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Minimal free resolutions of the tangent cones for Gorenstein monomial curves

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Abstract: We study the minimal free resolution of the tangent cone of Gorenstein monomial curves in affine 4-space. We give the explicit minimal free resolution of the tangent cone of noncomplete intersection Gorenstein monomial curve whose tangent cone has five minimal generators and show that the possible Betti sequences are $(1, 5, 6, 2)$ and $(1, 5, 5, 1)$. Moreover, we compute the Hilbert function of the tangent cone of these families as a result.

Key words: Gorenstein monomial curves, tangent cones, minimal free resolutions

1. Introduction

The minimal free resolution is a central topic in commutative algebra and is a very useful tool for extracting information about modules. Many algebraic invariants of the module such as Hilbert function and Betti numbers can be deduced from its minimal free resolution. When the module is associated to a geometric object, these invariants give useful geometric information about it. Since it is possible to calculate the Hilbert function in terms of the graded Betti numbers, free resolutions play an important role in the theory of Hilbert series. Although the Hilbert function of a standard graded algebra over a field k is well known in the Cohen-Macaulay case, in general, very little is known in local algebra. The problem which is due to M.E. Rossi [10] asks whether the Hilbert function of a Gorenstein local ring of dimension one is nondecreasing. Recently, it has been shown that there are many families of monomial curves giving negative answer to this problem [9]. However, it is still open for Gorenstein local rings associated to monomial curves in affine $d-$space for $3 < d < 10$ and our main aim is to understand the Hilbert function when $d = 4$.

Let R be the polynomial ring $k[x_1, \ldots, x_d]$ over an arbitrary field k. A monomial affine curve C has a parametrization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \ldots,\ x_d = t^{n_d} \tag{1.1}$$

where $n_1, n_2, \ldots, n_d$ are positive integers with $\gcd(n_1, n_2, ..., n_d) = 1$ and $n_1, n_2, \ldots, n_d$ is a minimal set of generators for the numerical semigroup

$$S =< n_1, n_2, ..., n_d >= \{n \mid n = \textstyle\sum_{i=1}^{d} a_i n_i,\ a_i\text{'s are nonnegative integers}\}.$$

The semigroup ring $k[t^{n_1}, \ldots, t^{n_d}]$ of S is isomorphic to the coordinate ring $k[x_1, \ldots, x_d]/I(C)$ and the coordinate ring $G = gr_m(k[[t^{n_1}, \ldots, t^{n_d}]])$ of the tangent cone of a monomial curve C at the origin is isomorphic to the ring $k[x_1, \ldots, x_d]/I(C)_*$. Here, $I(C)_*$ is generated by the polynomials f_*, the homogeneous

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summand of f of the least degree, for f in $I(C)$ where $I(C)$ is the defining ideal of C. A monomial curve given by the parametrization in (1.1) is called a Gorenstein monomial curve, if the associated local ring $k[[t^{n_1}, t^{n_2}, ..., t^{n_d}]]$ is Gorenstein. $k[[t^{n_1}, t^{n_2}, ..., t^{n_d}]]$ is Gorenstein if and only if the corresponding numerical semigroup $S =< n_1, n_2, ..., n_d >$ is symmetric [8].

Let $S =< n_1, n_2, n_3, n_4 >$ be a 4-generated numerical semigroup. If S is symmetric and complete intersection, then the Betti sequence of the corresponding semigroup ring is $(1, 3, 3, 1)$, see [11]. Barucci, Fröberg and Şahin [3] described the minimal free resolution of the semigroup ring of S, when S is symmetric and not complete intersection and showed that the Betti sequence is (1,5,5,1). When S is 4-generated symmetric and noncomplete intersection semigroup, the minimal free resolution and the list of possible Betti sequences of $G = gr_m(k[[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}]])$ is still unknown [11]. For pseudosymmetric numerical semigroups, see [12, 13]. If S and its tangent cone have the same Betti sequence, then S is of homogeneous type. For a homogeneous type semigroup, the Betti sequence of its Cohen-Macaulay tangent cone can be obtained from a minimal free resolution of its semigroup ring. For details, see [7]. In this article, we study the minimal free resolution of the tangent cone of Gorenstein noncomplete intersection monomial curve C in affine 4-space when the minimal number of generators of its tangent cone is five. Since homogeneous type semigroups have Cohen-Macaulay tangent cones and the Cohen-Macaulayness of tangent cones of these families of curves was shown in [2], here we consider only 5-generated tangent cones. Based on the Buchsbaum-Eisenbud Theorem [5] and knowing the minimal generators of the defining ideal of the tangent cone in four cases [2], we give the minimal free resolution of the tangent cone explicitly. Then, we compute the Hilbert function of the tangent cone for these families as corollaries. All computations have been done using SINGULAR. *

2. Bresinsky's theorem

In [4], Bresinsky gives the explicit description of the defining ideal of a noncomplete intersection Gorenstein monomial curve with embedding dimension four by the following theorem.

Theorem 2.1 *Let C be a monomial curve having the parametrization*

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3},\ x_4 = t^{n_4},$$

where S is a numerical semigroup minimally generated by n_1, n_2, n_3, n_4. S is symmetric and C is a noncomplete intersection monomial curve if and only if $I(C)$ is generated by the set

$$G = \{f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}},$$
$$f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

where the polynomials f_i's are unique up to isomorphism with $0 < \alpha_{ij} < \alpha_j$ with $\alpha_i n_i \in < n_1, \ldots, \hat{n}_i, \ldots, n_4 >$ such that α_i's are minimal for $1 \leq i \leq 4$, where $\hat{n}_i$ denotes that $n_i \notin < n_1, \ldots, \hat{n}_i, \ldots, n_4 >$.

