Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 10 Issue 6 (2019), Pages 23-31.

MAXIMAL CONVERGENCE OF FABER SERIES IN SMIRNOV-ORLICZ CLASSES

BURÇİN OKTAY

Abstract. It is known that Faber series are used for solving many problems in mechanical science, such as the problems on the stress analysis in the piezo-electric plane and the problems on the analysis of electro-elastic fields and thermo-elastic fields. In this paper, we consider that G is a complex domain bounded by a curve which belongs to a special subclass of smooth curves and the function f is analytic in the canonical domain G_R, $R > 1$. We research the rate of convergence to the function f by the partial sums of Faber series of the function f on the domain $\overline{G}$. We obtain results on the maximal convergence of the partial sums of the Faber series of the function f which belongs to the Smirnov-Orlicz class $E_M(G_R)$, $R > 1$.

1. Introduction and new results

Let G be a simply connected domain in the complex plane $\mathbb{C}$ bounded by a rectifiable curve Γ such that the complement of the closed domain $\overline{G}$ is a simply connected domain G' containing the point of infinity $z = \infty$. By the Riemann conformal mapping theorem there exists a unique function $w = \varphi(z)$ meromorphic in G' which maps the domain G' conformally and univalently onto the domain $|w| > 1$ and satisfies the conditions

$$\varphi(\infty) = \infty, \varphi'(\infty) = \gamma > 0, \tag{1.1}$$

where γ is the capacity of G. Let ψ be the inverse to φ and let ψ_0 be the mapping which maps the unit disk onto the domain G under the conditions $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. We define Γ_r to be the image of the circle $|w| = r, 0 < r < 1$, under the mapping ψ_0. If a function f, analytic on a domain G, satisfies the inequality

$$\int_{\Gamma_r} |f(z)|^p \, |dz| \leq M, \quad p > 0$$

for any r such that $0 < r < 1$, then f belongs to the Smirnov class $E_p(G)$ (see, e.g., [18, p. 77]).

2010 *Mathematics Subject Classification.* 30E10, 41A10, 41A58, 46E30.
Key words and phrases. Smirnov-Orlicz Classes, Faber polynomials, Faber Series, Smooth Curves, Maximal Convergence.

Submitted November 18, 2018. Published September 9, 2019.
Author was supported through the program BAP by Balikesir University with the project numbered 2018/073.
Communicated by N. Braha.

A function $M : (-\infty, \infty) \to (0, \infty)$ is called an N-function if it has the representation

$$M(u) = \int_0^{|u|} p(t)dt,$$

where the function p is right continuous and positive for $t \geq 0$ and strictly positive for $t > 0$, such that

$$p(0) = 0, p(\infty) = \lim_{t\to\infty} p(t) = \infty.$$

The function

$$N(v) := \int_0^{|v|} q(s)ds,$$

where

$$q(s) = \sup_{p(t)\leq s} t\,, \quad s \geq 0$$

is defined as complementary function of M [10, p. 11]. By $L_M(\Gamma)$ we denote the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha\,|f(z)|]\,|dz| < \infty$$

for some $\alpha > 0$. The space $L_M(\Gamma)$ becomes a Banach space with the norm

$$\|f\|_{L_M(\Gamma)} := \sup\left\{ \int_\Gamma |f(z)g(z)|\,|dz| : \quad g \in L_N(\Gamma), \rho(g; N) \leq 1 \right\},$$

where

$$\rho(g; N) := \int_\Gamma N[|g(z)|]\,|dz| < \infty.$$

The norm $\|\cdot\|_{L_M(\Gamma)}$ is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space. It is known that [16, p. 50]

$$L_M(\Gamma) \subset L_1(\Gamma).$$

Definition 1. *Let M be an N-function. If an analytic function f in G satisfies the condition*

$$\int_{\Gamma_r} M[|f(z)|]\,|dz| < \infty$$

uniformly in r, $0 < r < 1$, then it belongs to the Smirnov-Orlicz class $E_M(G)$.

