Available online: December 24, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 1, Pages 1019–1029 (2019)
DOI: 10.31801/cfsuasmas.501449
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ON MINIMAL FREE RESOLUTION OF THE ASSOCIATED GRADED RINGS OF CERTAIN MONOMIAL CURVES : NEW PROOFS IN $\mathbb{A}^4$

PINAR METE AND ESRA EMINE ZENGİN

Abstract. In this article, even if it is known for general case in [17], we give the explicit minimal free resolution of the associated graded ring of certain affine monomial curves in affine 4-space based on the standard basis theory. As a result, we give the minimal graded free resolution and the Hilbert function of the tangent cone of these families in $\mathbb{A}^4$ in the simple form according to [17].

1. Introduction

Let k be a field. The associated graded ring $G = gr_m(A) = \bigoplus_{i=0}^{\infty}(m^i/m^{i+1})$ of the local ring A with maximal ideal m is a standard graded k-algebra. Since it corresponds to the important geometric construction, it has been studied to get comprehensive information on the local ring (see [14, 13, 7, 8, 9]). Because the minimal finite free resolution of a finitely generated k-algebra is a very useful tool to extract information about the algebra, finding an explicit minimal free resolution of a standard k-algebra is a basic problem. This difficult problem has been extensively studied in the case of affine monomial curves [17, 15, 4, 10, 2].

We recall that a monomial affine curve C has a parametrization

$$x_0 = t^{m_0},\ x_1 = t^{m_1},\ \dots,\ x_n = t^{m_n}$$

where $m_0, m_1, \dots, m_n$ are positive integers with $gcd(m_0, m_1, ..., m_n) = 1$. The defining ideal $I(C)$ is the kernel of the k-algebra homomorphism $\phi : k[x_0, \dots x_n] \mapsto k[t]$ given by $x_i \mapsto t^{m_i}$, $i = 0, \dots, n$.

The additive semigroup, which is denoted by

$$< m_0, m_1, ..., m_n >= \{ \sum_{0\leq i\leq n} \mathbb{N}m_i \ \mid \ \mathbb{N} = \{0, 1, 2, \dots\}\}$$

Received by the editors: December 22, 2017; Accepted: June 18, 2018.

2010 *Mathematics Subject Classification.* Primary 13H10, 14H20; Secondary 13P10.

Key words and phrases. Minimal free resolution, monomial curve, Cohen-Macaulayness, Hilbert function of a local ring, tangent cone.

generated minimally by $m_0, m_1, ..., m_n$, i.e., $m_j \notin \sum_{0\leq i\leq n; i\neq j} \mathbb{N}m_i$ for $i \in \{0, \ldots, n\}$.

Assume that $m_0, m_1, \ldots, m_n$ be positive integers such that $0 < m_0 < m_1 < \ldots < m_n$ and $m_i = m_0 + id$ for every $1 \leq i \leq n$, where d is the common difference, i.e. the integers m_i's form an arithmetic progression. The monomial curve which is defined parametrically by

$$x_0 = t^{m_0},\ x_1 = t^{m_1},\ \ldots,\ x_n = t^{m_n}$$

such that $0 < m_0 < m_1 < \ldots < m_n$ form an arithmetic progression is called a certain monomial curve.

In order to study the associated graded ring of a monomial curve C at the origin, it is possible to consider either the associated graded ring of $A = k[[t^{m_0}, t^{m_1}, ..., t^{m_n}]]$ with respect to the maximal ideal $m = (t^{m_0}, t^{m_1}, ..., t^{m_n})$ which is denoted by $gr_m(k[[t^{m_0}, t^{m_1}, ..., t^{m_n}]])$, or the ring $k[x_0, x_1, ..., x_n]/I(C)^*$, where $I(C)^*$ is the ideal generated by polynomials f^*s which are smallest homogeneous summands of generators of the defining ideal $I(C)$. We recall that $I(C)^*$ is the defining ideal of the tangent cone of the curve C at the origin.