In Theorem 2.1, there is no restriction on the order of $n_1, \ldots, n_4$ and the set G is valid for only a permutation of these numbers. If we assume that $n_1 < n_2 < n_3 < n_4$, then we have to revise the set G with respect to the correct permutation of the variables x_1, x_2, x_3, x_4. Thus, there are six isomorphic possible permutations which can be considered within three cases:

*SINGULAR 2.0. A Computer Algebra System for Polynomial Computations. Available at http://www.singular.uni-kl.de.

1. $f_1 = (1,(3,4))$
   (a) $f_2 = (2,(1,4)), f_3 = (3,(1,2)), f_4 = (4,(2,3)), f_5 = ((1,3),(2,4))$
   (b) $f_2 = (2,(1,3)), f_3 = (3,(2,4)), f_4 = (4,(1,2)), f_5 = ((1,4),(2,3))$
2. $f_1 = (1,(2,3))$
   (a) $f_2 = (2,(3,4)), f_3 = (3,(1,4)), f_4 = (4,(1,2)), f_5 = ((2,4),(1,3))$
   (b) $f_2 = (2,(1,4)), f_3 = (3,(2,4)), f_4 = (4,(1,3)), f_5 = ((1,3),(4,2))$
3. $f_1 = (1,(2,4))$
   (a) $f_2 = (2,(1,3)), f_3 = (3,(1,4)), f_4 = (4,(2,3)), f_5 = ((1,2),(3,4))$
   (b) $f_2 = (2,(3,4)), f_3 = (3,(1,2)), f_4 = (4,(1,3)), f_5 = ((2,3),(1,4))$

Here, the notations $f_i = (i,(j,k))$ and $f_5 = ((i,j),(k,l))$ denote the generators $f_i = x_i^{\alpha_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}}$ and $f_5 = x_i^{\alpha_{ki}} x_j^{\alpha_{lj}} - x_k^{\alpha_{jk}} x_l^{\alpha_{il}}$ Thus, if we have the extra condition $n_1 < n_2 < n_3 < n_4$, then the generator set of its defining ideal is exactly given by one of these six permutations.

In [2], Arslan and Mete observed that the generator set of each of these curves turned out to be a standard basis with respect to the negative degree reverse lexicographical ordering in the following cases:

- In Case 1(a) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$,
- In Case 1(b) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$.
- In Case 2(b) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$.
- In Case 3(a) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{31} + \alpha_{34}$.

And in all above cases, the minimal number of generators of the tangent cone of a Gorenstein noncomplete intersection monomial curve is five. One can also see [1].

3. Minimal free resolutions

In this section, we give the minimal free resolution of the tangent cone of Gorenstein noncomplete intersection monomial curve C in embedding dimension four when the minimal number of generators of the tangent cone of C is five.

Case 1(a) : Let

$$f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$$

and

$$f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{24}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{13} + \alpha_{14}$, $\alpha_4 < \alpha_{42} + \alpha_{43}$ and $\alpha_3 < \alpha_{31} + \alpha_{32}$. Since the extra condition $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ and using Lemma 5.5.1 in [6], the defining ideal $I(C)_*$ of the tangent cone is generated by the following sets:

- *Case* $1(a1)$: $I(C)_* = (x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}}x_4^{\alpha_{14}})$
- *Case* $1(a2)$: $I(C)_* = (x_3^{\alpha_{13}}x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}}x_4^{\alpha_{14}})$

Theorem 3.1 *In Case* $1(a1)$ *and Case* $1(a2)$*, the sequence of R-modules*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

is a minimal free resolution for the tangent cone of C*, where*

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}}x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{32}}x_4^{\alpha_{14}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_3^{\alpha_{43}} & 0 & x_4^{\alpha_{24}} & x_2^{\alpha_{32}} & 0 & 0 \\ 0 & x_3^{\alpha_3} & 0 & 0 & 0 & x_4^{\alpha_{14}} \\ -x_4^{\alpha_{14}} & -x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3^{\alpha_{13}} & 0 & x_2^{\alpha_{32}} & 0 \\ 0 & 0 & 0 & -x_3^{\alpha_{13}} & -x_4^{\alpha_{24}} & -x_2^{\alpha_{42}} \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} & 0 \\ -x_4^{\alpha_{14}} & 0 \\ 0 & x_2^{\alpha_{32}} \\ -x_2^{\alpha_{42}}x_3^{\alpha_{43}} & -x_4^{\alpha_{24}} \\ 0 & x_3^{\alpha_{13}} \\ x_3^{\alpha_3} & 0 \end{pmatrix}$$

or

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}}x_4^{\alpha_{14}} & x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{32}}x_4^{\alpha_{14}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_4^{\alpha_{24}} & x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4^{\alpha_{14}} & 0 & x_3^{\alpha_3} \\ 0 & 0 & -x_4^{\alpha_{14}} & 0 & 0 & -x_2^{\alpha_2} + x_1^{\alpha_{21}}x_4^{\alpha_{24}} \\ -x_3^{\alpha_{13}} & 0 & 0 & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} & 0 \\ 0 & -x_3^{\alpha_{13}} & 0 & -x_2^{\alpha_{42}} & -x_4^{\alpha_{24}} & 0 \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_2^{\alpha_{32}} & x_1^{\alpha_{21}}x_3^{\alpha_{43}} \\ -x_4^{\alpha_{24}} & -x_2^{\alpha_{42}}x_3^{\alpha_{43}} \\ 0 & x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}} \\ 0 & x_3^{\alpha_3} \\ x_3^{\alpha_{13}} & 0 \\ 0 & -x_4^{\alpha_{14}} \end{pmatrix}$$

respectively.

