq 0$ and $\beta = 0$ along the curve.*)

Proof. Let γ have C-parallel mean curvature vector field. From (3.6), we have

$$-\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1\kappa_2 E_3 = \lambda\xi. \tag{3.30}$$

If $\alpha_0 = \frac{\pi}{2}$, we find $\kappa_1 = 0$, which is a contradiction. Thus, $\alpha_0 \neq \frac{\pi}{2}$.

Let $\beta \neq 0$. So M is an (α, β)-trans-Sasakian or a β-Kenmotsu manifold. (3.30) gives us $\xi \in \operatorname{span}\{E_1, E_2, E_3\}$. Thus, we can write

$$\xi = \cos\alpha_0 E_1 + \sin\alpha_0 \left(\cos\theta E_2 + \sin\theta E_3\right), \tag{3.31}$$

where θ is the angle function between E_2 and the orthogonal projection of ξ onto $\operatorname{span}\{E_2, E_3\}$. From (3.30) and (3.31), we find

$$\cos\theta = \frac{-\beta\sin\alpha_0}{\kappa_1}, \quad \sin\theta = \frac{-\kappa_2\cot\alpha_0}{\kappa_1}.$$

So we obtain (3.28). We also have (3.29) using (3.30). Since $\lambda\eta(E_2) = \kappa_1'$, we can calculate

$$\kappa_1' = \kappa_1\beta\sin\alpha_0\tan\alpha_0, \tag{3.32}$$

which gives us (3.26). Using (3.32) in (3.30), we find (3.27).

Now, let $\alpha \neq 0$, $\beta = 0$ along the curve. Since $\eta(E_2) = 0$, (3.30) and (3.31) give us $\kappa_1 > 0$ is a constant, $\theta = \frac{\pi}{2}$ and

$$-\kappa_1^2 E_1 + \kappa_1\kappa_2 E_3 = \lambda(\cos\alpha_0 E_1 + \sin\alpha_0 E_3). \tag{3.33}$$

From (3.33), we find $\kappa_2 = -\kappa_1\tan\alpha_0$. So κ_2 is also a constant. Hence γ is a helix.

Finally, let $\alpha = \beta = 0$ along the curve. In this case, (3.30) and (3.31) give us

$$-\kappa_1^2 E_1 + \kappa_1\kappa_2 E_3 = \lambda\xi, \tag{3.34}$$

$$\xi = \cos\alpha_0 E_1 + \sin\alpha_0 E_3. \tag{3.35}$$

Differentiating (3.35) along γ, we have

$$\frac{\kappa_2}{\kappa_1} = \cot\alpha_0. \tag{3.36}$$

From (3.34), we get

$$\frac{\kappa_2}{\kappa_1} = -\tan\alpha_0. \tag{3.37}$$

By the use of (3.36) and (3.37), we obtain $\cot\alpha_0 = -\tan\alpha_0$, which has no solution. The converse statement is clear. □

Using Theorem 3.4, we give the following corollary:

Corollary 3.2. *There does not exist any non-geodesic slant curve of order* 3 *with C-parallel mean curvature vector field in a cosymplectic manifold.*

In the normal bundle, we can state the following theorem:

Theorem 3.5. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order* 3 *with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if*

i) it is a Legendre curve with

$$\kappa_1 \neq \text{constant},$$

$$\kappa_2 = \frac{\kappa_1'\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1\beta},$$

$$\xi = \frac{-\beta}{\kappa_1}E_2 - \frac{\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1}E_3 \tag{3.38}$$

and

$$\lambda = \frac{-\kappa_1'\kappa_1}{\beta},$$

(in this case, M becomes an (α,β)-trans-Sasakian or a β-Kenmotsu manifold); or

ii) it is a Legendre helix with

$$\xi = E_3, \quad \kappa_2 = \alpha > 0, \quad \lambda = \kappa_1\kappa_2,$$

(in this case, M becomes an α-Sasakian or an (α,β)-trans-Sasakian manifold).

