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GENERALIZED PELL SEQUENCES RELATED TO THE EXTENDED GENERALIZED HECKE GROUPS $\overline{H}_{3,q}$ AND AN APPLICATION TO THE GROUP $\overline{H}_{3,3}$

Furkan Birol, Özden Koruoğlu*, Recep Sahin, and Bilal Demir

Abstract. We consider the extended generalized Hecke groups $\overline{H}_{3,q}$ generated by $X(z) = -(z-1)^{-1}$, $Y(z) = -(z+\lambda_q)^{-1}$ with $\lambda_q = 2\cos(\frac{\pi}{q})$ where $q \geq 3$ an integer. In this work, we study the generalized Pell sequences in $\overline{H}_{3,q}$. Also, we show that the entries of the matrix representation of each element in the extended generalized Hecke Group $\overline{H}_{3,3}$ can be written by using Pell, Pell-Lucas and modified-Pell numbers.

1. Introduction

The Pell, Pell-Lucas and modified Pell numbers respectively satisfy the recurrence relation with initial conditions

$$\begin{aligned} P_n &= 2P_{n-1} + P_{n-2}\ , \qquad P_o = 0 \text{ and } P_1 = 1 \\ Q_n &= 2Q_{n-1} + Q_{n-2}\ , \qquad Q_0 = Q_1 = 2 \\ q_n &= 2q_{n-1} + q_{n-2}\ , \qquad q_0 = q_1 = 1 \end{aligned}$$

The nth Pell, Pell-Lucas and modified-Pell numbers are explicitly given by the Binet-type formulas

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

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$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$$

$$q_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}$$

It is easy to see that

$$P_n + P_{n-1} = q_n = \frac{Q_n}{2}.$$

There are many generalizations of Pell, Pell-Lucas and modified-Pell sequences in the literature. For example, in [3], Horadam defined a second-order linear recurrence sequence $W_{n+1} = pW_n + qW_{n-1}, W_o = a$ and $W_1 = b$, (where a, b, p and q are arbitrary real numbers for $n > 0$). In [15], Bicknell studied the generalized Pell sequence $U_n = bU_{n-1} + U_{n-2}$. Here if $b = 2$, we get classic Pell sequence. Similarly, in [18], Catarino defined the generalized Pell sequences that are $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$ for $n \geq 1$ and $k > 0$ ($P_{k,o} = 0$ and $P_{k,1} = 1$). Serkland, in his master thesis [5], used a matrix generator of Pell sequence firstly, that is $M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}$. In [11], Ercalano obtained the matrix generators of Pell-Lucas sequences similarly. Also, there are many studies related to the usual and the generalized Pell sequences in [1, 2, 24].

On the other hand, in [12], Lehner introduced the generalized Hecke groups $H_{p,q}$, by taking

$$X = \frac{-1}{z - \lambda_p} \text{ and } V = z + \lambda_p + \lambda_q,$$

where $2 \leq p \leq q \leq \infty$, $p + q > 4$, $\lambda_p = 2\cos(\frac{\pi}{p})$, $\lambda_q = 2\cos(\frac{\pi}{q})$ (p and q are integers). Here if we take $Y = XV = -\frac{1}{z+\lambda_q}$, then we get the presentation,

$$H_{p,q} =< X, Y : X^p = Y^q = I > \simeq C_p * C_q.$$

In fact, generalized Hecke groups $H_{p,q}$ are the groups $G_{m,n}$ studied by Calta and Schmidt in [13] and [14].

In [4], the authors defined extended generalized Hecke groups $\overline{H}_{p,q}$, by adding the reflection $R(z) = \frac{1}{\bar{z}}$ to the generators of $H_{p,q}$ with presentation;

$\overline{H}_{p,q} =< X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I >\cong D_p *_{\mathbb{Z}_2} D_q$. Extended generalized Hecke groups $\overline{H}_{p,q}$ are the groups generated by $< A, B, C >$ in [27, pp.2665]

Here, if $p = 2$, we get the Hecke groups $H_{2,q} = H_q$ and the extended Hecke groups $\overline{H}_{2,q} = \overline{H}_q$ respectively. All Hecke groups H_q are included in generalized Hecke groups $H_{p,q}$. We know from [12] that $|H_q : H_{q,q}| = 2$. Then, we have $H_{3,3} \leq \Gamma$ and $\overline{H}_{3,3} \leq \overline{\Gamma}$. The most studied Hecke groups in the literature are modular group $\Gamma = H_3$ and extended modular group $\overline{\Gamma} = \overline{H}_3$. As the coefficents of all elements are integers in Γ and $\overline{\Gamma}$, there are many studies in the literature about these groups [7],[8],[9],[19],[22] and [23].