If $M(x) = M(x, p) := x^p$, $1 < p < \infty$, then the Smirnov-Orlicz class $E_M(G)$ coincides with the usual Smirnov class $E_p(G)$. Every function in the class $E_M(G)$ has non-tangential boundary values a.e. on Γ and the boundary function belongs to $L_M(\Gamma)$, and hence for $f \in E_M(G)$ we can define the norm $E_M(G)$ as:

$$\|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}\,.$$

Now we define the best approximation error for the function $f \in E_M(G)$ as:

$$\begin{aligned} E_n^M(f, G) :&= \inf \|f - p_n\|_{L_M(\Gamma)} \\ &= \inf\left\{ \sup\left\{ \int_\Gamma |(f(\zeta) - p_n(\zeta))g(\zeta)|\,|d\zeta| : \quad g \in L_N(\Gamma), \rho(g, N) \leq 1 \right\} \right\}, \end{aligned}$$

where inf is taken over the polynomials p_n of degree at most n.

Since φ is analytic in the domain G' without the point $z=\infty$, it has only the pole at $z=\infty$. Therefore its Laurent expansion in some neighborhood of the point $z=\infty$ has the form

$$\varphi(z)=\gamma z+\gamma_0+\frac{\gamma_1}{z}+\frac{\gamma_2}{z^2}+\cdots+\frac{\gamma_n}{z^n}+\cdots. \tag{1.2}$$

For a non-negative integer k, we set

$$\varphi^k(z)=\Big(\gamma z+\gamma_0+\frac{\gamma_1}{z}+\frac{\gamma_2}{z^2}+\cdots+\frac{\gamma_n}{z^n}+\cdots\Big)^k. \tag{1.3}$$

The polynomial part (i.e., the "principal part at infinity") in the Laurent series expansion of φ^k is called Faber polynomial on the domain G of order k. We use the notation

$$\varphi_k(z)=\gamma^k z^k+a_{k-1}^{(k)}z^{k-1}+a_{k-2}^{(k)}z^{k-2}+\cdots+a_1^{(k)}z+a_0^{(k)}. \tag{1.4}$$

For the sum of the terms containing negative powers of z in the expansion (1.3) we use the notation

$$-E_k(z)=\frac{b_1^{(k)}}{z}+\frac{b_2^{(k)}}{z^2}+\cdots+\frac{b_n^{(k)}}{z^n}+\cdots.$$

Hence the identity

$$\varphi_k(z)=\varphi^k(z)+E_k(z),\quad z\in\overline{G'} \tag{1.5}$$

holds in the sense of convergence. Now we define for $R>1$

$$\Gamma_R:=\{z\in\operatorname{ext}\Gamma:\quad |\varphi(z)|=R\},\quad G_R:=\operatorname{int}\Gamma_R.$$

If $R=1$, the curve Γ_1 is the boundary Γ of the domain G. Faber polynomials have the following integral representation

$$\varphi_k(z)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varphi^k(\varsigma)}{\varsigma-z}d\varsigma,\quad z\in G_R. \tag{1.6}$$

Instead of the closure of the simply connected domain G, if we consider a non-degenerate bounded continuum K with the simply connected complement G', all the definitions and formulae are unchanged. Thus Faber polynomials may be defined by (1.4) or (1.6) for any nondegenerate bounded continuum K with a simply connected complement. If a function f is analytic on a continuum K, then the following expansion holds

$$f(z)=\sum_{k=0}^{\infty}a_k\varphi_k(z),\quad z\in K$$

and the series converges absolutely and uniformly on K, where

$$a_k:=\frac{1}{2\pi i}\int_{|t|=1}\frac{f(\psi(t))}{t^{k+1}}dt,\quad k=0,1,2,\dots \tag{1.7}$$

are called Faber coefficients of the function f with respect to K. More detailed information about Faber polynomials, Faber series and their approximation properties can be found in [18]. In this paper, we study the remainder term

$$R_n(z,f)=f(z)-\sum_{k=0}^{n}a_k\varphi_k(z)=\sum_{k=n+1}^{\infty}a_k\varphi_k(z). \tag{1.8}$$

Suppose that f is analytic in the canonical domain $G_R, R > 1$. If $|f(z)| \le M$ for $z \in G_R$, we have the following formula for the Faber coefficients

$$a_k := \frac{1}{2\pi i}\int_{|t|=R} \frac{f(\psi(t))}{t^{k+1}}dt, \quad k = 0, 1, 2, \ldots. \tag{1.9}$$