Proof *Case* $1(a1)$: It is easy to show that $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$ proving that the sequence above is a complex. To prove the exactness, we use Buchsbaum–Eisenbud criterion [5]. Therefore, first we need to check that

$$rank(\varphi_1) + rank(\varphi_2) = 1 + 4 = 5 \text{ and } rank(\varphi_2) + rank(\varphi_3) = 4 + 2 = 6.$$

Clearly, $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. Since every 5×5 minors of φ_2 is zero, by McCoy's Theorem $rank(\varphi_2) \leq 4$. In matrix φ_2, deleting the 1st and the 3rd columns, and the 2nd row, we have $-x_2^{2\alpha_2+\alpha_{32}}$ and similarly, deleting the 3rd row, and the 5th and the 6th columns, we obtain $x_3^{2\alpha_3+\alpha_{13}}$ as $4 \times 4-$minors of φ_2. Thus, $rank(\varphi_2) = 4$. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. Among the 2-minors of φ_3, we have $x_2^{\alpha_2+\alpha_{32}}$, $x_3^{\alpha_3+\alpha_{13}}$ and $x_4^{\alpha_4}$ and this is a regular sequence, since $\{x_2, x_3, x_4\}$ is a regular sequence. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $1(a2)$: Similar to the first case, it is clear that $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$. $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$ are trivial. In matrix φ_2, deleting the 3rd row, and the 4th and the 5th columns, we have $x_3^{2\alpha_3+\alpha_{13}}$ and similarly, deleting the 2nd and the 6th columns, and the 4th row, we obtain $x_4^{2\alpha_4}$ and these determinants are relatively prime. 2-minors of φ_3 are

$$-x_3^{\alpha_{43}}f_2,\ x_2^{\alpha_{32}}f_2,\ x_2^{\alpha_{32}}x_3^{\alpha_3},\ -x_1^{\alpha_{21}}x_3^{\alpha_3},\ -x_2^{\alpha_{32}}x_4^{\alpha_{14}},\ -x_4^{\alpha_{24}}f_2,\ -x_4^{\alpha_{24}}x_3^{\alpha_3},\ x_2^{\alpha_{42}}x_3^{\alpha_3},\ x_4^{\alpha_4},\ -x_3^{\alpha_{13}}f_2,\ -x_3^{\alpha_3+\alpha_{13}},\\ -x_3^{\alpha_{13}}x_4^{\alpha_{14}}.$$

Among these 2-minors of φ_3, we have $\{x_2^{\alpha_{32}} f_2,\, -x_3^{\alpha_3+\alpha_{13}},\, x_4^{\alpha_4}\}$. Since x_2 is a nonzero divisor modulo $\{x_3, x_4\}$, $I(\varphi_3)$ contains a regular sequence of length 3. $\square$

Case 1(b) : Let

$$f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, f_3 = x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}} x_2^{\alpha_{42}}$$

and

$$f_5 = x_2^{\alpha_{42}} x_3^{\alpha_{13}} - x_1^{\alpha_{21}} x_4^{\alpha_{34}}$$

Here, $\alpha_1 = \alpha_{21} + \alpha_{41}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$, $\alpha_4 = \alpha_{14} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{13} + \alpha_{14}$, and $\alpha_4 < \alpha_{41} + \alpha_{42}$. The extra condition $\alpha_2 \le \alpha_{21} + \alpha_{23}$ and $\alpha_3 \le \alpha_{32} + \alpha_{34}$ again using Lemma 5.5.1 in [6] imply that the defining ideal $I(C)_*$ of the tangent cone is generated by the following sets:

- $Case\ 1(b1)\ : I(C)_* = (x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{42}} x_3^{\alpha_{13}})$
- $Case\ 1(b2)\ : I(C)_* = (x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{42}} x_3^{\alpha_{13}})$
- $Case\ 1(b3)\ : I(C)_* = (x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_2^{\alpha_{42}} x_3^{\alpha_{13}})$
- $Case\ 1(b4)\ : I(C)_* = (x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_1^{\alpha_{21}} x_4^{\alpha_{34}} - x_2^{\alpha_{42}} x_3^{\alpha_{13}})$

