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Rotational submanifolds in Euclidean spaces

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The rotational embedded submanifold was first studied by Kuiper as a submanifold in $\mathbb{E}^{n+d}$. The generalized Beltrami submanifolds and toroidal submanifold are the special examples of these kind of submanifolds. In this paper, we consider 3-dimensional rotational embedded submanifolds in Euclidean 5-space $\mathbb{E}^5$. We give some basic curvature properties of this type of submanifolds. Further, we obtain some results related with the scalar curvature and mean curvature of these submanifolds. As an application, we give an example of rotational submanifold in $\mathbb{E}^5$.

Keywords: Rotational submanifolds; scalar curvature; mean curvature.

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1. Introduction

The Gaussian curvature and mean curvature of the surfaces in Euclidean spaces play an important role in differential geometry. Especially, surfaces with constant Gaussian curvature [22], and constant mean curvature form nice classes of surfaces which are important for surface modeling [9]. Surfaces with constant negative curvature are known as pseudo-spherical surfaces (see, [16]). Rotational surfaces in Euclidean spaces are also an important subject of differential geometry. The rotational surfaces in $\mathbb{E}^3$ are called surfaces of revolution. Recently, Velickovic classified all rotational surfaces in $\mathbb{E}^3$ with constant Gaussian curvature [21]. Rotational

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surfaces in $\mathbb{E}^4$ were first introduced by Moore in 1919. In the recent years, some mathematicians have taken an interest in the rotational surfaces in $\mathbb{E}^4$, for example Ganchev and Milousheva [15], Dursun and Turgay [14], Arslan *et al.* [3]. The rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4$ are studied in [4]. Arslan *et al.* in [3] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical. They also gave some special classes of generalized rotational surfaces as examples. See also [5, 7, 8, 11, 23] for the rotational surfaces (with constant Gaussian curvature) in Euclidean 4 -space $\mathbb{E}^4$. For higher dimensional case, Arslan *et al.* defined rotational embedded surfaces in Euclidean spaces [6].

In [16], Gorkavyi and Nevmerzhitskaya introduced a special class of curves in $\mathbb{E}^n$ called generalized tractrices. Then, by applying special motions in $\mathbb{E}^n$ to generalized tractrices, they construct a special class of pseudo-spherical surfaces in $\mathbb{E}^n$ called generalized Beltrami surfaces.

In [18], Kuiper considered a unit speed regular curve γ in $\mathbb{E}^{n+1}$ and a vector function ρ represents either a unit speed curve $\rho = \rho(u)$ or a $(n-1)$-dimensional submanifold W^{n-1} in $S^{n-1} \subset \mathbb{E}^n$. Then, the rotation of γ around ρ give rise a submanifold M^n in $\mathbb{E}^{n+d}$, which is called *rotational submanifold*. Generalized Beltrami submanifolds and toroidal submanifolds [2, 20] are the special examples of these kind of submanifolds. See also [12, 13, 17, 19] for rotational submanifolds in higher dimensional case.

This paper is organized as follows: In Sec. 2, we give some basic concepts of the second fundamental form and curvatures of the submanifolds in $\mathbb{E}^{n+d}$. In Sec. 3, we consider 3-dimensional rotational submanifolds in $\mathbb{E}^5$. Further, we give some basic curvature properties of two types of rotational submanifolds $\mathbb{E}^5$. Consequently, we obtain some results related with the mean curvature and scalar curvature of 3-dimensional rotational submanifolds in $\mathbb{E}^5$.