In this paper we assume that the boundary Γ of the domain G is of the class $\mathfrak{B}(\alpha,\beta)$, $\alpha \in (0,1]$, $\beta \in [0,\infty)$ which is a special subclass of smooth curves defined in [6]. In this case we estimate the remainder term $R_n(z,f)$, for $z \in \Gamma$ for functions f belonging to the Smirnov-Orlicz class $E_M(G_R)$.
Let $\theta(s)$ denote the angle between the positive direction of the real axis and the tangent to the curve Γ at a point M at a distances s traveled counterclockwise from a fixed point on Γ.
The definition of the class $\mathfrak{B}(\alpha,\beta)$ is as the follows.

Definition 2. *([6]). If the inequality*

$$\omega(\theta,\delta) := \sup_{|h|\le\delta} \|\theta(\cdot) - \theta(\cdot + h)\|_{[0,2\pi]} \le c\delta^{\alpha}\ln^{\beta}\frac{4}{\delta}, \quad \delta \in (0,\pi] \tag{1.10}$$

holds for some parameters $\alpha \in (0,1]$, $\beta \in [0,\infty)$ and for a positive constant c independent of δ, then $\Gamma \in \mathfrak{B}(\alpha,\beta)$.

In this definition, the norm $\|\cdot\|_{[0,2\pi]}$ means the maximum norm over the interval $[0,2\pi]$.

In particular, the class $\mathfrak{B}(\alpha,0)$ coincides with the class of Lyapunov curves. Furthermore, the class $\mathfrak{B}(\alpha,\beta)$ is a subclass of Dini-smooth curves; i.e., $\int_0^c \frac{\omega(\theta,t)}{t}dt < \infty$ for some $c > 0$. For a proof of this result and additional information about the class $\mathfrak{B}(\alpha,\beta)$ see [6].

Now we give our main result.

Theorem 1.1. *If G is a domain bounded by a curve Γ of the class $\mathfrak{B}(\alpha,\beta)$, $\alpha \in (0,1]$, $\beta \in [0,\infty)$ and f a function in $E_M(G_R)$, then the remainder $R_n(z,f)$ satisfies the inequality*

$$|R_n(z,f)| \le c\frac{E_n^M(f,G_R)}{R^{n+1}(R-1)},$$

for $z \in \Gamma$. Here, $c > 0$ is a universal constant independent of n and z.

In the case that f belongs to Smirnov-Orlicz class and z belongs to the continuum K, maximal convergence of Faber series was studied in Theorem 1.4 in [5]. In that result for the boundary Γ of the continuum K there is no assumption. Theorem 1 given above characterizes the maximal convergence of Faber series in the Smirnov-Orlicz classes under the assumption that G is a domain bounded by a curve of the class $\mathfrak{B}(\alpha,\beta)$, $\alpha \in (0,1]$ and $\beta \in [0,\infty)$. The result given in Theorem 1 is an improvement of the result given Theorem 1.4 in [5].
There are some results related to maximal convergence in literature. Firstly, Bernstein and Walsh (see [18, p. 54-59]) studied the maximal convergence of polynomials. They also obtained direct and inverse theorems when the function f is analytic on canonical domain G_R. Walsh (see, e.g., [2, p. 27]) proved also some results on maximal convergence of Fourier series. Many results about maximal convergence of Faber series were proved by Suetin. In [18, Chapter X] he obtained results on

maximal convergence of Faber series of functions f analytic on the canonical domain G_R and continuous on $\overline{G}_R$ and when f belongs to the Smirnov class $E_p(G_R)$. He assumed that the boundary of G belongs to the class of Al'per curves which are larger than the class $\mathfrak{B}(\alpha,\beta)$, $\alpha\in(0,1]$, $\beta\in[0,\infty)$. He also proved some results on maximal convergence for the case of a continuum K.