Theorem 3.2 *In Case* $1(b1)$*, the minimal free resolution for the tangent cone of* C *is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}} x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{42}} x_3^{\alpha_{13}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_4^{\alpha_{34}} & 0 & -x_3^{\alpha_{23}} & 0 & x_2^{\alpha_{42}} & 0 \\ 0 & -x_4^{\alpha_4} & 0 & 0 & 0 & -x_3^{\alpha_{13}} \\ 0 & 0 & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 \\ x_3^{\alpha_{13}} & x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_3^{\alpha_{23}} & -x_4^{\alpha_{14}} & x_2^{\alpha_{32}} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} & 0 \\ -x_3^{\alpha_{13}} & 0 \\ 0 & x_2^{\alpha_{42}} \\ 0 & -x_4^{\alpha_{14}} \\ x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_3^{\alpha_{23}} \\ x_4^{\alpha_4} & 0 \end{pmatrix},$$

in Case $1(b2)$*, the minimal free resolution for the tangent cone of* C *is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}} x_4^{\alpha_{14}} & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{42}} x_3^{\alpha_{13}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_4^{\alpha_{34}} & 0 & -x_3^{\alpha_{23}} & 0 & 0 & x_2^{\alpha_{42}} \\ 0 & -x_4^{\alpha_4} & 0 & 0 & x_3^{\alpha_{13}} & 0 \\ 0 & 0 & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & x_1^{\alpha_{21}} & -x_4^{\alpha_{14}} \\ x_3^{\alpha_{13}} & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_3^{\alpha_{23}} & -x_2^{\alpha_{32}} & 0 \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} & 0 \\ -x_3^{\alpha_{13}} & 0 \\ x_1^{\alpha_{21}} x_4^{\alpha_{34}} & x_2^{\alpha_{42}} \\ 0 & -x_4^{\alpha_{14}} \\ -x_4^{\alpha_4} & 0 \\ x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_3^{\alpha_{23}} \end{pmatrix},$$

in Case $1(b3)$,

$$0 \to R^1 \xrightarrow{\varphi_3} R^5 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to G \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}} x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_2^{\alpha_{42}} x_3^{\alpha_{13}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_3^{\alpha_{23}} & 0 & x_4^{\alpha_{34}} & x_2^{\alpha_{42}} & 0 \\ 0 & x_4^{\alpha_{34}} & 0 & 0 & x_3^{\alpha_{13}} \\ x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 & 0 \\ x_2^{\alpha_{32}} & 0 & -x_3^{\alpha_{13}} & 0 & 0 \\ 0 & -x_3^{\alpha_{23}} & 0 & -x_4^{\alpha_{14}} & -x_2^{\alpha_{32}} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_2^{\alpha_{42}} x_3^{\alpha_{13}} \\ -x_3^{\alpha_{13}} x_4^{\alpha_{14}} \\ x_2^{\alpha_2} \\ x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ x_4^{\alpha_4} \end{pmatrix},$$

and in Case $1(b4)$*, then the minimal free resolution of the tangent cone of* C *is*

$$0 \to R^1 \xrightarrow{\varphi_3} R^5 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to G \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}} x_4^{\alpha_{14}} & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} & x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_2^{\alpha_{42}} x_3^{\alpha_{13}} - x_1^{\alpha_{21}} x_4^{\alpha_{34}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_4^{\alpha_{34}} & x_3^{\alpha_{23}} & -x_2^{\alpha_{42}} & 0 & 0 \\ 0 & 0 & 0 & x_4^{\alpha_{34}} & -x_3^{\alpha_{13}} \\ 0 & -x_4^{\alpha_{14}} & 0 & x_2^{\alpha_{42}} & -x_1^{\alpha_{21}} \\ x_3^{\alpha_{13}} & -x_2^{\alpha_{32}} & x_1^{\alpha_{21}} & 0 & 0 \\ 0 & 0 & x_4^{\alpha_{14}} & -x_3^{\alpha_{23}} & x_2^{\alpha_{32}} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} \\ x_2^{\alpha_{42}} x_3^{\alpha_{13}} - x_1^{\alpha_{21}} x_4^{\alpha_{34}} \\ x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ x_3^{\alpha_{13}} x_4^{\alpha_{14}} \\ x_4^{\alpha_4} \end{pmatrix}.$$

Proof *Case* $1(b1)$: $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. As in the $Case 1(a)$, in matrix φ_2, deleting the 2nd and the 5th columns, and the 3rd row, we have $x_3^{2\alpha_3}$ and similarly, deleting the 2nd row, and the 1st and the 3rd columns, we obtain $-x_2^{2\alpha_2+\alpha_{42}}$ as $4 \times 4-$ minors of φ_2. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. Among the 2-minors of φ_3, we have $x_2^{\alpha_2+\alpha_{42}}$, $-x_3^{\alpha_3}$ and $x_4^{\alpha_4+\alpha_{14}}$ and these three determinants constitute a regular sequence. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $1(b2)$: It is clear that $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In matrix φ_2, deleting the 3rd row, and the 2nd and the 6th columns, we have $-x_3^{2\alpha_3}$ and deleting the 4th and the 6th columns, and the 4th row, we obtain $-x_2^{\alpha_{32}} x_4^{2\alpha_4}$ and these determinants are relatively prime. Among the 2-minors of φ_3, we have $-x_3^{\alpha_3}$, $-x_4^{\alpha_4+\alpha_{14}}$ and $x_2^{\alpha_{42}} f_2$. Since x_2 is a nonzero divisor modulo $\{x_3, x_4\}$, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $1(b3)$: Clearly, $rank(\varphi_1) = rank(\varphi_3) = 1$. In matrix φ_2, deleting the 2nd row and the 3rd column, we have $x_2^{2\alpha_2}$ and deleting the 4th row and the 5th column, we get $x_4^{2\alpha_4}$. Since these determinants are powers of different variables, they constitute a regular sequence of length 2.