2. Basic Concepts

Let M^n be an n-dimensional smooth submanifold in $\mathbb{E}^{n+d}$ given with the isometric immersion (position vector), $X(s, u_1, \ldots, u_{n-1}) : (s, u_1, \ldots, u_{n-1}) \in U \subset \mathbb{E}^n$. The tangent space to M^n at an arbitrary point $p = X(s, u_1, \ldots, u_{n-1})$ of M^n span$\{X_s, \ldots, X_{u_{n-1}}\}$. In the chart $(s, u_1, \ldots, u_{n-1})$ the coefficients of the first fundamental form of M^n are given by

$$g_{ij} = \langle X_{u_i}, X_{u_j} \rangle, \quad u_0 = s, \quad 0 \leq i, \ j \leq n-1, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product [1]. Let $\chi(M^n)$ and $\chi^{\perp}(M^n)$ be the space of the smooth vector fields tangent and normal to M^n, respectively. Given any local orthonormal vector fields $X_1, X_2, \ldots, X_n$ tangent to M^n, consider the second fundamental map $h : \chi(M^n) \times \chi(M^n) \to \chi^{\perp}(M^n)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j, \quad 1 \leq i, \ j \leq n. \tag{2}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M^n and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear [10].

For any arbitrary orthonormal normal frame field $\{N_1, N_2, \ldots, N_d\}$ of M^n, recall the shape operator $A : \chi^{\perp}(M^n) \times \chi(M^n) \to \chi(M^n)$;

$$A_{N_\alpha} X_j = -(\widetilde{\nabla}_{X_j} N_\alpha)^T, \quad 1 \le \alpha \le d, \ \ X_j \in \chi(M^n). \tag{3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_\alpha} X_j, X_i \rangle = \langle h(X_i, X_j), N_\alpha \rangle = h_{ij}^\alpha, \quad 1 \le i, \ \ j \le n; \ 1 \le \alpha \le d, \tag{4}$$

where h_{ij}^α are the coefficients of the second fundamental form. The Eq. (2) is called *Gaussian formula*, and

$$h(X_i, X_j) = \sum_{\alpha=1}^{d} h_{ij}^\alpha N_\alpha, \quad 1 \le i, j \le n. \tag{5}$$

holds. Then the *mean curvature vector* $\overrightarrow{H}$ of M is given by

$$\overrightarrow{H} = \frac{1}{n} \sum_{k=1}^{n} h(X_k, X_k). \tag{6}$$

The norm of the mean curvature vector $H = \|\overrightarrow{H}\|$ is called the *mean curvature* of M^n.

We denote R and $R^{\perp}$ the curvature tensors associated with ∇ and D, respectively;

$$R(X_i, X_j)X_k = \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k - \nabla_{[X_i, X_j]} X; \quad 1 \le i, \ \ j, k \le n,$$

$$R^{\perp}(X_i, X_j) N_\alpha = h(X_i, A_{N_\alpha} X_j) - h(X_j, A_{N_\alpha} X_i); \quad 1 \le \alpha \le d.$$

The *equation of Gauss* and *Ricci* are given, respectively by

$$\begin{aligned} R_{ijkl} &= \langle R(X_i, X_j)X_k, X_l \rangle \\ &= \langle h(X_i, X_l), h(X_j, X_k) \rangle - \langle h(X_i, X_k), h(X_j, X_l) \rangle, \end{aligned} \tag{7}$$

$$\langle R^{\perp}(X_i, X_j) N_\alpha, N_\beta \rangle = \langle [A_{N_\alpha}, A_{N_\beta}] X_i, X_j \rangle, \tag{8}$$

for the vector fields X_i, X_j, X_k and X_l tangent to M^n and N_α, N_β normal to M^n. We observe that the normal connection D of M^n is flat if and only if all the shape operators A_{N_α} of M^n are diagonalizable [10]. Consequently, the *Ricci curvature* R_{ij} and *scalar curvature* r of M^n are defined, respectively as follows:

$$R_{ij} = \sum_{k=1}^{n} R_{ikjk},$$

$$r = \sum_{i=1}^{n} R_{ii}.$$

From the equation of Gauss, it is possible to find the scalar curvature r satisfy the following relation:

$$r = n^2|H|^2 - S, \tag{9}$$

where

$$S = \sum_{\alpha=1}^{d} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2. \tag{10}$$

is the square of the second fundamental form and H is the mean curvature of M^n [10].