2. Auxiliary Results

From (1.8) and (1.9) we obtain,

$$R_n(z,f)=\frac{1}{2\pi i}\int_{|t|=R} f(\psi(t))\left[\sum_{k=n+1}^{\infty}\frac{\varphi_k(z)}{t^{k+1}}\right]dt. \tag{2.1}$$

Let P_n be the polynomial of the best uniform approximation of the function f in the closed domain $\overline{G}_R$, then the formula (2.1) implies

$$R_n(z,f)=\frac{1}{2\pi i}\int_{|t|=R}\{f(\psi(t))-P_n(\psi(t))\}\sum_{k=n+1}^{\infty}\frac{\varphi_k(z)}{t^{k+1}}dt. \tag{2.2}$$

From (1.5), we can write

$$\sum_{k=n+1}^{\infty}\frac{\varphi_k(z)}{t^{k+1}}=\sum_{k=n+1}^{\infty}\frac{\varphi^k(z)}{t^{k+1}}+\sum_{k=n+1}^{\infty}\frac{E_k(z)}{t^{k+1}},\quad z=\psi(w). \tag{2.3}$$

The function $E_k(\psi(\omega))$ is given by

$$E_k(\psi(\omega))=\frac{1}{2\pi i}\int_{|\tau|=1}\tau^k F(\tau,\omega)d\tau,\quad |\omega|\geq 1, \tag{2.4}$$

where

$$F(\tau,\omega)=\frac{\psi'(\tau)}{\psi(\tau)-\psi(\omega)}-\frac{1}{\tau-\omega}=\sum_{k=0}^{\infty}\frac{E_k(\psi(\omega))}{t^{k+1}} \tag{2.5}$$

If Γ is sufficiently smooth, then this expansion converges in the closed domain $|\tau|\geq 1, |\omega|\geq 1$ [18, p. 156].

For $|w|\geq 1$ and $|t|=R$, we can write

$$\sum_{k=n+1}^{\infty}\frac{E_k(\psi(\omega))}{t^{k+1}}=\frac{1}{2\pi i}\int_{|\tau|=1}F(\tau,\omega)\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}d\tau. \tag{2.6}$$

If one wants to estimate the remainder term $R_n(z,f)$ for $z\in\Gamma$ when f is analytic in G_R, $R>1$, from (2.2), (2.3) and (2.6), it is necessary to prove that the integral

$$\int_{|\tau|=1}|F(\tau,\omega)|\,|d\tau|$$

is finite for all $|w|\geq 1$, according to the geometric properties of the boundary Γ of the domain G. In [14] we proved that the integral above is finite in the case that the boundary of the domain G is of the class $\mathfrak{B}(\alpha,\beta)$, $\alpha\in(0,1]$, $\beta\in[0,\infty)$. The related theorem is as the following.

Theorem 2.1. *([14]) If G is a domain bounded by a curve of the class $\mathfrak{B}(\alpha,\beta)$, $\alpha\in(0,1]$, $\beta\in[0,\infty)$, then there exists a constant $c>0$ such that for all $|w|\geq 1$ the following inequality holds*

$$\int_{|\tau|=1}|F(\tau,w)|\,|d\tau|=\int_{|\tau|=1}\left|\frac{\psi'(\tau)}{\psi(\tau)-\psi(w)}-\frac{1}{\tau-w}\right||d\tau|\leq c<\infty$$

and this integral converges uniformly with respect to $|w| \geq 1$.

If Γ belongs to the class $\mathfrak{B}(\alpha, \beta)$, $\alpha \in (0,1]$, $\beta \in [0, \infty)$ then the Al'per condition, i.e., the condition $\int_0^1 \omega(\theta, t) \frac{1}{t} \ln \frac{1}{t} dt < \infty$ holds. If the Al'per condition holds, then the inequality

$$0 < c_1 \leq |\psi'(w)| \leq c_2 < \infty, \quad |w| \geq 1 \tag{2.7}$$

is valid for some positive constants c_1 and c_2 [18, p. 141]. Hence this property is also valid for φ' on Γ and Γ_R, $R > 1$.

Also, the following two theorems are useful for the proof of our main result.

Theorem 2.2. *(*[10, p. 74]*). For every pair of real valued functions* $u \in L_M(\Gamma)$, $v \in L_N(\Gamma)$, *the inequality*

$$\left| \int_\Gamma u(z) v(z) dz \right| \leq \|u\|_{L_M(\Gamma)} \|v\|_{L_N(\Gamma)}$$

holds.

