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A new monoid construction under crossed products

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Abstract

In this paper we define a new monoid construction under crossed products for given monoids. We also present a generating set and a relator set for this product. Finally, we give the necessary and sufficient conditions for the regularity of it.

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1 Introduction and preliminaries

In [1], some conditions for the regularity of the semi-direct product are given. Moreover, in [2], a new monoid construction under semi-direct product and Schützenberger product for any two monoids is defined, and its regularity is examined. Also, in [3], necessary and sufficient conditions for this new product to be strongly π -inverse are determined. The regularity and π -inverse property for the Schützenberger product are studied in [4]. By using similar methods as in these above papers, the purpose of this paper is to define a new monoid construction under a crossed product and to give its regularity.

Definition 1 A crossed system of monoids is a quadruple (A, B, α, f) , where A and B are two monoids, and $f : B \times B \rightarrow A$ and $\alpha : B \rightarrow \text{End}(A)$, where $\text{End}(A)$ denotes the collection of endomorphism of A , are two maps such that the following conditions hold:

$$\alpha_{b_1}(\alpha_{b_2}(a))f(b_1, b_2) = f(b_1, b_2)\alpha_{b_1 b_2}(a), \tag{1}$$

$$f(b_1, b_2)f(b_1 b_2, b_3) = \alpha_{b_1}(f(b_2, b_3))f(b_1, b_2 b_3) \tag{2}$$

for all $b_1, b_2, b_3 \in B, a \in A$. The crossed system (A, B, α, f) is called normalized if $f(1_B, 1_B) = 1_A$. The map $\alpha : B \rightarrow \text{End}(A)$ is called weak action and $f : B \times B \rightarrow A$ is called an α -cocycle.

If (A, B, α, f) is a normalized crossed system, then we have $f(1_B, b) = f(b, 1_B) = 1_A$ and $\alpha_{1_B}(a) = a$ by [5].

Let A and B be monoids, and let $f : B \times B \rightarrow A$ and $\alpha : B \rightarrow \text{End}(A)$ be two maps. Let $A\#_{\alpha}^f B := A \times B$ as a set with a binary operation defined by the formula

$$(a_1, b_1)(a_2, b_2) := (a_1(\alpha_{b_1}(a_2))f(b_1, b_2), b_1 b_2)$$

for all $b_1, b_2 \in B, a_1, a_2 \in A$. Then $(A\#_{\alpha}^f B, \cdot)$ is a monoid with unit $1_{A\#_{\alpha}^f B} = (1_A, 1_B)$ if and only if (A, B, α, f) is a normalized crossed system. In this case, the monoid $A\#_{\alpha}^f B$ is called

a crossed product of A and B associated to the crossed system (A, B, α, f) [5]. The reader is referred to [6] and [5] for more details on this material.

Definition 2 Let A and B be monoids. For a subset P of $A \times B$ and $a \in A$, $b \in B$, we let define $Pb = \{(c, db); (c, d) \in P\}$ and $aP = \{(ac, d); (c, d) \in P\}$. Then the Schützenberger product of the monoids A and B , denoted by $A \diamond B$, is the set $A \times \wp(A \times B) \times B$ (where $\wp(\cdot)$ denotes the power set) with the multiplication given by

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2).$$

It is known that $A \diamond B$ is a monoid with identity $(1_A, \emptyset, 1_B)$, where \emptyset is an empty set (see [7]).

2 A new monoid construction

In this section, as one of the main results of the paper, we define a new monoid construction under a crossed product and the Schützenberger product by considering the definitions given in the above section. In order to do that, firstly we give the definition of this new product and then we define its presentation.

Definition 3 Let A and B be monoids. For $P \subseteq A \times B$ and $b \in B$, we define

$$Pb = \{(a, db); (a, d) \in P\}.$$

Let us consider the following multiplication:

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 \alpha_{b_1}(a_2) f(b_1, b_2), P_1 b_2 \cup P_2, b_1 b_2)$$

on the set $A \times \wp(A \times B) \times B$, where $f : B \times B \rightarrow A$ and $\alpha : B \rightarrow \text{End}(A)$ are given in Definition 1.