Case $1(b4)$: As in the above case, $rank(\varphi_1) = rank(\varphi_3) = 1$. In matrix φ_2, deleting 2nd row and 1st column, we have ${f_2}^2$ and deleting 4th row and 5th column, we obtain $x_4^{2\alpha_4}$. These two determinans are relatively prime, they constitute a regular sequence. $I(\varphi_2)$ contains a regular sequence of length 2. $\square$

Case 2(b) : Let

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}} x_3^{\alpha_{43}}$$

and

$$f_5 = x_1^{\alpha_{41}} x_2^{\alpha_{32}} - x_3^{\alpha_{13}} x_4^{\alpha_{24}}.$$

Here, $\alpha_1 = \alpha_{21} + \alpha_{41}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{24} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{12} + \alpha_{13}$ and $\alpha_4 < \alpha_{41} + \alpha_{43}$. Since the extra condition $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$ and Lemma 5.5.1 in [6], the defining ideal $I(C)_*$ of the tangent cone is generated by the following sets:

- *Case* $2(b1)$: $I(C)_* = (x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_3^{\alpha_{13}} x_4^{\alpha_{24}})$
- *Case* $2(b2)$: $I(C)_* = (x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_3^{\alpha_{13}} x_4^{\alpha_{24}})$
- *Case* $2(b3)$: $I(C)_* = (x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2}, x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_3^{\alpha_{13}} x_4^{\alpha_{24}})$
- *Case* $2(b4)$: $I(C)_* = (x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_3^{\alpha_{13}} x_4^{\alpha_{24}})$

Theorem 3.3 *In Case $2(b1)$, the minimal free resolution for the tangent cone of C is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}} x_3^{\alpha_{13}} & x_2^{\alpha_2} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_3^{\alpha_{13}} x_4^{\alpha_{24}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_4^{\alpha_{24}} & x_3^{\alpha_{43}} & x_2^{\alpha_{32}} & 0 & 0 & 0 \\ 0 & 0 & -x_3^{\alpha_{13}} & 0 & 0 & -x_4^{\alpha_4} \\ 0 & -x_2^{\alpha_{12}} & 0 & 0 & -x_4^{\alpha_{24}} & 0 \\ 0 & 0 & 0 & -x_3^{\alpha_{13}} & 0 & x_2^{\alpha_2} \\ -x_2^{\alpha_{12}} & 0 & 0 & x_4^{\alpha_{34}} & x_3^{\alpha_{43}} & 0 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_3^{\alpha_{43}} & x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ -x_4^{\alpha_{24}} & 0 \\ 0 & -x_4^{\alpha_4} \\ 0 & x_2^{\alpha_2} \\ x_2^{\alpha_{12}} & 0 \\ 0 & x_3^{\alpha_{13}} \end{pmatrix},$$

in Case $2(b2)$, the minimal free resolution for the tangent cone of C is

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}} x_3^{\alpha_{13}} & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_3^{\alpha_{13}} x_4^{\alpha_{24}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_4^{\alpha_{24}} & x_3^{\alpha_{43}} & x_2^{\alpha_{32}} & 0 & 0 & 0 \\ 0 & 0 & -x_3^{\alpha_{13}} & 0 & 0 & -x_4^{\alpha_4} \\ 0 & -x_2^{\alpha_{12}} & 0 & 0 & -x_4^{\alpha_{24}} & 0 \\ 0 & 0 & 0 & -x_3^{\alpha_{13}} & 0 & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}} \\ -x_2^{\alpha_{12}} & 0 & -x_1^{\alpha_{21}} & x_4^{\alpha_{34}} & x_3^{\alpha_{43}} & 0 \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_3^{\alpha_{43}} & x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ -x_4^{\alpha_{24}} & 0 \\ 0 & -x_4^{\alpha_4} \\ 0 & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}} \\ x_2^{\alpha_{12}} & 0 \\ 0 & x_3^{\alpha_{13}} \end{pmatrix},$$

in Case $2(b3)$, the minimal free resolution of the tangent cone of C is

$$0 \to R^1 \xrightarrow{\varphi_3} R^5 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}} x_3^{\alpha_{13}} & x_2^{\alpha_2} & x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_3^{\alpha_{13}} x_4^{\alpha_{24}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_4^{\alpha_{24}} & 0 & 0 & -x_2^{\alpha_{32}} & -x_3^{\alpha_{43}} \\ 0 & 0 & 0 & x_3^{\alpha_{13}} & x_4^{\alpha_{34}} \\ 0 & -x_4^{\alpha_{24}} & 0 & 0 & x_2^{\alpha_{12}} \\ 0 & -x_2^{\alpha_{32}} & -x_3^{\alpha_{13}} & 0 & 0 \\ x_2^{\alpha_{12}} & x_3^{\alpha_{43}} & x_4^{\alpha_{34}} & 0 & 0 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ -x_2^{\alpha_{12}} x_3^{\alpha_{13}} \\ x_2^{\alpha_2} \\ x_4^{\alpha_4} \\ -x_3^{\alpha_{13}} x_4^{\alpha_{24}} \end{pmatrix},$$

lastly, in Case $2(b4)$, *the minimal free resolution for the tangent cone of* C *is*