3. Rotational Submanifolds in $\mathbb{E}^5$

Let

$$f : W^d \to \mathbb{E}^p; \quad f(x) = (f_1(x), \ldots, f_p(x)), \ \ x \in W^d$$

be an isometric immersion of d-dimensional Riemannian manifold W^d into p-dimensional Euclidean space $\mathbb{E}^p$. Consider the standard immersion $g : S^{q-1} \to \mathbb{E}^q$ onto unit sphere S^{q-1}. By rotating the submanifold W^d around S^{q-1} one can obtain a rotational submanifold M given with the isometric immersion

$$X : M \to \mathbb{E}^{p+q-1}; \quad X(x,y) = (f_1(x), \ldots, f_{p-1}(x), f_p(x)g(y)), \tag{11}$$

where the last component $g(y)$, being the position vector in $\mathbb{E}^q$ and $f_p(x) > 0$ for all $x \in W^d$, $y \in S^{q-1}$ (see [18, p. 218]).

If we choose W^d as the regular curve $\gamma(I)$, $I \subset \mathbb{R}$, in p-dimensional Euclidean space $\mathbb{E}^p$ then the resultant rotational submanifold M which lies in ambient space $\mathbb{E}^{p+q-1}$ will be represented by the isometric immersion

$$X(s,y) = (f_1(s), \ldots, f_{p-1}(s), f_p(s)g(y)). \tag{12}$$

where the last component $g(y)$ represent either a unit speed spherical curve or a spherical submanifold of $\mathbb{E}^q$.

In the sequel we consider 3-dimensional rotational submanifolds in 5-dimensional Euclidean space $\mathbb{E}^5$. We have the following two possible cases;

Case I. For $p = 2$ and $q = 4$, the isometric immersion

$$X(s,u,v) = (f_1(s), f_2(s)g(u,v)) \tag{13}$$

with

$$g(u,v) = (0; a_1 \cos u, a_1 \sin u, a_2 \cos v, a_2 \sin v) \tag{14}$$

describes a rotational submanifold M^3 in 5-dimensional Euclidean space $\mathbb{E}^5$. The surface given with the position vector (14) is a Clifford torus T^2 in E^4, such that $a_1, a_2 \in \mathbb{R}$ are real constants satisfying $a_1^2 + a_2^2 = 1$.

Differentiating (13) with respect to s, u and v we obtain

$$\begin{aligned} X_s &= (f_1', a_1 f_2' \cos u, a_1 f_2' \sin u, a_2 f_2' \cos v, a_2 f_2' \sin v), \\ X_u &= (0, -a_1 f_2 \sin u, a_1 f_2 \cos u, 0, 0), \\ X_v &= (0, 0, 0, -a_2 f_2 \sin v, a_2 f_2 \cos v), \end{aligned} \tag{15}$$

respectively.

We can find the coefficients of the first fundamental form as follows:

$$\begin{aligned} &g_{11} = 1, \quad g_{22} = a_1^2 f_2^2, \quad g_{33} = a_2^2 f_2^2, \\ &g_{12} = g_{13} = g_{23} = 0. \end{aligned} \tag{16}$$

Consequently, if we take the arc-length of the curve γ as the parameter s the first fundamental form of M^3 becomes

$$I = ds^2 + f_2^2(a_1^2 du^2 + a_2^2 dv^2).$$

The normal space of M^3 is spanned by the following vector fields:

$$\begin{aligned} N_1 &= \frac{1}{\kappa}(f_1'', a_1 f_2'' \cos u, a_1 f_2'' \sin u, a_2 f_2'' \cos v, a_2 f_2'' \sin v), \\ N_2 &= (0, a_2 \cos u, a_2 \sin u, -a_1 \cos v, -a_1 \sin v), \end{aligned} \tag{17}$$

where $\kappa > 0$ is the curvature of the profile curve γ defined by

$$\kappa(s) = \|\gamma''(s)\| = \sqrt{f_1''(s))^2 + (f_2''(s))^2}. \tag{18}$$

The second partial derivatives of X are expressed as follows:

$$\begin{aligned} X_{ss} &= (f_1'', a_1 f_2'' \cos u, a_1 f_2'' \sin u, a_2 f_2'' \cos v, a_2 f_2'' \sin v), \\ X_{uu} &= (0, -a_1 f_2 \cos u, -a_1 f_2 \sin u, 0, 0), \\ X_{vv} &= (0, 0, 0, -a_2 f_2 \cos v, -a_2 f_2 \sin v), \\ X_{su} &= (0, -a_1 f_2' \sin u, a_1 f_2' \cos u, 0, 0), \\ X_{sv} &= (0, 0, 0, -a_2 f_2' \sin v, a_2 f_2' \cos v), \\ X_{uv} &= (0, 0, 0, 0, 0). \end{aligned} \tag{19}$$