Let us show the associative property:

$$\begin{aligned} & ((a_1, P_1, b_1)(a_2, P_2, b_2))(a_3, P_3, b_3) \\ &= (a_1 \alpha_{b_1}(a_2) f(b_1, b_2), P_1 b_2 \cup P_2, b_1 b_2)(a_3, P_3, b_3) \\ &= (a_1 \alpha_{b_1}(a_2) f(b_1, b_2) \alpha_{b_1 b_2}(a_3) f(b_1 b_2, b_3), (P_1 b_2 \cup P_2) b_3 \cup P_3, b_1 b_2 b_3) \\ &= (a_1 \alpha_{b_1}(a_2) \alpha_{b_1}(\alpha_{b_2}(a_3)) f(b_1, b_2) f(b_1 b_2, b_3), P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3) \\ &= (a_1 \alpha_{b_1}(a_2) \alpha_{b_1}(\alpha_{b_2}(a_3)) \alpha_{b_1}(f(b_2, b_3)) f(b_1, b_2 b_3), P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3) \end{aligned}$$

and

$$\begin{aligned} & (a_1, P_1, b_1)((a_2, P_2, b_2)(a_3, P_3, b_3)) \\ &= (a_1, P_1, b_1)(a_2 \alpha_{b_2}(a_3) f(b_2, b_3), P_2 b_3 \cup P_3, b_2 b_3) \\ &= (a_1 \alpha_{b_1}(a_2 \alpha_{b_2}(a_3) f(b_2, b_3)) f(b_1, b_2 b_3), P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3) \\ &= (a_1 \alpha_{b_1}(a_2) \alpha_{b_1}(\alpha_{b_2}(a_3)) \alpha_{b_1}(f(b_2, b_3)) f(b_1, b_2 b_3), P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3). \end{aligned}$$

Let us denote this new product by $A_{cp} \#_{\alpha}^f B$. Then, by the above argument, we say that $A_{cp} \#_{\alpha}^f B$ is a monoid with the identity $(1_A, \emptyset, 1_B)$.

By the following remark, we explain why this new product is worked on in this paper.

Remark 1 In [8–13] the authors give some new results about the p -Cockcroft property of some extensions. So, by using these papers, one can also work on this subject by using this new product. So, one can give some new efficient (equivalently, p -Cockcroft) presentation examples. By the way, one can also do further algebraic works on this new product. For instance, in this paper, we give necessary and sufficient conditions for this new product to be regular.

3 Presentation of $A_{cp} \#_{\alpha}^f B$

Let us consider Remark 1. In order to do such algebraic work, we need to define the presentation of this new product. So, in the following theorem, we give a presentation of $A_{cp} \#_{\alpha}^f B$ as one of the main results of this paper.

Theorem 1 *Let us suppose that the monoids A and B are defined by presentations $[X; R]$ and $[Y; S]$, respectively. Then $A_{cp} \#_{\alpha}^f B$ is defined by generators*

$$Z = X \cup Y \cup \{z_{a,b}; a \in A, b \in B\}$$

and the relations

$$R, \tag{3}$$

$$S = W_S, \tag{4}$$

$$yx = \alpha_y(x)y \quad (x \in X, y \in Y), \tag{5}$$

$$z_{a,b}^2 = z_{a,b}, \quad z_{a,b}z_{c,d} = z_{c,d}z_{a,b} \quad (a, c \in A, b, d \in B), \tag{6}$$

$$z_{a,b}y = yz_{a,b}, \quad xz_{a,b} = z_{a,b}x \quad (x \in X, y \in Y, a \in A, b \in B), \tag{7}$$

where W_S is the word on X .

Proof Let us denote the set of all words in Z by Z^* . Let

$$\psi : Z^* \longrightarrow A_{cp} \#_{\alpha}^f B$$

be a homomorphism defined by $\psi(x) = (x, \emptyset, 1_B)$, $\psi(y) = (1_A, \emptyset, y)$ and $\psi(z_{a,b}) = (1_A, \{(a, b)\}, 1_B)$, where $x \in X, y \in Y, a \in A$ and $b \in B$. Also, we can easily see that we have

$$(a_1, \emptyset, 1_B)(a_2, \emptyset, 1_B) = (a_1a_2, \emptyset, 1_B), \tag{8}$$

$$(1_A, \emptyset, b_1)(1_A, \emptyset, b_2) = (f(b_1, b_2), \emptyset, b_1b_2), \tag{9}$$

$$(1_A, P_1, 1_B)(1_A, P_2, 1_B) = (1_A, P_1 \cup P_2, 1_B), \tag{10}$$

$$(a, \emptyset, 1_B)(1_A, \emptyset, b)(1_A, P, 1_B) = (a, P, b)$$

for $a, a_1, a_2 \in A, b, b_1, b_2 \in B$ and $P_1, P_2 \subseteq A \times B$. This says that ψ is onto. Now let us show that $A_{cp} \#_{\alpha}^f B$ satisfies relations (3)-(7). Let $R = x_1x_2 \cdots x_s$, where $x_1, x_2, \dots, x_s \in X$. Thus the