$$0 \to R^1 \xrightarrow{\varphi_3} R^5 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}} x_3^{\alpha_{13}} & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}} & x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_3^{\alpha_{13}} x_4^{\alpha_{24}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_4^{\alpha_{24}} & x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & 0 & 0 \\ 0 & -x_3^{\alpha_{13}} & -x_4^{\alpha_{34}} & 0 & 0 \\ 0 & 0 & -x_2^{\alpha_{12}} & 0 & -x_4^{\alpha_{24}} \\ 0 & 0 & -x_1^{\alpha_{21}} & -x_3^{\alpha_{13}} & -x_2^{\alpha_{32}} \\ -x_2^{\alpha_{12}} & -x_1^{\alpha_{21}} & 0 & x_4^{\alpha_{34}} & x_3^{\alpha_{43}} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}} \\ x_4^{\alpha_4} \\ -x_3^{\alpha_{13}} x_4^{\alpha_{24}} \\ -x_2^{\alpha_2} + x_1^{\alpha_{21}} x_4^{\alpha_{24}} \\ x_2^{\alpha_{12}} x_3^{\alpha_{13}} \end{pmatrix}.$$

Proof *Case* $2(b1)$: $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. Since every 5×5 minors of φ_2 is zero, $rank(\varphi_2) \leq 4$. In matrix φ_2, deleting the 1st and the 6th columns, and the 3rd row, we have $x_3^{2\alpha_3}$ and deleting the 2nd row, and the 4th and the 5th columns, we get $x_2^{2\alpha_2+\alpha_{12}}$ as $4\times4-$minors of φ_2. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. Among the 2-minors of φ_3, we have $-x_2^{\alpha_2+\alpha_{12}}$, $x_3^{\alpha_3}$ and $x_4^{\alpha_4+\alpha_{24}}$. Since these are powers of different variables, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $2(b2)$: $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In matrix φ_2, deleting the 3rd row, and the 1st and the 6th columns, we have $x_3^{2\alpha_3}$ and deleting the 2nd and the 3rd columns, and the 4th row, we obtain $-x_4^{2\alpha_4+\alpha_{24}}$ and these determinants are relatively prime. Among the 2-minors of φ_3, we have $-x_2^{\alpha_{12}} f_2$, $x_3^{\alpha_3}$ and $x_4^{\alpha_4+\alpha_{24}}$. Since x_2 is a nonzero divisor modulo $\{x_3, x_4\}$, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $2(b3)$: Clearly, $rank(\varphi_1) = rank(\varphi_3) = 1$. In matrix φ_2, deleting the 3rd column and the 2nd row, we have $-x_2^{2\alpha_2}$ and deleting the 4th row and the 4th column, we obtain $x_4^{2\alpha_4}$ and these are relatively prime.

Case $2(b4)$: As in the above case, $rank(\varphi_1) = rank(\varphi_3) = 1$. In the matrix φ_2, deleting the 1st row and the 5th column, we have $-x_2^{2\alpha_{12}} x_3^{2\alpha_{13}}$ and deleting the 4th row and the 2nd column, we obtain $-x_4^{2\alpha_4}$ and they are relatively prime. □

Case 3(a) : In this case,

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_4^{\alpha_{34}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$$

and

$$f_5 = x_1^{\alpha_{31}} x_2^{\alpha_{42}} - x_3^{\alpha_{23}} x_4^{\alpha_{14}}$$

Here, $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{42}$, $\alpha_3 = \alpha_{23} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ gives $\alpha_1 > \alpha_{12} + \alpha_{14}$ and $\alpha_4 < \alpha_{42} + \alpha_{43}$. The extra conditions $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{31} + \alpha_{34}$ and Lemma 5.5.1 in [6] imply that the defining ideal $I(C)_*$ of the tangent cone is generated by the following sets:

- $Case\ 3(a1)\ : I(C)_* = (x_2^{\alpha_{12}}x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_3^{\alpha_{23}}x_4^{\alpha_{14}})$
- $Case\ 3(a2)\ : I(C)_* = (x_2^{\alpha_{12}}x_3^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_3^{\alpha_{23}}x_4^{\alpha_{14}})$
- $Case\ 3(a3)\ : I(C)_* = (x_2^{\alpha_{12}}x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3} - x_1^{\alpha_{31}}x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_3^{\alpha_{23}}x_4^{\alpha_{14}})$
- $Case\ 3(a4)\ : I(C)_* = (x_2^{\alpha_{12}}x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}}x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_3^{\alpha_{23}}x_4^{\alpha_{14}})$

Theorem 3.4 *In Case* $3(a1)$*, then the minimal free resolution for the tangent cone of* C *is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}}x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_3^{\alpha_{23}}x_4^{\alpha_{14}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} 0 & 0 & x_3^{\alpha_{23}} & x_4^{\alpha_{34}} & x_2^{\alpha_{42}} & 0 \\ 0 & -x_3^{\alpha_3} & 0 & 0 & -x_4^{\alpha_{14}} & 0 \\ x_4^{\alpha_{14}} & x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2^{\alpha_{12}} & 0 & -x_3^{\alpha_{23}} \\ -x_3^{\alpha_{43}} & 0 & -x_2^{\alpha_{12}} & 0 & 0 & x_4^{\alpha_{34}} \end{pmatrix},\ \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} & 0 \\ -x_4^{\alpha_{14}} & 0 \\ -x_2^{\alpha_{42}}x_3^{\alpha_{43}} & x_4^{\alpha_{34}} \\ 0 & -x_3^{\alpha_{23}} \\ x_3^{\alpha_3} & 0 \\ 0 & x_2^{\alpha_{12}} \end{pmatrix},$$

