REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 53, No. 2, 2012, 25–30

SOME RESULTS ON O*-GROUPS

SEBAHATTIN İKIKARDES AND RECEP SAHIN

ABSTRACT. A compact Klein surface with boundary of algebraic genus $g \geq 2$ has at most $12(g-1)$ automorphisms. When a surface attains this bound, it has maximal symmetry, and the group of automorphisms is then called an M^*-group. If a finite group G of odd order acts on a bordered Klein surface X of algebraic genus $g \geq 2$, then $|G| \leq 3(g-1)$. If G acts with the largest possible order $3(g-1)$, then G is called an O^*-group. In this paper, using the results about some normal subgroups of the extended modular group $\overline{\Gamma}$, we obtain some results about O^*-groups. Also, we give the relationships between O^*-groups and M^*-groups.

1. Introduction

Let X be a compact bordered Klein surface of algebraic genus $g \geq 2$. In [8], May proved that the automorphism group G of X is finite, and the order of G is at most $12(g-1)$. Groups isomorphic to the automorphism group of such a compact bordered Klein surface with this maximal number of automorphisms are called M^*-groups [9]. These groups were investigated intensively [1–3, 5, 8–10, 15, 16] . The first important result about M^*-groups was that they must have a certain partial presentation. An M^*-group is generated by three distinct elements α, β, γ obeying nontrivially the following relations,

$$\alpha^2 = \beta^2 = \gamma^2 = (\beta\gamma)^2 = (\alpha\gamma)^3 = I.$$

The order q of $\alpha\beta$ is called an index of the presentation G. There is a nice connection between the index and the action of G on X. If G is an M^*-group with an index q, then G is the group automorphisms of a Klein surface X and the number of boundary components of X equals $|G|/2q$.

In [9], May proved that there is a relationship between the extended modular group and M^*-groups which says a finite group of order at least twelve is an M^*-group if and only if it is the homomorphic image of the extended modular group. In fact, by using known results about normal subgroups of the extended modular group, he found an infinite family of M^*-groups.

Additionally, in [11, Theorem 1] May showed that an M^*-group G is supersolvable if and only if the order of G is 4.3^r for some positive integer r.

2000 *Mathematics Subject Classification.* 30F50;11F06; 20H10.

Key words and phrases. O^*-groups, Extended modular group, M^*-groups.

On the other hand, if a finite group G of odd order acts on a bordered Klein surface X of algebraic genus $g \geq 2$, then $|G| \leq 3(g-1)$. If G acts with the largest possible order $3(g-1)$, then G is called an O^*-group (see, [13] and [14]). A noncyclic group of odd order G is an O^*-group if and only if it is generated by two elements of order 3, (see [13]).

If G is an O^*-group and N is a normal subgroup of G with $[G:N] > 3$, then the quotient group G/N is an O^*-group, [13]. Of course, an O^*-group may have quotient groups of order 3 or trivial, but these are not O^*-groups. For example, $\mathbb{Z}_3$ is not an O^*-group, even though it clearly acts on a bordered surface of genus 2 (a sphere with three holes or a torus with one hole). Finally, O^*-groups are solvable, since all groups of odd order are solvable by the Feit-Thompson Theorem [5].

In this paper, we obtain some results about O^*-groups using the results about normal subgroups of the extended modular group $\overline{\Gamma}$. Also, we give some relationships between O^*-groups and M^*-groups.

2. The extended modular group and related results

The extended modular group $\overline{\Gamma} = PGL(2,\mathbb{Z})$ has a presentation

$$\overline{\Gamma} = \langle r_1,\ r_2,\ r_3 \mid r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^3 = I\ \rangle \cong D_2 *_{\mathbb{Z}_2} D_3$$

where

$$r_1(z) = \frac{1}{\overline{z}}, \qquad r_2(z) = -\overline{z}, \qquad \text{and} \qquad r_3(z) = \frac{-\overline{z}}{\overline{z}+1},$$

and the signature of $\overline{\Gamma}$ is $(0;+;[2,3];\{(-)\})$ (see [7]). The modular group $\Gamma = PSL(2,\mathbb{Z})$ is a subgroup of index 2 in the extended modular group $\overline{\Gamma}$.

It is known that the first commutator subgroup $\overline{\Gamma}'$ of $\overline{\Gamma}$ is a free product of two cyclic groups of order three, i.e.,

$$\overline{\Gamma}' = \langle r_1r_3, r_2r_3r_1r_2 \mid (r_1r_3)^3 = (r_2r_3r_1r_2)^3 = I \rangle \cong \mathbb{Z}_3 * \mathbb{Z}_3,$$

and the signature of $\overline{\Gamma}'$ is $(0;+;[3,3];\{(-)\})$. The orders of the commutator quotient groups $\overline{\Gamma}/\overline{\Gamma}'$ and $\overline{\Gamma}'/\overline{\Gamma}''$ are 4 and 9, respectively. $\overline{\Gamma}' = \overline{\Gamma}^2 = \Gamma^2$ where $\overline{\Gamma}^m$ is called the m-th power subgroup — the subgroup of $\overline{\Gamma}$ generated by the m-th powers of all elements of $\overline{\Gamma}$.