Proof. From (3.8), we have

$$\kappa_1' E_2 + \kappa_1\kappa_2 E_3 = \lambda\xi. \tag{3.39}$$

Then we get

$$\eta(E_1) = 0,$$

$$\kappa_1\eta(E_2) = -\beta. \tag{3.40}$$

Firstly, let $\beta \neq 0$. Then M is an (α,β)-trans-Sasakian or a β-Kenmotsu manifold. From (3.39) and (3.40), we have

$$\lambda = \frac{-\kappa_1'\kappa_1}{\beta},$$

which gives us $\kappa_1 \neq$ constant. We also have

$$\eta(E_3) = \frac{-\beta\kappa_2}{\kappa_1'}. \tag{3.41}$$

By the use of (3.40) and (3.41), we can write

$$\xi = \frac{-\beta}{\kappa_1}E_2 - \frac{\beta\kappa_2}{\kappa_1'}E_3. \tag{3.42}$$

Since ξ is a unit vector field, we obtain

$$\kappa_2 = \frac{\kappa_1'\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1\beta}. \tag{3.43}$$

Finally, let $\beta = 0$ along the curve. Then (3.40) gives us $\eta(E_2) = 0$. From (3.39), we find $\kappa_1 =$ constant, $\xi = E_3$ and $\lambda = \kappa_1\kappa_2$. Differentiating $\xi = E_3$ along the curve γ, we get $\kappa_2 = \alpha$. Thus γ is a Legendre helix. Since $\kappa_2 = \alpha > 0$, M cannot be cosymplectic.

The converse statement is trivial. □

Case III. The osculating order $r \geq 4$.

For non-geodesic slant curves of osculating order $r \geq 4$, we give the following theorem:

Theorem 3.6. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle α_0 in a trans-Sasakian manifold with $\dim M \geq 5$. Then γ has C-parallel mean curvature vector field if and only if it satisfies*

$$\kappa_1 = \text{constant},$$
$$\kappa_2 = -\kappa_1 \tan\alpha_0 = \text{constant},$$
$$\kappa_3 = \frac{-\alpha g(\varphi E_1, E_4)}{\sin\alpha_0} = \sqrt{\alpha^2 - \frac{4\kappa_1^2}{\sin^2(2\alpha_0)}} = \text{constant},$$
$$\xi = \cos\alpha_0 E_1 + \sin\alpha_0 E_3,$$
$$\varphi E_1 \in \operatorname{span}\{E_2, E_4\}, \quad g(\varphi E_1, E_4) \neq 0$$

and

$$\lambda = \frac{-\kappa_1^2}{\cos\alpha_0} = \text{constant}.$$

In this case, M becomes an α-Sasakian manifold.

Proof. Let γ be a curve with C-parallel mean curvature vector field. From (3.6), we have

$$-\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1\kappa_2 E_3 = \lambda\xi. \tag{3.44}$$

Moreover, from Proposition 2.1, M is either α-Sasakian, β-Kenmotsu or cosymplectic. Firstly, let us consider α-Sasakian case. We have

$$\eta(E_2) = 0, \tag{3.45}$$
$$\nabla_T \xi = -\alpha\varphi E_1. \tag{3.46}$$

(3.44) and (3.45) give us κ_1 is a constant. The Legendre case causes a contradiction with γ being non-geodesic; so, $\alpha_0 \neq \frac{\pi}{2}$. From (3.44), we obtain

$$\lambda = \frac{-\kappa_1^2}{\cos\alpha_0} = \text{constant}, \tag{3.47}$$
$$\xi = \cos\alpha_0 E_1 + \sin\alpha_0 E_3. \tag{3.48}$$

Differentiating (3.48) and using (3.46), we get

$$-\alpha\varphi E_1 = (\kappa_1\cos\alpha_0 - \kappa_2\sin\alpha_0)E_2 + \kappa_3\sin\alpha_0 E_4, \tag{3.49}$$

which gives us

$$\varphi E_1 \in \operatorname{span}\{E_2, E_4\}, \tag{3.50}$$
$$\kappa_3 = \frac{-\alpha g(\varphi E_1, E_4)}{\sin\alpha_0}. \tag{3.51}$$

Since $\kappa_3 > 0$, we have $g(\varphi E_1, E_4) \neq 0$. Using (3.44), (3.47) and (3.48), we find

$$\kappa_2 = -\kappa_1 \tan\alpha_0 = \text{constant}. \tag{3.52}$$

Thus, from (3.49) and (3.52), we get

$$\kappa_1\cos\alpha_0 - \kappa_2\sin\alpha_0 = \frac{\kappa_1}{\cos\alpha_0}$$

and

$$-\alpha\varphi E_1 = \frac{\kappa_1}{\cos\alpha_0}E_2 + \kappa_3\sin\alpha_0 E_4. \tag{3.53}$$

Since $g(\varphi E_1, \varphi E_1) = \sin^2 \alpha_0$, using equation (3.53), we have

$$\kappa_3 = \sqrt{\alpha^2 - \frac{4\kappa_1^2}{\sin^2(2\alpha_0)}} = \text{constant}.$$

So the necessity condition is proved. Conversely, if γ is the above curve, (3.44) is satisfied.