There are strong connections between the modular, extended modular group and the recurrence number sequences that are Fibonacci, Pell and Pell- Lucas. In [20, 21], Mushtaq and Hayat obtained the relations between the generalized Pell sequence and the coset diagrams in modular group Γ. In [16], Yilmaz studied the relations between generators $\begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix}$ in Hecke groups $H(\sqrt{q})$ ($q \geq 5$ prime numbers) and the generalized Fibonacci and Lucas sequences. In [25], the authors defined the generalized Pell sequences $U_k = 2\sqrt{m}U_{k-1} + U_{k-2}$ for $k \geq 2$ and related these sequences and they gave some relations with the principle subgroups $H_2(\sqrt{m})$ of the Hecke groups $H(\sqrt{m})$. Also Jones and Thornton showed in [10] that there is a relationship between Fibonacci numbers and the entries of a matrix representation of the element

$$f = RXY = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \overline{\Gamma},$$

Here, the k^{th} power of f is

$$f^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix}$$

where f_k is the Fibonacci sequence. Using these results, Koruoğlu and Şahin obtained the generalized Fibonacci sequences in $\overline{\Gamma}$ [17]. Then, they got all the elements of the extended modular group $\overline{\Gamma}$ by using Fibonacci numbers.

In this paper, we obtain a recurrence sequence which is a generalized Pell sequence using the elements $RXY, RX^2Y, RXY^2, RX^2Y^2$ in the group $\overline{H}_{3,q}$. In these sequences, we get the Pell sequence if $q = 3$. Then, we give an application using these results to the group $\overline{H}_{3,3}$. In that, we prove that the matrix entries of the each element of the group $\overline{H}_{3,3}$ can be written with Pell, Pell-Lucas and modified-Pell numbers.

2. Generalized Pell sequences in the extended Hecke groups $\overline{H}_{3,q}$

The group $\overline{H}_{3,q}$ is generated the following generators

$$X = \frac{-1}{z-1},\ Y = -\frac{1}{z+\lambda_q} \text{ and } R(z) = \frac{1}{\overline{z}}$$

where $\lambda_q = 2\cos(\frac{\pi}{q})$, q an integer $3 \leq q$ .Then we get the presentation of the group $\overline{H}_{3,q}$,

$$\overline{H}_{3,q} =< X, Y, R : X^3 = Y^q = R^2 = (XR)^2 = (YR)^2 = I > .$$

Throughout this paper, we identify matrix represention of any elements in $\overline{H}_{3,q}$. We use only the matrix representation A, because $\pm A$ represent the same transformation. Hence, we write the generators as

$$X = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.1. *Consider the following block forms in $\overline{H}_{3,q}$:*

$$\begin{aligned} XYR &= \begin{pmatrix} \lambda_q & 1 \\ 1+\lambda_q & 1 \end{pmatrix}, RXY = \begin{pmatrix} 1 & 1+\lambda_q \\ 1 & \lambda_q \end{pmatrix}, \\ X^2YR &= \begin{pmatrix} 1+\lambda_q & 1 \\ 1 & 0 \end{pmatrix}, RX^2Y = \begin{pmatrix} 0 & 1 \\ 1 & 1+\lambda_q \end{pmatrix} \end{aligned}$$

Using these block forms, we have the followings.