in Case $3(a2)$*, the minimal free resolution for the tangent cone of* C *is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}}x_4^{\alpha_{14}} & x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_3^{\alpha_{23}}x_4^{\alpha_{14}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} 0 & 0 & x_3^{\alpha_{23}} & x_4^{\alpha_{34}} & x_2^{\alpha_{42}} & 0 \\ 0 & -x_3^{\alpha_3} & 0 & 0 & -x_4^{\alpha_{14}} & 0 \\ x_4^{\alpha_{14}} & x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2^{\alpha_{12}} & 0 & -x_3^{\alpha_{23}} \\ -x_3^{\alpha_{43}} & 0 & -x_2^{\alpha_{12}} & 0 & -x_1^{\alpha_{21}} & x_4^{\alpha_{34}} \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}} & 0 \\ -x_4^{\alpha_{14}} & 0 \\ -x_2^{\alpha_{42}}x_3^{\alpha_{43}} & x_4^{\alpha_{34}} \\ 0 & -x_3^{\alpha_{23}} \\ x_3^{\alpha_3} & 0 \\ 0 & x_2^{\alpha_{12}} \end{pmatrix},$$

in Case $3(a3)$*, the minimal free resolution of the tangent cone of* C *is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}}x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_3^{\alpha_3} - x_1^{\alpha_{31}}x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_3^{\alpha_{23}}x_4^{\alpha_{14}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} 0 & 0 & x_3^{\alpha_{23}} & x_4^{\alpha_{34}} & x_2^{\alpha_{42}} & 0 \\ 0 & -x_3^{\alpha_3} + x_1^{\alpha_{31}}x_4^{\alpha_{34}} & 0 & 0 & -x_4^{\alpha_{14}} & 0 \\ x_4^{\alpha_{14}} & x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ x_1^{\alpha_{31}} & 0 & 0 & -x_2^{\alpha_{12}} & 0 & -x_3^{\alpha_{23}} \\ -x_3^{\alpha_{43}} & 0 & -x_2^{\alpha_{12}} & 0 & 0 & x_4^{\alpha_{34}} \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_2^{\alpha_2} & 0 \\ -x_4^{\alpha_{14}} & 0 \\ -x_2^{\alpha_{42}}x_3^{\alpha_{43}} & x_4^{\alpha_{34}} \\ x_1^{\alpha_{31}}x_2^{\alpha_{42}} & -x_3^{\alpha_{23}} \\ x_3^{\alpha_3} - x_1^{\alpha_{31}}x_4^{\alpha_{34}} & 0 \\ 0 & x_2^{\alpha_{12}} \end{pmatrix},$$

lastly, in Case $3(a4)$*, then the minimal free resolution of the tangent cone is*

$$0 \to R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R^1 \to R/I(C)_* \to 0$$

where

$$\varphi_1 = \left(x_2^{\alpha_{12}} x_4^{\alpha_{14}} \quad x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} \quad x_3^{\alpha_3} - x_1^{\alpha_{31}} x_4^{\alpha_{34}} \quad x_4^{\alpha_4} \quad x_3^{\alpha_{23}} x_4^{\alpha_{14}}\right),$$

$$\varphi_2 = \begin{pmatrix} x_3^{\alpha_{23}} & x_2^{\alpha_{42}} & x_4^{\alpha_{34}} & 0 & 0 & 0 \\ 0 & -x_4^{\alpha_{14}} & 0 & 0 & 0 & x_3^{\alpha_3} - x_1^{\alpha_{31}} x_4^{\alpha_{34}} \\ 0 & 0 & 0 & -x_4^{\alpha_{14}} & 0 & -x_2^{\alpha_2} + x_1^{\alpha_{21}} x_3^{\alpha_{23}} \\ 0 & 0 & -x_2^{\alpha_{12}} & -x_1^{\alpha_{31}} & -x_3^{\alpha_{23}} & 0 \\ -x_2^{\alpha_{12}} & -x_1^{\alpha_{21}} & 0 & x_3^{\alpha_{43}} & x_4^{\alpha_{34}} & 0 \end{pmatrix}, \varphi_3 = \begin{pmatrix} x_4^{\alpha_{34}} & x_2^{\alpha_{42}} x_3^{\alpha_{43}} \\ 0 & -x_3^{\alpha_3} + x_1^{\alpha_{31}} x_4^{\alpha_{34}} \\ -x_3^{\alpha_{23}} & -x_1^{\alpha_{31}} x_2^{\alpha_{42}} \\ 0 & x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}} \\ x_2^{\alpha_{12}} & x_1^{\alpha_1} \\ 0 & -x_4^{\alpha_{14}} \end{pmatrix}.$$