Now, let us consider the β-Kenmotsu case. The proof is done as in the proof of Theorem 3.4 and same results are found with some extra conditions which cause contradiction. Firstly, we have

$$\kappa_1 \eta(E_2) = -\beta \sin^2 \alpha_0, \tag{3.54}$$

and

$$\nabla_T \xi = \beta[T - \cos \alpha_0 \xi]. \tag{3.55}$$

Since $\xi \in \text{span}\{E_1, E_2, E_3\}$, we can write

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 \{\cos \theta E_2 + \sin \theta E_3\}, \tag{3.56}$$

where $\theta = \theta(s)$ is the angle function between E_2 and the orthogonal projection of ξ onto span $\{E_2, E_3\}$. Since $\kappa_3 > 0$ and $\sin\alpha_0 \neq 0$; differentiating (3.56) and using (3.55), one can easily find that $\sin \theta = 0$. So we have

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_2. \tag{3.57}$$

From (3.44) and (3.57), we have $\kappa_2 = 0$, a contradiction.

Finally, let us consider the cosymplectic case. In this case, we have

$$\eta(E_2) = 0, \tag{3.58}$$

$$\nabla_T \xi = 0. \tag{3.59}$$

(3.44) and (3.58) give us

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_2, \tag{3.60}$$

$$\kappa_1 = \text{constant}.$$

Differentiating (3.60) and using (3.59), we obtain $\kappa_3 = 0$, which is also a contradiction. □

The following corollaries are direct consequences of Theorem 3.6:

Corollary 3.3. *If the osculating order $r = 4$ in Theorem 3.6, then γ is a helix.*

Corollary 3.4. *There does not exist a non-geodesic slant curve of osculating order $r \geq 4$ with C-parallel mean curvature vector field in a β-Kenmotsu or a cosymplectic manifold.*

In the normal bundle, we can state the following theorem:

Theorem 3.7. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle α_0 in a trans-Sasakian manifold with $\dim M \geq 5$. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if it is a Legendre curve with*

$$\kappa_1 = \text{constant},$$

$$\kappa_2 = \alpha g(\varphi E_1, E_2), \tag{3.61}$$

$$\kappa_3 = -\alpha g(\varphi E_1, E_4), \tag{3.62}$$

$$\kappa_2^2 + \kappa_3^2 = \alpha, \tag{3.63}$$

$$\lambda = \kappa_1 \kappa_2,$$

$$\xi = E_3, \quad \alpha \neq 0$$

and

$$\varphi E_1 = \frac{\kappa_2}{\alpha} E_2 - \frac{\kappa_3}{\alpha} E_4. \tag{3.64}$$

In this case, M becomes an α-Sasakian manifold.

Proof. From (3.8), we have

$$\kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \xi. \tag{3.65}$$

Then we get

$$\eta(E_1) = 0,$$

$$\kappa_1 \eta(E_2) = -\beta. \tag{3.66}$$

Firstly, let $\beta = 0$. Then, from (3.65) and (3.66),

$$\eta(E_2) = 0,$$

$$\lambda = \kappa_1 \kappa_2,$$

$$\xi = E_3. \tag{3.67}$$

Differentiating (3.67), we find

$$-\alpha \varphi E_1 = -\kappa_2 E_2 + \kappa_3 E_4,$$

which gives us (3.61), (3.62), (3.63) and (3.64), where $\alpha \neq 0$, that is, M is an α-Sasakian manifold.

Now, let us assume that $\beta \neq 0$. We have same results in Theorem 3.5, but some extra calculations lead to a contradiction. Since $\xi \in \operatorname{span}\{E_2, E_3\}$, we can write

$$\xi = \cos\theta E_2 + \sin\theta E_3, \tag{3.68}$$

where $\theta = \theta(s)$ is the angle function between ξ and E_2. Differentiating (3.68), we find

$$\kappa_3 = \frac{-\alpha g(\varphi E_1, E_4)}{\sin\theta},$$

which gives us $\alpha \neq 0$. Since $\dim M \geq 5$, this contradicts Proposition 2.1. □

4. Slant curves with C-proper mean curvature vector field

We consider the following cases:

Case I. The osculating order $r = 2$.