Now we give some results related with the normal subgroups of $\overline{\Gamma}$.

Theorem 2.1. *i) There are exactly 2 normal subgroups of index 12 in $\overline{\Gamma}$ where Γ' is the first commutator subgroup of Γ and Γ_2 is the principal congruence subgroup of Γ of level 2. Explicitly these are $\Gamma' = \langle r_2r_3r_2r_3r_1r_3,\ r_2r_3r_1r_3r_2r_3\rangle$ and $\Gamma_2 = \langle r_2r_3r_2r_3,$ $(r_2r_3r_1r_3)^2\rangle$.*

ii) $\left|\overline{\Gamma}' : \Gamma'\right| = 3$ and $\left|\overline{\Gamma}' : \Gamma_2\right| = 3$.

iii) $\overline{\Gamma}' = \overline{\Gamma}^2 = \Gamma^2$ where $\overline{\Gamma}^m$ is called the m-th power subgroup — the subgroup of $\overline{\Gamma}$ generated by the m-th powers of all elements of $\overline{\Gamma}$.

Proof. See [15] and [16]. □

The study of O^*-groups lies in the study of factor groups of the first commutator subgroup $\overline{\Gamma}'$ of $\overline{\Gamma}$. Thus we give the following theorem.

Theorem 2.2. *A noncyclic group G of odd order is an O^*-group if and only if G is a homomorphic image of the first commutator subgroup $\overline{\Gamma}'$ of $\overline{\Gamma}$.*

Proof. It is clear that O^*-groups are finite quotient groups of $\overline{\Gamma}'$. On the other hand, let $G \simeq \overline{\Gamma}'/N$ be a factor group of odd order larger than 3. It is easy to see that if some of the elements r_1r_3 and $r_2r_3r_1r_2$ belong to N then $\left|\overline{\Gamma}' : N\right| \leq 3$. Therefore the images R_1R_3 and $R_2R_3R_1R_2$ of r_1r_3 and $r_2r_3r_1r_2$, make $\overline{\Gamma}'/N$ an O^*-group. □

Now we get the following results.

Corollary 2.3. *i) The quotient groups $\overline{\Gamma}'/\overline{\Gamma}''$ and $\overline{\Gamma}'/\ (\overline{\Gamma}')^3$ are O^*-groups.*

ii) If G is an O^*-group, then $|G : G'| = 3$ or $|G : G'| = 9$.

Proof. i) It is easy to see from the Theorem 2.1 and $\left|\overline{\Gamma}' : \overline{\Gamma}''\right| = 9$.

ii) Let G be an O^*-group. We know that the order of the quotient group $\overline{\Gamma}'/\overline{\Gamma}''$ is 9. By the Theorem 2.1, the index $|G : G'|$ divides 9 Then $|G : G'| = 1$, 3 or 9. Since the order of G is odd and all groups of odd order are solvable, the index $|G : G'|$ is not equal to 1. Thus $|G : G'| = 3$ or 9. □

Theorem 2.4. *Any O^*-group G possesses either one or two subgroups of index 3. An O^*-group G possesses at most one normal subgroup of index 9.*

Proof. Let G be an O^*-group. G can have at most two subgroups of index 3 since $\overline{\Gamma}'$ has exactly two subgroups of index 3 (namely Γ' and Γ_2). Since G is a homomorphic image of $\overline{\Gamma}'$, the subgroups of G corresponding to each of Γ' and Γ_2 are $G_1 = \langle R_2R_3R_2R_3R_1R_3,\ R_2R_3R_1R_3R_2R_3\rangle$ and $G_2 = \langle R_2R_3R_2R_3, (R_2R_3R_1R_3)^2\rangle$. Also, G can have at most one subgroup of index 9 since $\overline{\Gamma}'$ has a unique normal subgroup of index 9 (namely $\overline{\Gamma}''$). Then if G has a subgroup G_3 of index 9, then G_3 is a homomorphic image of the subgroup $\overline{\Gamma}''$ of $\overline{\Gamma}'$. □

Notice that $\Gamma^2 = \overline{\Gamma}'$ is the only normal subgroup of the modular group Γ of index 2 [4, Table 1]. Using the results in [13] and the table for normal subgroups of Γ in [4], we give the following result.

Corollary 2.5. *Let N be a normal subgroup of Γ such that $N \subset \Gamma^2$.*

i) If $|\Gamma : N| = 2.3p$ for an odd prime p such that 3 divides $p-1$, then Γ^2/N is an O^*-group.

ii) If $|\Gamma : N| = 2.3p^2$ for an odd prime p such that 3 divides $p-1$, then Γ^2/N is an O^*-group.

iii) If $|\Gamma : N| = 2.3p^2$ for an odd prime p such that 3 divides $p+1$, then Γ^2/N is an O^*-group.