Using (17) and (19) we can get the coefficients of the second fundamental form h as follows:

$$\begin{aligned} L_{11}^1 &= \langle X_{ss}, N_1 \rangle = \kappa(s), \\ L_{22}^1 &= \langle X_{uu}, N_1 \rangle = -\frac{a_1^2 f_2(s) f_2''(s)}{\kappa(s)}, \\ L_{22}^2 &= \langle X_{uu}, N_2 \rangle = -a_1 a_2 f_2(s), \\ L_{33}^1 &= \langle X_{vv}, N_1 \rangle = -\frac{a_2^2 f_2(s) f_2''(s)}{\kappa(s)}, \end{aligned}$$

$$
\begin{aligned}
L_{33}^2 &= \langle X_{vv}, N_2 \rangle = a_1 a_2 f_2(s), \\
L_{11}^2 &= L_{12}^1 = L_{12}^2 = L_{13}^1 = L_{13}^2 = L_{23}^1 = L_{23}^2 = 0.
\end{aligned}
\tag{20}
$$

Furthermore, the orthonormal frame field tangent to M^3 is given by

$$
\begin{aligned}
X_1 &= \frac{X_s}{\|X_s\|} = (f_1', a_1 f_2' \cos u, a_1 f_2' \sin u, a_2 f_2' \cos v, a_2 f_2' \sin v), \\
X_2 &= \frac{X_u}{\|X_u\|} = \frac{X_u}{a_1 f_2} = (0, -\sin u, \cos u, 0, 0), \\
X_3 &= \frac{X_v}{\|X_v\|} = \frac{X_v}{a_2 f_2} = (0, 0, 0, -\sin v, \cos v).
\end{aligned}
\tag{21}
$$

With respect to this frame we can obtain the second fundamental maps;

$$
\begin{aligned}
h(X_1, X_1) &= \frac{1}{\|X_s\|^2}(L_{11}^1 N_1 + L_{11}^2 N_2) = \kappa N_1, \\
h(X_2, X_2) &= \frac{1}{\|X_u\|^2}(L_{22}^1 N_1 + L_{22}^2 N_2) = -\frac{f_2''}{\kappa f_2} N_1 - \frac{a_2}{a_1 f_2} N_2, \\
h(X_3, X_3) &= \frac{1}{\|X_v\|^2}(L_{33}^1 N_1 + L_{33}^2 N_2) = -\frac{f_2''}{\kappa f_2} N_1 + \frac{a_1}{a_2 f_2} N_2, \\
h(X_1, X_2) &= h(X_1, X_3) = h(X_2, X_3) = 0.
\end{aligned}
\tag{22}
$$

Consequently, by the use of (22), (5) with (10) the square length of the second fundamental form h becomes

$$
S = \kappa^2 + \frac{1}{f_2^2}\left(\frac{2(f_2'')^2}{\kappa^2} + \frac{a_1^4 + a_2^4}{a_1^2 a_2^2}\right). \tag{23}
$$

Further, substituting (22) into (6) the mean curvature vector $\overrightarrow{H}$ of M^3 becomes

$$
\overrightarrow{H} = \frac{1}{3}\left\{\left(\kappa - \frac{2 f_2''}{\kappa f_2}\right) N_1 + \left(\frac{a_1^2 - a_2^2}{a_1 a_2 f_2}\right) N_2\right\}. \tag{24}
$$

Summing up the above relations we obtain the following result.