For this case, we have the following theorems:

Theorem 4.1. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 2 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-proper mean curvature vector field if and only if $\alpha = 0$ and $\beta \neq 0$ along the curve and*

i) γ is a Legendre circle with $\kappa_1 = \mp\beta = \text{constant}$, $\xi = \pm E_2$, $\lambda = -\beta^3$; or

ii) γ is a non-Legendre slant curve with

$$\begin{aligned} \kappa_1 &= \mp\beta \sin\alpha_0, \\ \kappa_1'' - \kappa_1^3 &= \pm 3\kappa_1'\kappa_1 \tan\alpha_0, \\ \xi &= \cos\alpha_0 E_1 \pm \sin\alpha_0 E_2 \end{aligned} \tag{4.1}$$

and

$$\lambda = \frac{3\kappa_1'\kappa_1}{\cos\alpha_0}. \tag{4.2}$$

Proof. Let γ have C-proper mean curvature vector field. From (3.7), we have

$$3\kappa_1\kappa_1' E_1 + \left(\kappa_1^3 - \kappa_1''\right) E_2 = \lambda\xi. \tag{4.3}$$

Thus, $\xi \in \text{span}\{E_1, E_2\}$. So we can write

$$\xi = \cos\alpha_0 E_1 \pm \sin\alpha_0 E_2. \tag{4.4}$$

Differentiating (4.4) and using (3.12), we find

$$-\alpha\varphi E_1 + \beta\sin^2\alpha_0 E_1 \mp \beta\cos\alpha_0 \sin\alpha_0 E_2 = \mp\kappa_1 \sin\alpha_0 E_1 + \kappa_1 \cos\alpha_0 E_2. \tag{4.5}$$

(4.4) and (4.5) give us $\alpha = 0$ along the curve. We have $\beta \neq 0$, since $\kappa_1 = \mp\beta\sin\alpha_0$. If $\alpha_0 = \frac{\pi}{2}$, then γ is a Legendre curve with $\kappa_1 = \mp\beta = \text{constant}$, $\xi = \pm E_2$, $\lambda = -\beta^3$. Let $\alpha_0 \neq \frac{\pi}{2}$. Then, by the use of (4.3) and (4.4), we obtain (4.1) and (4.2). □

In the normal bundle, we can state the following theorem:

Theorem 4.2. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 2 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with*

$$\kappa_1 = \mp\beta, \quad \xi = \pm E_2, \quad \lambda = \beta'', \tag{4.6}$$

and $\beta(s) \neq as + b$, where a and b are arbitrary constants. In this case, $\alpha = 0$ along the curve.

Proof. Let γ have C-proper mean curvature vector field in the normal bundle. From (3.9) and Theorem 3.1, γ is a Legendre curve with

$$-\kappa_1'' E_2 = \lambda\xi.$$

So we have

$$\lambda = \pm\kappa_1'$$

and

$$\xi = \pm E_2. \tag{4.7}$$

Differentiating (4.7), we find

$$-\alpha\varphi E_1 + \beta E_1 = \mp\kappa_1 E_1. \tag{4.8}$$

(4.8) gives us (4.6) and $\alpha = 0$ along the curve, which completes the proof. □

Case II. The osculating order $r = 3$.

For this case, we have the following theorems:

Theorem 4.3. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 3 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-proper mean curvature vector field if and only if*

i) it satisfies

$$\kappa_2 = \kappa_1 + \alpha,$$
$$2\kappa_1^3 - \kappa_1'' = 0,$$
$$\alpha_0 = \frac{\pi}{4},$$
$$\xi = \frac{\sqrt{2}}{2}\left(E_1 - E_3\right),$$
$$\lambda = 3\sqrt{2}\kappa_1\kappa_1'$$

and

$$\kappa_1 \neq \text{constant},$$

(in this case, M becomes an α-Sasakian or a cosymplectic manifold); or

ii) it satisfies

$$3\kappa_1\kappa_1' = \lambda\cos\alpha_0,$$
$$\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = \lambda\eta(E_2),$$
$$-\left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right) = \lambda\eta(E_3)$$

and

$$\eta(E_2)^2 + \eta(E_3)^2 = \sin^2\alpha_0.$$

(In this case, M becomes an (α, β)-trans-Sasakian or a β-Kenmotsu manifold.)