Proof *Case* $3(a1)$: Clearly, $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In matrix φ_2, deleting the 4th and the 5th columns, and the 3rd row, we have $-x_3^{2\alpha_3+\alpha_{23}}$ and similarly, deleting the 2nd and the 3rd columns, and the 4th row, we obtain $x_4^{2\alpha_4}$ as $4\times 4-$minors of φ_2. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. Among the 2-minors of φ_3, we have $x_2^{\alpha_2+\alpha_{12}}$, $x_3^{\alpha_3+\alpha_{23}}$ and $-x_4^{\alpha_4}$. Since they are powers of different variables, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $3(a2)$: $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In matrix φ_2, deleting the 3rd row, and the 4th and the 5th columns, we have $-x_3^{2\alpha_3+\alpha_{23}}$ and deleting the 2nd and the 3rd columns, and the 4th row, we obtain $-x_4^{2\alpha_4}$ and these determinants are relatively prime. Among the 2-minors of φ_3, we have $x_2^{\alpha_{12}} f_2$, $x_3^{\alpha_3+\alpha_{23}}$ and $-x_4^{\alpha_4}$. Since x_2 is a nonzero divisor modulo $\{x_3, x_4\}$, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $3(a3)$: $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In matrix φ_2, deleting the 1st and the 6th columns, and the 2nd row, we have $x_2^{2\alpha_2+\alpha_{12}}$ and similarly, deleting the 2nd and the 3rd columns, and the 4th row, we obtain $-x_4^{2\alpha_4}$ and these determinants are relatively prime. Among the 2-minors of φ_3, we have $x_2^{\alpha_2+\alpha_{12}}$, $x_3^{\alpha_{23}} f_3$ and $-x_4^{\alpha_4}$ and x_3 is a nonzero divisor modulo $\{x_2, x_4\}$. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

Case $3(a4)$: As in the above cases, $rank(\varphi_1) = 1$ and $rank(\varphi_3) = 2$. In the matrix φ_2, deleting the 1st row, and the 5th and the 6th columns, we obtain $-x_2^{2\alpha_{12}} x_4^{2\alpha_{14}}$ and deleting the 3rd row, and the 2nd and the 3rd columns, we have $x_3^{\alpha_{23}} {f_3}^2$ and they are relatively prime. Among the 2-minors of φ_3, we have $-x_2^{\alpha_{12}} f_2$, $-x_3^{\alpha_{23}} f_3$ and $-x_4^{\alpha_4}$ and x_4 is a nonzero divisor modulo $\{x_2^{\alpha_{12}} f_2, x_3^{\alpha_{23}} f_3\}$. Thus, $I(\varphi_3)$ contains a regular sequence of length 3. □

4. Hilbert function

In [2], Arslan and Mete showed that the Hilbert function is nondecreasing for local Gorenstein rings with embedding dimension four associated to noncomplete intersection monomial curve C in all above cases. In this section, we compute the Hilbert function of the tangent cone of C, if C is a Gorenstein noncomplete intersection monomial curve in $\mathbb{A}^4$ as in Case 1(a). For the other aforementioned cases, one can get similar results.

Theorem 3.1 implies that the tangent cone of C has the following graded minimal free resolution:

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \to R/I(C)_* \to 0$$

where $F_1 = \bigoplus_{i=1}^{5} R(-b_i)$, $F_2 = \bigoplus_{i=1}^{6} R(-c_i)$ and $F_3 = \bigoplus_{i=1}^{2} R(-d_i)$. Here, the numbers b_i are called the 1st Betti degrees, c_i are called 2nd Betti degrees and d_i are called 3rd Betti degrees.

Corollary 4.1 *Under the hypothesis of Theorem 3.1, Betti degrees of the minimal graded free resolution of the tangent cone of C is given by*

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \to R/I(C)_* \to 0$$

where $B_1 = \{b_1, b_2, b_3, b_4, b_5\}$, $B_2 = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $B_3 = \{d_1, d_2\}$ *and*

$$b_1 = \alpha_{13} + \alpha_{14}\,,\ b_2 = \alpha_2\,,\ b_3 = \alpha_3\,,\ b_4 = \alpha_4\,,\ b_5 = \alpha_{32} + \alpha_{14}\,,$$

$$c_1 = \alpha_3 + \alpha_{14}\,,\ c_2 = \alpha_2 + \alpha_3\,,\ c_3 = \alpha_4 + \alpha_{13}\,,\ c_4 = \alpha_{32} + \alpha_{13} + \alpha_{14}\,,\ c_5 = \alpha_{32} + \alpha_4\,,\ c_6 = \alpha_{14} + \alpha_2$$

$$d_1 = \alpha_2 + \alpha_3 + \alpha_{14}\,,\ d_2 = \alpha_4 + \alpha_{13} + \alpha_{32}\,.$$

The following corollary stems from the well-known fact that

$$H_G(i) = H_R(i) - H_{F_1}(i) + H_{F_2}(i) - H_{F_3}(i).$$

Corollary 4.2 *Under the hypothesis of Theorem 3.1., the Hilbert function of the tangent cone of C is given by*

$$\begin{aligned} H_G(i) = &\binom{i+3}{3} - \binom{i-b_1+3}{3} - \binom{i-b_2+3}{3} - \binom{i-b_3+3}{3} \\ &- \binom{i-b_4+3}{3} - \binom{i-b_5+3}{3} + \binom{i-c_1+3}{3} \\ &+ \binom{i-c_2+3}{3} + \binom{i-c_3+3}{3} + \binom{i-c_4+3}{3} \\ &+ \binom{i-c_5+3}{3} + \binom{i-c_6+3}{3} - \binom{i-d_1+3}{3} \\ &- \binom{i-d_2+3}{3}, \end{aligned}$$

for $i \geq 0$.

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