Theorem 1. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (13). *Then the mean curvature H and the scalar curvature r of M^3 become*

$$
3H = \sqrt{\left(\kappa - \frac{2 f_2''(s)}{\kappa f_2(s)}\right)^2 + \left(\frac{a_1^2 - a_2^2}{a_1 a_2 f_2(s)}\right)^2}, \quad f_2(s) \neq 0, \tag{25}
$$

and

$$
r = \frac{2}{f_2^2(s)}\left(\frac{(f_2''(s))^2}{\kappa^2} - 2 f_2(s) f_2''(s) - 1\right), \tag{26}
$$

respectively, where, $\kappa > 0$ is the curvature of the profile curve γ and $a_1, a_2 \in \mathbb{R}$ are real constants satisfying $a_1^2 + a_2^2 = 1$.

Since the profile curve γ has unit speed parametrization

$$f_1'(s) = \sqrt{1 - (f_2'(s))^2}. \tag{27}$$

holds. So, after some computation we have

$$\kappa^2 = \|\gamma''(s)\|^2 = \frac{(f_2''(s))^2}{1 - (f_2'(s))^2}. \tag{28}$$

Consequently, substituting (28) into (26) we obtain the following result.

Corollary 2. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (13). *Then the scalar curvature r of M^3 becomes*

$$r = \frac{-2}{(f_2)^2}\{(f_2')^2 + 2f_2 f_2''\}. \tag{29}$$

For the case of vanishing scalar curvature we have the following result.

Corollary 3. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (13). *Then M^3 has vanishing scalar curvature if and only if*

$$f_1(s) = \pm\frac{1}{a}(as+b)\left(1 - a^2\left(\frac{3}{2}(as+b)\right)^{-2/3}\right)^{3/2}, \tag{30}$$

$$f_2(s) = \left(\frac{3}{2}(as+b)\right)^{\frac{2}{3}}, \tag{31}$$

holds.

Proof. Assume that M^3 has vanishing scalar curvature then

$$(f_2')^2 + 2f_2 f_2'' = 0$$

holds. This differential equation has a non-trivial solution

$$f_2(s) = \left(\frac{3}{2}(as+b)\right)^{\frac{2}{3}}.$$

So, differentiating $f_2(s)$ and using (27) we obtain (30). This completes the proof of the corollary. □

For the minimal case we have;

Corollary 4. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (13). *Then M^3 is minimal if and only if*

$$f_2 f_2'' + 2(f_2')^2 - 2 = 0 \quad \text{and} \quad a_1 = \pm a_2 = \frac{1}{\sqrt{2}} \tag{32}$$

holds.

Proof. Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (13). If M^3 is a minimal submanifold then $\kappa^2 = \frac{2f_2''}{f_2}$ and $a_1 = \pm a_2 = \frac{1}{\sqrt{2}}$

holds. So, using (28) we obtain the following differential equation:

$$f_2''(f_2 f_2'' + 2(f_2')^2 - 2) = 0.$$

If $f_2''(s) = 0$ holds then $\kappa = 0$ which gives a contradiction. So the differential equation $f_2 f_2'' + 2(f_2')^2 - 2 = 0$ holds. This gives the proof of the result. □

Case II. For $p = 3$ and $q = 3$, the isometric immersion (12) describes a rotational submanifold M^3 in $\mathbb{E}^5$ given with the parametrization

$$X(s, u, v) = (f_1(s), f_2(s), f_3(s)g(u, v)), \tag{33}$$

where

$$g(u, v) = (0, 0; \cos u, \sin u \cos v, \sin u \sin v), \tag{34}$$

is the position vector of the unit sphere $S^2 \subset \mathbb{E}^3$.