Theorem 2.3. *(*[10, p. 67]*). For every pair of real valued functions* $u \in L_M(\Gamma)$, $v \in L_N(\Gamma)$, *the inequality*

$$\int_\Gamma u(z) v(z) dz \leq \rho(u; M) + \rho(\nu; N)$$

holds.

3. Proofs of Main Result

3.1. **Proof of Theorem 1.** Let $z \in \Gamma$ and let P_n be the best approximating polynomial of degree at most n to the function $f \in E_M(G_R)$. From the relations (2.2) and (2.3) we get

$$|R_n(z,f)| \leq \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt|$$

$$+ \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k(\psi(w))}{t^{k+1}} \right| |dt|,$$

where $w = \varphi(z)$ and $E_k(\psi(w))$ was defined in (2.4).

Let

$$I_1 := \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt|.$$

Now taking $\zeta = \psi(t)$ and taking into account (2.7), we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{\Gamma_R} |f(\zeta)-P_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\zeta)]^{k+1}} \right| |\varphi'(\zeta)|\,|d\zeta| \\ &\le \frac{c_1}{2\pi}\int_{\Gamma_R} |f(\zeta)-P_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\zeta)]^{k+1}} \right| |d\zeta| \\ &\le \frac{c_1}{2\pi}\int_{\Gamma_R} |f(\zeta)-P_n(\zeta)| \frac{|\varphi(z)|^{n+1}}{|\varphi(\zeta)|^{n+1}\,|\varphi(\zeta)-\varphi(z)|}\,|d\zeta| \\ &\le \frac{c_1}{2\pi R^{n+1}(R-1)}\int_{\Gamma_R} |f(\zeta)-P_n(\zeta)|\,|d\zeta|, \end{aligned}$$

since $z\in\Gamma$. By Theorem 3,

$$I_1 \le \frac{c_1}{2\pi R^{n+1}(R-1)} \left\{ \sup \int_{\Gamma_R} |f(\zeta)-P_n(\zeta)|\,|g(\zeta)|\,|d\zeta| \right\} \cdot \left\{ \sup \int_{\Gamma_R} 1\cdot |h(\zeta)|\,|d\zeta| \right\},$$

where the suprema are taken over all functions $g \in L_N(\Gamma)$ with $\rho(g;N)\le 1$ and $h\in L_M(\Gamma)$ with $\rho(h;M)\le 1$, respectively. Hence the last inequality implies that

$$I_1 \le \frac{c_1}{2\pi R^{n+1}(R-1)} E_n^M(f,G_R) \left\{ \sup \int_{\Gamma_R} |h(\zeta)|\,|d\zeta|\,;\rho(h;M)\le 1 \right\},$$

where

$$\left\{ \sup \int_{\Gamma_R} |h(\zeta)|\,|d\zeta|\,;\rho(h;M)\le 1 \right\} \le 1 + N(1)\mathrm{MEAS}(\Gamma_R) \le c_2$$

because of Theorem 4.

Hence I_1 is estimated as

$$I_1 \le \frac{c_3}{2\pi R^{n+1}(R-1)} E_n^M(f,G_R). \tag{3.1}$$

Now we set

$$I_2 := \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k(\psi(w))}{t^{k+1}} \right| |dt|.$$

Using the representation of $E_k(\psi(w))$ given in (2.4) and using Theorem 2 and Theorem 3, we estimate

$$\begin{aligned} I_2 &\le \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t))-P_n(\psi(t))| \frac{1}{2\pi}\int_{|\tau|=1} \left| \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} F(\tau,w) \right| |d\tau|\,|dt| \\ &\le \frac{1}{2\pi}\int_{|\tau|=1} |F(\tau,w)| \left\{ \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t))-P_n(\psi(t))| \left| \frac{\tau^{n+1}}{t^{n+1}(t-\tau)} \right| |dt| \right\} |d\tau| \\ &\le \frac{1}{2\pi R^{n+1}(R-1)} \int_{|\tau|=1} |F(\tau,w)|\,|d\tau| \left\{ \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t))-P_n(\psi(t))|\cdot 1\,|dt| \right\} \\ &\le \frac{c_4}{2\pi R^{n+1}(R-1)} \left\{ \sup \int_{\Gamma_R} |f(\zeta)-P_n(\zeta)|\,|g(\zeta)|\,|d\zeta| \right\} \end{aligned}$$