Proof. Let γ have C-proper mean curvature vector field. Then, from (3.7), we have

$$3\kappa_1\kappa_1' E_1 + \left(\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1''\right) E_2 - (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 = \lambda\xi. \tag{4.9}$$

Now, let us assume that $\beta = 0$. Then we have $\eta(E_2) = 0$, so we can write

$$\xi = \cos\alpha_0 E_1 - \sin\alpha_0 E_3. \tag{4.10}$$

We cannot choose $\eta(E_3) = \sin\alpha_0$, because it leads to a contradiction. Differentiating (4.10), we have

$$-\alpha\varphi E_1 = (\kappa_1\cos\alpha_0 - \kappa_2\sin\alpha_0)E_2, \tag{4.11}$$

which gives us

$$\kappa_2 = \kappa_1\cot\alpha_0 + \alpha. \tag{4.12}$$

Since α is a constant, we obtain

$$\kappa_2' = \kappa_1' \cot \alpha_0. \tag{4.13}$$

From (4.9), we can write

$$3\kappa_1 \kappa_1' = \lambda \cos \alpha_0, \tag{4.14}$$

$$\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa_1'' = 0 \tag{4.15}$$

and

$$2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' = \lambda \sin \alpha_0. \tag{4.16}$$

By the use of (4.12) in (4.15), we get

$$\kappa_1^3 - \sin^2 \alpha_0 \kappa_1'' = 0. \tag{4.17}$$

So we have $\kappa_1 \neq$ constant and $\alpha_0 \neq \frac{\pi}{2}$. In view of (4.12), (4.13), (4.14) and (4.16), we find $\cos 2\alpha_0 = 0$, which means that $\alpha_0 = \frac{\pi}{4}$. Hence, taking $\alpha_0 = \frac{\pi}{4}$ in above equations, the proof is done for α-Sasakian and cosymplectic manifolds.

Now, let us assume that $\beta \neq 0$. (4.9) gives us $\xi \in \operatorname{span}\{E_1, E_2, E_3\}$. So we can write

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 \{\cos \theta E_2 + \sin \theta E_3\}, \tag{4.18}$$

where $\theta = \theta(s)$ is the angle function between E_2 and the orthogonal projection of ξ onto $\operatorname{span}\{E_2, E_3\}$. Using (4.9) and (4.18), the proof is completed. □

In the normal bundle, we can give the following result:

Theorem 4.4. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 3 with contact angle α_0 in a trans-Sasakian manifold. Then γ has C-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with*

i)

$$\kappa_1 = c_1 e^{\alpha s} + c_2 e^{-\alpha s}, \tag{4.19}$$
$$\kappa_2 = \alpha,$$
$$\xi = E_3, \quad \varphi E_1 = E_2$$

and

$$\lambda = -2\alpha^2 (c_1 e^{\alpha s} - c_2 e^{-\alpha s}), \tag{4.20}$$

where c_1 and c_2 are arbitrary constants, (in this case, M becomes an α-Sasakian manifold); or

ii)

$$\lambda = \frac{\kappa_1 \kappa_1'' - \kappa_1^2 \kappa_2^2}{\beta}, \tag{4.21}$$

$$\xi = \frac{-\beta}{\kappa_1} E_2 \pm \frac{\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1} E_3 \tag{4.22}$$

and

$$\pm \left(\kappa_1'' - \kappa_1 \kappa_2^2\right) \sqrt{\kappa_1^2 - \beta^2} = 2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'. \tag{4.23}$$

In this case, M becomes an (α, β)-trans-Sasakian or a β-Kenmotsu manifold.

Proof. Let γ have C-proper mean curvature vector field in the normal bundle. From (3.9) and Theorem 3.1, γ is a Legendre curve with

$$\left(\kappa_1\kappa_2^2 - \kappa_1''\right) E_2 - \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right) E_3 = \lambda\xi. \tag{4.24}$$

Let $\beta = 0$. Then we find $\eta(E_2) = 0$, which gives us

$$\kappa_1\kappa_2^2 - \kappa_1'' = 0, \tag{4.25}$$

$$\xi = E_3 \tag{4.26}$$

and

$$\lambda = -\left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right). \tag{4.27}$$

Differentiating (4.26), we have

$$\kappa_2 = \alpha \tag{4.28}$$

and

$$\varphi E_1 = E_2.$$

Since α is a non-zero constant, by the use of (4.25) and (4.28), we find (4.19). Using (4.19), (4.27) and (4.28), we obtain (4.20).

Now, let $\beta \neq 0$. Then (3.11) and (4.24) give us (4.21). Since the unit vector field $\xi \in \operatorname{span}\{E_2, E_3\}$, using (3.11), we find (4.22). By the use of (4.21), (4.22) and (4.24), we obtain (4.23). Since $\beta \neq 0$, M is an (α, β)-trans-Sasakian or a β-Kenmotsu manifold. □

Case III. The osculating order $r \geq 4$.