relation (3) follows from $(x_1, \emptyset, 1)(x_2, \emptyset, 1) \cdots (x_s, \emptyset, 1_B) = (R, \emptyset, 1_B) = (1_A, \emptyset, 1_B)$. Also, let $S = y_1 y_2 \cdots y_k$, where $y_1, y_2, \dots, y_k \in Y$, then the relation (4) follows from

$$(1_A, \emptyset, y_1)(1_A, \emptyset, y_2) \cdots (1_A, \emptyset, y_k) = (W_S, \emptyset, S) = (W_S, \emptyset, 1_B),$$

where $W_S = f(y_1, y_2)f(y_1 y_2, y_3)f(y_1 y_2 y_3, y_4) \cdots f(y_1 y_2 \cdots y_{k-1}, y_k)$. For relations (5), we have

$$(1_A, \emptyset, y)(x, \emptyset, 1_B) = (\alpha_y(x)f(y, 1_B), \emptyset, y) = (\alpha_y(x), \emptyset, 1_B)(1_A, \emptyset, y).$$

In fact the relations given in (6) follow from (8), (9) and (10). Now let us show that relations (7) hold by the following:

$$\begin{aligned} (1_A, \{(a, b)\}, 1_B)(1_A, \emptyset, y) &= (1_A, \{(a, by)\}, y) \\ &= (1_A, \emptyset, y)(1_A, \{(a, by)\}, 1_B), \\ (x, \emptyset, 1_B)(1_A, \{(a, b)\}, 1_B) &= (x, \{(a, b)\}, 1_B) \\ &= (1_A, \{(a, b)\}, 1_B)(x, \emptyset, 1_B). \end{aligned}$$

Thus these above arguments say that ψ induces an epimorphism $\overline{\psi}$ from the monoid defined by (3)-(7), say M , onto $A_{cp} \#_{\alpha}^f B$.

Let us consider the relations (5) and (7). By using these relations, there exist words w_x in X^* , $w_y \in Y^*$ and $w_{a,b} \in \{z_{a,b} : a \in A, b \in B\}^*$ such that $w = w_x w_y w_{a,b}$ in M for $w \in Z^*$. Moreover, it can be noted that relations (6) can be used to prove that there exists a set $P(w) \subseteq A \times B$ such that $w_{a,b} = \prod_{(a,b) \in P(w)} z_{a,b}$. So, we have

$$\begin{aligned} \overline{\psi}(w) &= \psi(w) = \psi(w_x w_y w_{a,b}) = \psi(w_x)\psi(w_y)\psi(w_{a,b}) \\ &= (w_x, \emptyset, 1_B)(1_A, \emptyset, w_y)(1_A, P(w), 1_B) \\ &= (w_x, P(w), w_y) \end{aligned}$$

for any word $w \in Z^*$.

Now, let us take $w' = w'_x w'_y w'_{a,b}$ and $w'' = w''_x w''_y w''_{a,b}$ for some $w', w'' \in Z^*$. If $\psi(w') = \psi(w'')$, then, by the equality of these components, we deduce that $w'_x = w''_x$ in A , $w'_y = w''_y$ in B and $P(w') = P(w'')$. Relations (3) and (4) imply that $w'_x = w''_x$ and $w'_y = w''_y$ hold in M . So that $w' = w''$ holds. Thus $\overline{\psi}$ is injective. \square

4 Regularity of $A_{cp} \#_{\alpha}^f B$

Let A and B be monoids. As depicted in Remark 1, one can work on this new product to show some algebraic properties. To this end, in this section we define the necessary and sufficient conditions for $A_{cp} \#_{\alpha}^f B$ to be regular.

For an element a in a monoid M , let us take a^{-1} for the set of inverses of a in M , that is, $a^{-1} = \{b \in B : aba = a \text{ and } bab = b\}$. Hence M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the empty set.

The reader is referred to [2-4] and [1] for more details.

Let us consider the notations given in Definition 3. Then we have the following theorem as a final main result of this paper.

Theorem 2 *Let A and B be any monoids. The product $A_{cp} \#_{\alpha}^f B$ is regular if and only if A is a regular monoid and B is a group.*

Proof Let us suppose that $A_{cp} \#_{\alpha}^f B$ is regular. Thus, for $(a, \{(1_A, 1_B)\}, 1_B) \in A_{cp} \#_{\alpha}^f B$, there exists (x, P, y) such that

$$\begin{aligned} (a, \{(1_A, 1_B)\}, 1_B) &= (a, \{(1_A, 1_B)\}, 1_B)(x, P, y)(a, \{(1_A, 1_B)\}, 1_B) \\ &= (ax, \{(1_A, 1_B)\}y \cup P, y)(a, \{(1_A, 1_B)\}, 1_B) \\ &= (ax\alpha_y(a), \{(1_A, 1_B)\}y \cup P \cup \{(1_A, 1_B)\}, y), \\ (x, P, y) &= (x, P, y)(a, \{(1_A, 1_B)\}, 1_B)(x, P, y) \\ &= (x\alpha_y(a), P \cup \{(1_A, 1_B)\}, y)(x, P, y) \\ &= (x\alpha_y(a)\alpha_y(x)f(y, y), Py \cup \{(1_A, 1_B)\}y \cup P, y^2). \end{aligned}$$