(i) $(XYR)^k = \begin{pmatrix} \lambda_q G_k + G_{k-1} & G_k \\ (1+\lambda_q)G_k & G_k + G_{k-1} \end{pmatrix}$

(ii) $(RXY)^k = \begin{pmatrix} G_k + G_{k-1} & (1+\lambda_q)G_k \\ G_k & \lambda_q G_k + G_{k-1} \end{pmatrix}$

(iii) $(X^2YR)^k = \begin{pmatrix} G_{k+1} & G_k \\ G_k & G_{k-1} \end{pmatrix}$

(iv) $(RX^2Y)^k = \begin{pmatrix} G_{k-1} & G_k \\ G_k & G_{k+1} \end{pmatrix}$

where $G_0 = 0$, $G_1 = 1$ *and* $G_n = (1+\lambda_q)G_{n-1} + G_{n-2}$ *for all* $n \geq 2$ *integers.*

Proof. (i) In order to prove, we use induction method. Firstly for $k = 2$,

$$\begin{aligned}(XYR)^2 &= \begin{pmatrix} \lambda_q & 1 \\ 1+\lambda_q & 1 \end{pmatrix}\begin{pmatrix} \lambda_q & 1 \\ 1+\lambda_q & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_q(1+\lambda_q)+1 & 1+\lambda_q \\ (1+\lambda_q)(1+\lambda_q) & 1+\lambda_q+1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_q G_2 + G_1 & G_2 \\ (1+\lambda_q)G_2 & G_2+G_1 \end{pmatrix}.\end{aligned}$$

Hence, we obtained correct result for $k = 2$.

Secondly, let us assume that

$$(XYR)^{k-1} = \begin{pmatrix} \lambda_q G_{k-1} + G_{k-2} & G_{k-1} \\ (1+\lambda_q)G_{k-1} & G_{k-1}+G_{k-2} \end{pmatrix}, \quad k-1 \in \mathbb{Z}^+.$$

Finally,

$$\begin{aligned}&(XYR)^k = (XYR)^{k-1}(XYR) \\ &= \begin{pmatrix} \lambda_q G_{k-1} + G_{k-2} & G_{k-1} \\ (1+\lambda_q)G_{k-1} & G_{k-1}+G_{k-2} \end{pmatrix}\begin{pmatrix} \lambda_q & 1 \\ 1+\lambda_q & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_q G_k + G_{k-1} & G_k \\ (1+\lambda_q)G_k & G_k+G_{k-1} \end{pmatrix}\end{aligned}$$

Therefore, we obtain a real number sequence G_n that contains the Pell-sequence. If we put $\lambda_q = 1$, we get the known sequence $G_n = 2G_{n-1} + G_{n-2}$.

The other cases of this theorem are easily proven similarly using the induction method. □

3. An application to the Generalized Hecke Group $\overline{H}_{3,3}$

Now we give an application to the group $\overline{H}_{3,3}$. The group $\overline{H}_{3,3}$ is a subgroup of the extended modular group $\overline{H}_3$ and there is a relationship between automorphism group of a compact bordered Klein surface with maximum odd order (O^*-group) and the group $\overline{H}_{3,3}$, [6] and [26].

Here, our aim is to find the entries of the matrix presentation of the elements in $\overline{H}_{3,3}$. For the purpose of that, we use the blocks in $\overline{H}_{3,3}$ as

$$XY = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, X^2Y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, XY^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, X^2Y^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

If we use the presentation of the group $\overline{H}_{3,3}$ and these blocks, we can express each reduced word in $\overline{H}_{3,3}$ as either

$$Y^a(X^{i_0}Y^{j_0})^{m_0}(X^{i_1}Y^{j_1})^{m_1}...(X^{i_n}Y^{j_n})^{m_n}X^b$$

or

$$Y^a(X^{i_0}Y^{j_0})^{m_0}(X^{i_1}Y^{j_1})^{m_1}...(X^{i_n}Y^{j_n})^{m_n}X^bR$$

where $a, b, i_c, j_c = 0, 1$ or 2 and m_c, n_c are positive integers $(0 \leq c \leq n)$.

Here, we take into account the matrix representations of four elements are

$$RXY = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, RX^2Y = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$
$$RXY^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, RX^2Y^2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Thus, we can find each element of this group by using RXY, RX^2Y, RXY^2, RX^2Y^2. Firstly we give the following the Lemma.