In this case, we can state the following theorem:

Theorem 4.5. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle α_0 in a trans-Sasakian manifold with $\dim M \geq 5$. Then γ has C-proper mean curvature vector field if and only if it satisfies*

$$3\kappa_1\kappa_1' = \lambda\cos\alpha_0,$$

$$\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = \lambda\eta(E_2),$$

$$-(2\kappa_1'\kappa_2 + \kappa_1\kappa_2') = \lambda\eta(E_3),$$

$$-\kappa_1\kappa_2\kappa_3 = \lambda\eta(E_4)$$

and

$$\eta(E_2)^2 + \eta(E_3)^2 + \eta(E_4)^2 = \sin^2\alpha_0,$$

where λ is a non-zero differentiable function on I.

Proof. Since ξ is a unit vector field, by the use of (3.7) and (3.10), the proof is completed. □

In the normal bundle, we can give the following theorem:

Theorem 4.6. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle α_0 in a trans-Sasakian manifold* $\dim M \geq 5$. *Then γ has C-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with*

$$\kappa_1\kappa_2^2 - \kappa_1'' = 0,$$
$$\kappa_2 = \alpha g(\varphi E_1, E_2),$$
$$\kappa_3 = -\alpha g(\varphi E_1, E_4),$$
$$\kappa_2^2 + \kappa_3^2 = \alpha,$$
$$\lambda = -2\kappa_1'\kappa_2 - \kappa_1\kappa_2',$$
$$\xi = E_3, \quad \alpha \neq 0$$

and

$$\varphi E_1 = \frac{\kappa_2}{\alpha}E_2 - \frac{\kappa_3}{\alpha}E_4.$$

In this case, M becomes an α-Sasakian manifold.

Proof. The proof is similar to the proof of Theorem 3.7. □

5. Examples

Example 1. Let us consider the 3-dimensional manifold

$$M = \left\{(x, y, z) \in \mathbb{R}^3 | z > 0\right\},$$

where (x, y, z) are the standard coordinates on $\mathbb{R}^3$ and the metric tensor field on M is given by

$$g = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z}$$

are g-orthonormal vector fields in $\chi(M)$. Let φ be the $(1,1)$-tensor field defined by

$$\varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \varphi e_3 = 0.$$

Let us define a 1-form $\eta(Z) = g(Z, e_3)$, for all $Z \in \chi(M)$ and the characteristic vector field $\xi = e_3$. In ([9], [13]), it was proved that $(M, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold. Thus, it is a trans-Sasakian manifold with $\alpha = 0$, $\beta = 1$.

The curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in M with contact angle α_0 if and only if the following equations are satisfied:

$$(\gamma_1')^2 + (\gamma_2')^2 = \sin^2\alpha_0(\gamma_3)^2,$$
$$\gamma_3 = c.e^{-s\cos\alpha_0},$$

where $c > 0$ is an arbitrary constant.

Let $\gamma : I \subseteq \mathbb{R} \to M$, $\gamma(s) = (as + b, ms + n, c)$ where $a, b, m, n, c \in \mathbb{R}$, $c > 0$, $a^2 + m^2 = c^2$ and s is the arc-length parameter on open interval I. The unit tangent vector field T along γ is

$$T = \frac{a}{c}e_1 + \frac{m}{c}e_2.$$

Then γ is a Legendre curve since $\eta(T) = 0$, that is, $\alpha_0 = \frac{\pi}{2}$. Using Koszul's formula, we get $\nabla_T T = -e_3$, which gives us $\kappa_1 = 1$, $E_2 = -e_3$. After simple calculations, we find $\nabla_T E_2 = -T$, that is, $\kappa_2 = 0$. Then γ is of osculating order $r = 2$. From Theorem 4.1 i), γ has C-proper mean curvature vector field in the tangent bundle with $\kappa_1 = \beta = 1$, $\xi = -E_2$, $\lambda = -\beta^3 = -1$. Hence, an explicit example of Theorem 4.1 i) in the given manifold M is $\gamma(s) = (3s, 4s, 5)$.

In the above example, if we take $e_3 = z\frac{\partial}{\partial z}$, $\xi = e_3$ and define the other structures in the same way, we have a trans-Sasakian manifold with $\alpha = 0$, $\beta = -1$ which was given in ([10], [13]). In this manifold, $\gamma(s) = (s, 0, 1)$ is another example of Theorem 4.1 i) with $\kappa_1 = -\beta = 1$, $\xi = E_2$, $\lambda = -\beta^3 = 1$.

We will use the following trans-Sasakian manifold given in [5] to construct new examples.