$$\cdot \left\{ \sup \int_{\Gamma_R} 1 \cdot |h(\zeta)| \, |d\zeta| \right\},$$

where the suprema are taken over all functions $g \in L_N(\Gamma)$ with $\rho(g; N) \leq 1$ and $h \in L_M(\Gamma)$ with $\rho(h; M) \leq 1$, respectively. If we continue similarly to the last part of the estimation of I_1, we obtain

$$I_2 \leq \frac{c_5}{2\pi R^{n+1}(R-1)} E_n^M(f, G_R). \tag{3.2}$$

Hence from (3.1) and (3.2), we finally conclude that

$$|R_n(z, f)| \leq I_1 + I_2 \leq c_6 \frac{E_n^M(f, G_R)}{R^{n+1}(R-1)}$$

with some constant $c_6 > 0$ independent of n and $z \in \Gamma$.

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped to improve this article.

References

[1] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, (1993).
[2] D. Gaier, *Lectures on Complex Approximation*, (translated from German by Renate McLaughlin), Boston, Birkhauser, (1987).
[3] A. Guven, D. M. Israfilov, *Polynomial Approximation in Smirnov-Orlicz Classes*, Comput. Methods Funct. Theory, **2 2** (2002) 509-517.
[4] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translation of Mathematical Monographs R.I.:AMS, Providence, **26** (1968).
[5] D. M. Israfilov, B. Oktay, R.Akgun, *Approximation in Smirnov-Orlicz Classes*, Glasnik Matematicki **40 60** (2005), 87-102.
[6] D. M. Israfilov, B. Oktay, *Approximation properties of the Bieberbach polynomials in closed Dini-smooth domains*, Bull. Belg. Math. Soc., **13 1** (2006), 91-99.
[7] D. M. Israfilov, B. Oktay, *Approximation properties of Julia polynomials*, Acta Math. Sin. (Engl.Ser.), **23 7** (2007) 1303-1310.
[8] V. Kokilashvili, *On approximation of analytic functions from Ep classes*,(Russian) Trudy Tbilisi. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **34** (1968) 82-102.
[9] V. Kokilashvili, *On analytic functions of Smirnov-Orlicz classes*, Studia Math, **31** (1968) 43-59.
[10] M. A. Krasnoselskii, Ta.B.Rutickii, *Convex Functions and Orlicz Spaces*, p.Noordhoff Ltd. Groningen, (1961).
[11] T. Kovari, Ch. Pommerenke, *On Faber polynomials and Faber expansions*, Mathematische Zeitschrift, **99 3** (1967) 193-206.
[12] S. N. Mergelyan, *Certain questions of the constructive theory of functions*, (Russian) Trudy Math, Inst. Steklov, **37** (1951), 1-91.
[13] B. Oktay, D. M. Israfilov, *An Approximation of Conformal Mappings on smooth domains.* Complex, Var. Elliptic Equ., **58 6**, (2013), 741-750.
[14] B. Oktay, *Convergence of Faber series in the Smirnov classes with variable exponent on canonical domains*, submitted for publication.
[15] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, Berlin, (1991).
[16] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, (1991).
[17] P. K. Suetin, *Polynomials Orthogonal over a Region and Bieberbach Polynomials*, Proceedings of the Steklov Institute of Mathematics, Amer. Math. Soc. Providence, Rhode Island, (1975).
[18] P. K. Suetin, *Series of Faber Polynomials*, Gordon and Breach Science Publishers, Amsterdam, (1998).
[19] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc. Colloq. Publ. 20 (Amer.Mathematical Soc., Providence, RI, 5th ed.), (1969).

Burcin Oktay
Faculty of Art and Science, Department of Mathematics, Balikesir University, 10145, Balikesir, Turkey
E-mail address: burcinokt@gmail.com