Lemma 3.1. *In $\overline{H}_{3,3}$, $RXY = X^2Y^2R$, $RX^2Y = XY^2R$, $RXY^2 = X^2YR$, $RX^2Y^2 = XYR$.*

Proof. Using the relations $X^3 = Y^3 = R^2 = (XR)^2 = (YR)^2 = I$ in $\overline{H}_{3,3}$, we obtain these equalities. □

Now we calculate the k^{th} powers of RXY, RX^2Y, RXY^2, RX^2Y^2. These matrix entries can be written as Pell, Pell-Lucas and modified Pell numbers so these results are valuable. We recall that P_k is the kth Pell number and Q_k is the kth Pell-Lucas number.

Lemma 3.2. *(i) For $m = RXY^2 = X^2YR$,*

$$m^k = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{pmatrix}$$

(ii) For $n = RX^2Y = XY^2R$,

$$n^k = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^k = \begin{pmatrix} P_{k-1} & P_k \\ P_k & P_{k+1} \end{pmatrix}$$

(iii) For $t = RX^2Y^2 = XYR$,

$$t^k = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^k = \begin{pmatrix} P_{k-1} + P_k & P_k \\ 2P_k & P_{k-1} + P_k \end{pmatrix} = \begin{pmatrix} \frac{Q_k}{2} & P_k \\ 2P_k & \frac{Q_k}{2} \end{pmatrix}$$

(iv) For $l = RXY = X^2Y^2R$,

$$l^k = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} P_{k-1} + P_k & 2P_k \\ P_k & P_{k-1} + P_k \end{pmatrix} = \begin{pmatrix} q_k & 2P_k \\ P_k & q_k \end{pmatrix}$$

Corollary 3.3. *Each reduced word in the group* $\overline{H}_{3,3}$ *can be written as product of the four elements* RXY, RX^2Y, RXY^2, RX^2Y^2. *Hence each matrix entries are written as Pell, Pell-Lucas and modified-Pell numbers.*

By using this Corollary, we give two examples.

Example 3.4. *Consider the reduced word*

$$W(X, Y, R) = RXRYYRXRYYXRYRXXY$$

in $\overline{H}_{3,3}$. *We write this word as*

$$\begin{aligned} W(X, Y, R) &= (RXR)YY(RX)RYYXR(YR)(XXY) \\ &= (X^2Y^2)(X^2Y^2)(XY^2)(X^2Y) \end{aligned}$$

If we use $X^2Y^2 = Rt = lR, XY^2 = Rm = nR, X^2Y = Rn$, *then we can write*

$$\begin{aligned} W(X, Y, R) &= ltn^2 \\ &= \begin{pmatrix} P_0 + P_1 & 2P_1 \\ P_1 & P_0 + P_1 \end{pmatrix} \begin{pmatrix} P_0 + P_1 & P_1 \\ 2P_1 & P_0 + P_1 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} \\ &= \begin{pmatrix} q_1 & 2P_1 \\ P_1 & q_1 \end{pmatrix} \begin{pmatrix} \frac{Q_1}{2} & P_1 \\ 2P_1 & \frac{Q_1}{2} \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}. \end{aligned}$$

Example 3.5. *Consider the reduced word*

$$W(X, Y, R) = XXYXYYXXYXXYXXYXY$$

in $\overline{H}_{3,3}$. *Similarly we write this word as*

$$
\begin{aligned}
W(X,Y,R) &= m^3nml \\
&= \begin{pmatrix} P_4 & P_3 \\ P_3 & P_2 \end{pmatrix}\begin{pmatrix} P_0 & P_1 \\ P_1 & P_2 \end{pmatrix}\begin{pmatrix} P_2 & P_1 \\ P_1 & P_0 \end{pmatrix}\begin{pmatrix} P_0+P_1 & 2P_1 \\ P_1 & P_0+P_1 \end{pmatrix} \\
&= \begin{pmatrix} P_4 & P_3 \\ P_3 & P_2 \end{pmatrix}\begin{pmatrix} P_0 & P_1 \\ P_1 & P_2 \end{pmatrix}\begin{pmatrix} P_2 & P_1 \\ P_1 & P_0 \end{pmatrix}\begin{pmatrix} q_1 & 2P_1 \\ P_1 & q_1 \end{pmatrix}.
\end{aligned}
$$

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