Let $M = N \times (a, b)$ where N is an open connected subset of $\mathbb{R}^2$ and (a, b) is an open interval in $\mathbb{R}$. Let (x, y, z) be the coordinate functions on M. Now let us take the functions

$$\omega_1, \omega_2 : N \to \mathbb{R}, \quad \sigma, f : M \to \mathbb{R}^*_+.$$

The normal almost contact metric structure (φ, ξ, η, g) on M is given by

$$\varphi = \begin{bmatrix} 0 & 1 & -\omega_2 \\ -1 & 0 & \omega_1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz + \omega_1 dx + \omega_2 dy,$$

$$g = \begin{bmatrix} \omega_1^2 + \sigma e^{2f} & \omega_1\omega_2 & \omega_1 \\ \omega_1\omega_2 & \omega_2^2 + \sigma e^{2f} & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{bmatrix}.$$

Let us choose g-orthonormal frame fields as follows:

$$H_1 = \frac{e^{-f}}{\sqrt{\sigma}}\left[\frac{\partial}{\partial x} - \omega_1\frac{\partial}{\partial z}\right], \quad H_2 = \frac{e^{-f}}{\sqrt{\sigma}}\left[\frac{\partial}{\partial y} - \omega_2\frac{\partial}{\partial z}\right], \quad H_3 = \xi = \frac{\partial}{\partial z}.$$

It is seen that M is a trans-Sasakian manifold with

$$\alpha = \frac{e^{-2f}}{2\sigma}\left(\frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x}\right), \quad \beta = \frac{1}{2\sigma}\frac{\partial \sigma}{\partial z} + \frac{\partial f}{\partial z}.$$

In [5], it is shown that $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in M with contact angle α_0 if and only if

$$(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2\alpha_0}{\sigma}e^{-2f},$$

$$\omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3' = \cos\alpha_0.$$

Using this method, we have the following examples:

Example 2. Let us consider the Legendre helix $\gamma(s) = (0, \frac{s}{2}, 2)$ in $(M, \varphi, \xi, \eta, g)$ where $\omega_1 = f = 0$, $\omega_2 = 2x$ and $\sigma = 2z$. Then M is a trans-Sasakian manifold of type $(\frac{-1}{2z}, \frac{1}{2z})$, that is,

$$\alpha = \frac{-1}{2z} = -\beta.$$

It was shown that $\kappa_1 = \kappa_2 = \frac{1}{4}$ (see [19]). Let us show that γ has C-proper mean curvature vector field in the tangent and normal bundles. After direct calculations, we obtain $T = H_2$, $\nabla_T T = \frac{-1}{4} H_3$. Then we have $\xi = H_3 = -E_2$. Finally, we get $\nabla_T E_2 = \frac{-1}{4} T + \frac{1}{4} H_1$. Hence $E_3 = H_1$. By the use of Theorems 4.3 and 4.4 respectively, we find that γ is a curve with C-proper mean curvature vector field in the tangent bundle with $\lambda = \frac{-1}{32}$ and in the normal bundle with $\lambda = \frac{-1}{64}$. Furthermore, in [5], the authors proved that γ has proper mean curvature vector field (in the tangent bundle) with $\lambda = \frac{1}{8}$.

Example 3. Let us choose $\omega_1 = f = 0$, $\omega_2 = -y$ and $\sigma = z$. So $\alpha = 0$ and $\beta = \frac{1}{2z}$. Thus M is a β-Kenmotsu manifold. Then $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in M if and only if

$$(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2 \alpha_0}{\gamma_3},$$

$$-\gamma_2\gamma_2' + \gamma_3' = \cos \alpha_0.$$

Let us take $\gamma(s) = (0, 2^{3/4}\sqrt{s}, \sqrt{2}s)$ in M. We find $\alpha_0 = \frac{\pi}{2}$, that is, γ is a Legendre curve. After some calculations, using Theorem 3.3, we find that γ is of osculating order $r = 2$ and it has C-parallel mean curvature vector field in the normal bundle with $\kappa_1 = \beta = \frac{\sqrt{2}}{4s}$, $\xi = -E_2$ and $\lambda = -\beta' = \frac{\sqrt{2}}{4s^2}$. Moreover, γ has C-proper mean curvature vector field in the normal bundle with $\lambda = \beta'' = \frac{\sqrt{2}}{2s^3}$ which verifies Theorem 4.2.