This result have impact upon the existence of O^*-groups of certain orders. Now we give the following example using the table in [4].

Example 2.1. *i) If $p = 7$ then $|N| = 42$ then Γ^2/N is an O^*-group.*
ii) If $p = 13$ then $|N| = 1014$ then Γ^2/N is an O^*-group.
iii) If $p = 11$ then $|N| = 726$ then Γ^2/N is an O^*-group.

3. Relationships between O^*-groups and M^*-groups

In this section, we examine some relationships between O^*-groups and M^*-groups. Firstly, we give the following theorem, which is based on a result of Bujalance, Cirre and Turbek in [2, Corollary 3.5].

Theorem 3.1. *If G is an O^*-group generated by two elements u and v each of order three and if G admits any two of the following automorphisms*

$$\delta_1 : u \to u^{-1},\ v \to v^{-1},\quad \delta_2 : u \to v,\ v \to u,\ \delta_3 : u \to v^{-1}, v \to u^{-1}$$

then the semidirect product group $H = G \rtimes_\delta (C_2 \times C_2)$ is an M^*-group.

Note that Corollary 3.5 of Bujalance, Cirre and Turbek in [2] characterized all M^*-groups H with a normal subgroup G of index 4. But, this normal subgroup G may not be an O^*-group. Since the order of any M^*-group H is $12(g-1)$, $g \geq 2$ integer, the order of G may be even number or 3. For example, the group $\mathbb{Z}_2 \times S_3$ is an (supersolvable) M^*-group and has a normal subgroup $\mathbb{Z}_3$ of index 4, but it is not an O^*-group. Now we have the following.

Corollary 3.2. *If a M^*-group H contains a normal subgroup G of index 4 and if G is generated by two elements of order 3 such that the order of G is an odd number larger than 3, then G is an O^*-group.*

Of course, there are several O^*- groups which can be obtained from M^*-groups. For example, if H is a supersolvable M^*-group of order 4.3^r, $r \geq 2$ and if G is a subgroup of index 4 in H, then H is an O^*-group. For $r = 2$, the supersolvable M^*-group $D_3 \times D_3$ acts on a torus with three holes and it has a normal subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3$ of index 4. Therefore $\mathbb{Z}_3 \times \mathbb{Z}_3$ is an (the smallest) O^*- group.

On the other hand, an M^*-group can be obtained from an O^*-group as in Theorem 3.1.

Example 3.1. *We consider the group M_p with presentation*

$$\langle u,\ v \mid u^3 = v^3 = (uv)^3 = (u^{-1}v)^p = I\rangle,\ (see,\ [12]).$$

This group has order $3p^2$. If p is an odd number greater than one or a power of 3, then each group M_p is an O^*-group. For example, the group M_3 is an O^*-group of order 27. From [2, Examples 3.6 (i)], each group M_p admits the above automorphisms δ_1 and δ_2. Therefore $M_p \rtimes_\delta (C_2 \times C_2)$ is an M^*-group. The associated Klein surface can be chosen to be a torus with p^2 boundary components.

Note that, in general, not every M^*-group can be obtained from an O^*-group as in Theorem 3.1.

Theorem 3.3. *Let p be a prime such that 3 divides $p - 1$. The O^*-group G of order $3p$ does not admit any two of the above automorphisms δ_1, δ_2 and δ_3.*

Proof. From [13, Theorem 7], it is known that the group G of order $3p$ is an O^*-group if p is a prime number such that 3 divides $p-1$. Of course, here p is greater than 5. Thus, if the group G admits any two the above automorphisms δ_1, δ_2 and δ_3, then the group $G \rtimes_\delta (C_2 \times C_2)$ of order $12p$ would be an M^*-group. But this is impossible, since there are no M^*-groups of order $12p$ for any prime $p > 5$, see [10, Lemma 4]. □

Finally, by using the Theorem 3.1 and some results of Bujalance, Cirre and Turbek in [3], we give the index of the presentation of an M^*-group which can be obtained from its commutator subgroup.

Corollary 3.4. *i) Let G be an O^*-group. If $|G : G'| \neq 3$, then G admits the automorphisms δ_1 and δ_2 and also $H = G \rtimes_\delta (C_2 \times C_2)$ is the only M^*-group which has G as its commutator subgroup.*

ii) If G is an O^*-group generated by two elements u and v each of order three and if $H = G \rtimes_\delta (C_2 \times C_2)$ is an M^*-group then the index of the presentation of H is $2\ ord(uv^{-1})$.

Proof. For i) and ii) see Corollary 3.12 and Proposition 3.15, in [3], respectively. □

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Sebahattin İkikardes and Recep Sahin
Balıkesir Üniversitesi,
Fen-Edebiyat Fakültesi,
Matematik Bölümü,
10145 Çağış Kampüsü, Balıkesir, Turkey.
skardes@balikesir.edu.tr and rsahin@balikesir.edu.tr

Recibido: 19 de octubre de 2010
Aceptado: 27 de diciembre de 2011