Thus we have $y = 1_B$. This gives that $a = axa$ and $x = xax$. Hence A is regular. By using the similar argument, for $(1_A, \{(1_A, 1_B)\}, b) \in A_{cp} \#_{\alpha}^f B$, there exists (x, P, y) such that

$$\begin{aligned} (1_A, \{(1_A, 1_B)\}, b) &= (1_A, \{(1_A, 1_B)\}, b)(x, P, y)(1_A, \{(1_A, 1_B)\}, b) \\ &= (\alpha_b(x)f(b, y), \{(1_A, 1_B)\}y \cup P, by)(1_A, \{(1_A, 1_B)\}, b) \\ &= (\alpha_b(x)f(b, y)f(by, b), \{(1_A, 1_B)\}yb \cup Pb \cup \{(1_A, 1_B)\}, byb), \\ (x, P, y) &= (x, P, y)(1_A, \{(1_A, 1_B)\}, b)(x, P, y) \\ &= (xf(y, b), Pb \cup \{(1_A, 1_B)\}, y)(x, P, y) \\ &= (xf(y, b)\alpha_{yb}(x)f(yb, y), Pby \cup \{(1_A, 1_B)\}y \cup P, yby). \end{aligned}$$

Here, since we have

$$\begin{aligned} \{(1_A, 1_B)\} &= \{(1_A, 1_B)\}yb \cup Pb \cup \{(1_A, 1_B)\}, \\ P &= Pby \cup \{(1_A, 1_B)\}y \cup P \end{aligned}$$

and, in particular, $(1_A, yb) = (1_A, 1_B)$ and $Pby = P$, we get $yb = by = 1_B$. This says that B is a group.

Suppose conversely that A is a regular monoid and B is a group. Let us take $(a, P, b) \in A_{cp} \#_{\alpha}^f B$. Since B is a group, then there exists y in B such that $by = yb = 1_B$. Also, since A is regular, we can take $c = \alpha_y(v)$ for some $f(b, y)v \in a^{-1}$ such that $v \in (af(b, y))^{-1}$. Now, let us consider the following:

$$\begin{aligned} a\alpha_b(c)f(b, y)\alpha_{by}(a)f(by, b) &= a\alpha_b(c)f(b, y)a = a\alpha_b(\alpha_y(v))f(b, y)a \\ &= af(b, y)\alpha_{by}(v)a = af(b, y)va = a, \\ c\alpha_y(a)f(y, b)\alpha_{yb}(c)f(yb, y) &= c\alpha_y(a)f(y, b)c = \alpha_y(v)\alpha_y(a)f(y, b)\alpha_y(v) \\ &= \alpha_y(v)\alpha_y(a)\alpha_y(f(b, y))\alpha_y(v) \\ &= \alpha_y(vaf(b, y)v) = \alpha_y(v) = c. \end{aligned}$$

Also, by choosing, $P_2 = P_1y \subseteq A \times B$, where $P_1 \subseteq A \times B$, we get

$$P_1yb \cup P_2b \cup P_1 = P_1 \cup P_1yb \cup P_1 = P_1 \cup P_1 \cup P_1 = P_1,$$

$$P_2by \cup P_1y \cup P_2 = P_1y \cup P_1y \cup P_1y = P_1y = P_2.$$

Consequently, for every $(a, P_1, b) \in A_{cp} \#_{\alpha}^f B$, there exists $(c, P_2, y) \in A_{cp} \#_{\alpha}^f B$ such that

$$\begin{aligned} (a, P_1, b)(c, P_2, y)(a, P_1, b) &= (\alpha_b(c)f(b, y)\alpha_{by}(a)f(by, b), P_1yb \cup P_2b \cup P_1, byb) \\ &= (a, P_1, b), \end{aligned}$$

$$\begin{aligned} (c, P_2, y)(a, P_1, b)(c, P_2, y) &= (c\alpha_y(a)f(y, b)\alpha_{yb}(c)f(yb, y), P_2by \cup P_1y \cup P_2, yby) \\ &= (c, P_2, y), \end{aligned}$$

where $P_2 = P_1y$, $by = yb = 1_B$ and $c = \alpha_y(v)$ for some $f(b, y)v \in a^{-1}$ such that $v \in (af(b, y))^{-1}$.
 Hence the result. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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