Differentiating (33) with respect to s, u and v we obtain

$$X_s = (f_1', f_2', f_3' \cos u, f_3' \sin u \cos v, f_3' \sin u \sin v),$$
$$X_u = (0, 0, -f_3 \sin u, f_3 \cos u \cos v, f_3 \cos u \sin v),$$
$$X_v = (0, 0, 0, -f_3 \sin u \sin v, f_3 \sin u \cos v),$$

respectively. We can find the coefficients of the first fundamental form as follows:

$$\begin{aligned} &g_{11} = 1, \quad g_{22} = f_3^2(s), \quad g_{33} = f_3^2(s)\sin^2 u, \\ &g_{12} = g_{13} = g_{23} = 0. \end{aligned} \tag{35}$$

Consequently, if we take the arc-length of the curve γ as the parameter s the first fundamental form of M^3 becomes

$$I = ds^2 + f_3^2(du^2 + \sin^2 u dv^2).$$

The second partial derivatives of X are expressed as follows:

$$\begin{aligned} X_{ss} &= (f_1'', f_2'', f_3'' \cos u, f_3'' \sin u \cos v, f_3'' \sin u \sin v), \\ X_{su} &= (0, 0, -f_3' \sin u, f_3' \cos u \cos v, f_3' \cos u \sin v), \\ X_{sv} &= (0, 0, 0, -f_3' \sin u \sin v, f_3' \sin u \cos v), \\ X_{uv} &= (0, 0, 0, -f_3 \cos u \sin v, f_3 \cos u \cos v), \\ X_{uu} &= (0, 0, -f_3 \cos u, -f_3 \sin u \cos v, -f_3 \sin u \sin v), \\ X_{vv} &= (0, 0, 0, -f_3 \sin u \cos v, -f_3 \sin u \sin v). \end{aligned} \tag{36}$$

The normal space of M^3 is spanned by the following vector fields:

$$\begin{aligned} N_1 &= \frac{1}{\kappa}(f_1'', f_2'', f_3'' \cos u, f_3'' \sin u \cos v, f_3'' \sin u \sin v), \\ N_2 &= \frac{1}{\kappa}(A, B, \kappa_1 \cos u, \kappa_1 \sin u \cos v, \kappa_1 \sin u \sin v), \end{aligned} \tag{37}$$

in such a way that

$$A = f_2' f_3'' - f_2'' f_3',$$
$$B = f_3' f_1'' - f_3'' f_1',$$

are smooth functions,

$$\kappa_1 = f_1' f_2'' - f_1'' f_2' \tag{38}$$

is the curvature of the projection of the curve γ on the Oe_1e_2-plane and

$$\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2} \tag{39}$$

is the curvature of the profile curve γ.

Using (36) and (37) we can get the coefficients of the second fundamental form as follows:

$$\begin{aligned}
L_{11}^1 &= \langle X_{ss}, N_1 \rangle = \kappa, \\
L_{22}^1 &= \langle X_{uu}, N_1 \rangle = -\frac{f_3'' f_3}{\kappa}, \quad \kappa \neq 0, \\
L_{22}^2 &= \langle X_{uu}, N_2 \rangle = -\frac{f_3 \kappa_1}{\kappa}, \\
L_{33}^1 &= \langle X_{vv}, N_1 \rangle = -\frac{f_3'' f_3}{\kappa} \sin^2 u, \\
L_{33}^2 &= \langle X_{vv}, N_2 \rangle = -\frac{f_3 \kappa_1}{\kappa} \sin^2 u, \\
L_{11}^2 &= L_{12}^1 = L_{12}^2 = L_{13}^1 = L_{13}^2 = L_{23}^1 = L_{23}^2 = 0.
\end{aligned} \tag{40}$$

Here, $\kappa \neq 0$, means that the profile curve $\gamma(s)$ is different from a straight line.

Furthermore, the orthonormal frame field tangent to M^3 is given by

$$\begin{aligned}
X_1 &= \frac{X_s}{\|X_s\|} = (f_1', f_2', f_3' \cos u, f_3' \sin u \cos v, f_3' \sin u \sin v), \\
X_2 &= \frac{X_u}{\|X_u\|} = (0, 0, -\sin u, \cos u \cos v, \cos u \sin v), \\
X_3 &= \frac{X_v}{\|X_v\|} = (0, 0, 0, -\sin v, \cos v).
\end{aligned} \tag{41}$$