Example 4. Let us choose $\omega_1 = f = 0$, $\omega_2 = y$ and $\sigma = z$. Then $\alpha = 0$ and $\beta = \frac{1}{2z}$. Hence M is a β-Kenmotsu manifold. Then $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in M if and only if

$$(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2 \alpha_0}{\gamma_3},$$

$$\gamma_2\gamma_2' + \gamma_3' = \cos \alpha_0.$$

Let us consider the non-Legendre slant curve $\gamma(s) = (\frac{4}{105} 7^{3/4}\sqrt{30s}, 0, \frac{\sqrt{7}s}{15})$ in M with contact angle $\alpha_0 = \arccos(\frac{\sqrt{7}}{15}) = \arcsin(\frac{2\sqrt{2}}{15})$. After some straightforward calculations, using Theorem 4.1 ii), we find that γ has C-proper mean curvature vector field (in the tangent bundle) with

$$\kappa_1 = \frac{\sqrt{14}}{7s},$$

$$\xi = \frac{\sqrt{7}}{15} E_1 - \frac{2\sqrt{2}}{15} E_2,$$

$$\beta = \frac{15\sqrt{7}}{14s},$$

and

$$\lambda = \frac{-90\sqrt{7}}{49s^3}.$$

It is easy to check that κ_1 satisfies

$$\kappa_1'' - \kappa_1^3 = -3\kappa_1'\kappa_1 \tan\alpha_0.$$

Acknowledgement

The authors are thankful to Professor Jun-ichi Inoguchi for his critical comments towards the improvement of the paper.

References

[1] Arroyo, J., Barros, M. and Garay, O. J., *A characterisation of helices and Cornu spirals in real space forms*, Bull. Austral. Math. Soc. **56** (1997), 37–49. MR 1464047.

[2] Baikoussis, C. and Blair, D. E., *On Legendre curves in contact* 3*-manifolds*, Geom. Dedicata **49** (1994), 135–142. MR 1266269.

[3] Blair, D. E., *The theory of quasi-Sasakian structures*, J. Diff. Geom. **1** (1967), 331–345. MR 0226538.

[4] Blair, D. E., *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhauser, Boston, 2002. MR 1874240.

[5] Călin, C. and Crasmareanu, M., *Slant curves in* 3*-dimensional normal almost contact geometry*, Mediterr. J. Math. **10** (2013), 1067–1077. MR 3045696.

[6] Călin, C., Crasmareanu, M., Munteanu, M. I., *Slant curves in three-dimensional* f*-Kenmotsu manifolds*, J. Math. Anal. Appl. **394** (2012), 400–407. MR 2926230.

[7] Chen, B. Y., *Some open problems and conjectures on submanifolds of finite type: recent development*, Tamkang J. Math. **45** (2014), 87–108. MR 3188077.

[8] Cho, J. T., Inoguchi, J. and Lee, J.-E., *On slant curves in Sasakian* 3*-manifolds*, Bull. Austral. Math. Soc. **74** (2006), 359–367. MR 2273746.

[9] De, U. C., Yıldız, A., Yalınız, A. F., *On* φ*-recurrent Kenmotsu manifolds*, Turkish J. Math. **33** (2009), 17–25. MR 2524112.

[10] De, U. C., Sarkar, A., *On three-dimensional trans-Sasakian manifolds,* Extracta Math. **23** (2008), 265–277. MR 2524542.

[11] Inoguchi J. and Lee, J-E., *Almost contact curves in normal almost contact* 3*-manifolds*, J. Geom. **103** (2012), 457–474. MR 3017056.

[12] Inoguchi J. and Lee, J-E., *On slant curves in normal almost contact metric* 3*-manifolds*, Beitr. Algebra Geom. **55** (2014), 603–620.

[13] Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. **24** (1972), 93–103. MR 0319102.

[14] Lee, J. E., Suh, Y. J. and Lee, H., C*-parallel mean curvature vector fields along slant curves in Sasakian* 3*-manifolds*, Kyungpook Math. J. **52** (2012), 49–59. MR 2908750.

[15] Ludden, G. D., *Submanifolds of cosymplectic manifolds*, J. Differential Geometry **4** (1970), 237–244. MR 0271883.

[16] Marrero, J. C., *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162** (1992), 77–86. MR 1199647.

[17] Oubiña, J. A., *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), 187–193. MR 0834769.

[18] Srivastava, S. K., *Almost contact curves in trans-Sasakian* 3*-manifolds*, preprint, arXiv:1401.6429 [math.DG], 2013.

[19] Welyczko, J., *On Legendre curves in* 3*-dimensional normal almost contact metric manifolds*, Soochow J. Math. **33** (2007), 929–937. MR 2404614.

Şaban Güvenç, Cihan Özgür
Department of Mathematics,
Balıkesir University,
10145, Çağış, Balıkesir, Turkey
cozgur@balikesir.edu.tr
sguvenc@balikesir.edu.tr

Received: April 8, 2013
Accepted: May 30, 2014