With respect to this frame we can obtain the second fundamental maps;

$$\begin{aligned}
h(X_1, X_1) &= \frac{1}{\|X_s\|^2} (L_{11}^1 N_1 + L_{11}^2 N_2) = \kappa N_1, \\
h(X_2, X_2) &= \frac{1}{\|X_u\|^2} (L_{22}^1 N_1 + L_{22}^2 N_2) = -\frac{f_3''}{\kappa f_3} N_1 - \frac{\kappa_1}{\kappa f_3} N_2, \\
h(X_3, X_3) &= \frac{1}{\|X_v\|^2} (L_{33}^1 N_1 + L_{33}^2 N_2) = -\frac{f_3''}{\kappa f_3} N_1 - \frac{\kappa_1}{\kappa f_3} N_2, \\
h(X_1, X_2) &= h(X_1, X_3) = h(X_2, X_3) = 0.
\end{aligned} \tag{42}$$

Consequently, by the use of (42), (5) with (10) the square length of the second fundamental form h becomes

$$S = \kappa^2 + \frac{2}{\kappa^2 f_3^2}((f_3'')^2 + \kappa_1^2). \tag{43}$$

Further, substituting (42) into (6) the mean curvature vector $\overrightarrow{H}$ of M^3 becomes

$$\overrightarrow{H} = \frac{1}{3}\left\{\left(\kappa - \frac{2f_3''}{\kappa f_3}\right)N_1 - \frac{2\kappa_1}{\kappa f_3}N_2\right\}. \tag{44}$$

Summing up the above relations we obtain the following result.

Theorem 5. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (33)*. Then the mean curvature H and the scalar curvature r of M^3 become*

$$3H = \sqrt{\left(\kappa - \frac{2f_3''}{\kappa f_3}\right)^2 + \frac{4\kappa_1^2}{\kappa^2 f_3^2}}, \tag{45}$$

and,

$$r = \frac{-4\kappa^2 f_3 f_3'' + 2(f_3'')^2 + 2\kappa_1^2}{\kappa^2 f_3^2}, \tag{46}$$

respectively. Here κ_1 and κ are curvature functions given by (38) *and* (39)*, respectively.*

We give the following example.

Example 6. Consider the rotational submanifold M^3 given with the parametrization

$$\begin{aligned} f_1(s) &= \pm\int\sqrt{1 - ae^{-2s}}ds + c,\\ f_2(s) &= \lambda e^{-s},\\ f_3(s) &= \mu e^{-s}, \end{aligned} \tag{47}$$

where

$$a = \lambda^2 + \mu^2$$

is the constant function. Further, substituting (47) into (46) and using (38) and (39) we obtain

$$r = \frac{2}{\mu^2 e^{-2s}} - 6. \tag{48}$$

For the case of vanishing scalar curvature we have the following result.

Corollary 7. *Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization* (33)*. Then M^3 has vanishing scalar curvature if and only if*

$$\kappa^2 = \frac{(f_3'')^2 + \kappa_1^2}{2f_3 f_3''} \tag{49}$$

holds.

For the minimal case we have;

Corollary 8. *Let* M^3 *be a rotational submanifold in* $\mathbb{E}^5$ *given with the parametrization* (33). *Then* M^3 *is minimal if and only if*

$$f_3 f_3'' + 2(f_3')^2 - 2 = 0, \tag{50}$$

holds.

Proof. Let M^3 be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (33). If M^3 is minimal then $\kappa^2 = \frac{2f_3''}{f_3}$ and $\kappa_1 = 0$ holds. So, using the Eq. (38) we get

$$f_1'(s) = \lambda f_2'(s). \tag{51}$$

Since the profile furve γ is given with arc-length parameter s, then using (51) we obtain

$$f_2'(s) = \frac{\sqrt{1-(f_3')^2}}{\sqrt{1+\lambda^2}}. \tag{52}$$

Consequently, differentiating (52) with respect to s and using (39) with $\kappa^2 = \frac{2f_3''}{f_3}$ we get the following differential equation:

$$f_3''(f_3 f_3'' + 2(f_3')^2 - 2) = 0.$$

If $f_3''(s) = 0$ holds then $\kappa = 0$ which gives a contradiction. So the differential equation $f_3 f_3'' + 2(f_3')^2 - 2 = 0$ holds. This gives the proof of the result. □

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