Our main aim is to give an explicit minimal free resolution of the associated graded ring for certain monomial curves in affine 4-space. Even if one can obtain the numerical invariants of the minimal free resolution of the tangent cone of this kind of curves by using the Theorem 4.1 and Proposition 4.6 in [17], we give the minimal free resolutions of their tangent cones in an explicit form by giving a new proof. Knowing the minimal generating set of the defining ideal of these curves from [11], we find the minimal generators and show the Cohen-Macaulayness of the tangent cone of these families of curves. See also [7]. We obtain explicit minimal free resolution by using Schreyer's theorem but prove it using the Buchsbaum-Eisenbud theorem [3]. Finally, we give the minimal graded free resolutions and as a corollary compute the Hilbert functions of the tangent cones for these families. All computations have been carried out using SINGULAR[6].

2. Minimal generators of the associated graded ring

In this section, we find the minimal generators of the tangent cone of the certain monomial curve C having the defining ideal as in Theorem 4.5 in [11] in affine 4-space. First, we recall the theorem which gives the construction of the minimal set of generators for the defining ideal of a certain affine monomial curve in $\mathbb{A}^4$.

Let $m_0 < m_1 < m_2 < m_3$ be positive integers with $gcd(m_0, m_1, m_2, m_3) = 1$ and assume that m_0, m_1, m_2, m_3 form an arithmetic progression with common difference d. Let $R = k[x_0, x_1, x_2, y]$ be a polynomial ring over a field k. We use y instead of x_3 by following the same notation in [15, 16, 11]. Let $\phi : R \to k[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}]$ be the k-algebra homomorphism defined by

$$\phi(x_0) = t^{m_0},\ \phi(x_1) = t^{m_1},\ \phi(x_2) = t^{m_2},\ \phi(y) = t^{m_3}$$

and $I(C) = Ker(\phi)$. Let us write $m_0 = 3a + b$ such that a and b are positive integers $a \geq 1$ and $b \in [1, 3]$. In [15], the following theorem is given as a definition.

Theorem 1. [11] *Let*

$$\begin{aligned}
&\xi_{11} := x_1^2 - x_0x_2,\\
&\varphi_i := x_{i+1}x_2 - x_iy,\ for\ 0 \leq i \leq 1.\\
&\psi_j := x_{b+j}y^a - x_0^{a+d}x_j,\ if\ 1 \leq b \leq 2\ \ and\ \ 0 \leq j \leq 2-b.\\
&\theta := y^{a+1} - x_0^{a+d}x_{3-b}.\\
&G := \begin{cases} \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0, \psi_1\} \cup \{\theta\} & if \quad b = 1,\\ \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0\} \cup \{\theta\} & if \quad b = 2,\\ \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\theta\} & if \quad b = 3.\end{cases}
\end{aligned}$$

then, G is a minimal generating set for the defining ideal $I(C)$.

Now, we recall the definition of the negative degree reverse lexicographical ordering among the other local orderings.

Definition 2. [5, p.14] *(negative degree reverse lexicographical ordering)*
$x^\alpha >_{ds} x^\beta :\Leftrightarrow degx^\alpha < degx^\beta$, *where* $degx^\alpha = \alpha_1 + ... + \alpha_n$,
or $(degx^\alpha = degx^\beta$ *and* $\exists 1 \leq i \leq n : \alpha_n = \beta_n, ..., \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i)$.

In the following Lemma, we show that the above set G is also standard basis with respect to $>_{ds}$.

Lemma 3. *The minimal set G is a standard basis with respect to the negative degree reverse lexicographical ordering $>_{ds}$ with $x_0 > x_1 > x_2 > y$.*

Proof. We apply the standard basis algorithm to the set G. We will prove for $b = 1, 2$, and 3, respectively. By using the notation in [5], we denote the leading monomial of a polynomial f by $LM(f)$, the S-polynomial of the polynomials f and g by *spoly*(f, g) and the Mora's polynomial weak normal form of f with respect to G by $NF(f \mid G)$.

Case $b = 1$.
From the minimal generating set G in Theorem 1, we obtain

$$G = \{\xi_{11} = x_1^2 - x_0x_2,\ \ \varphi_0 = x_1x_2 - x_0y,\ \ \varphi_1 = x_2^2 - x_1y,$$
$$\psi_0 = x_1y^a - x_0^{a+d+1},\ \ \psi_1 = x_2y^a - x_0^{a+d}x_1,\ \ \theta = y^{a+1} - x_0^{a+d}x_2\}.$$

Recalling that the ordering is the negative degree reverse lexicographical ordering, we have $LM(\xi_{11}) = x_1^2$, $LM(\varphi_0) = x_1x_2$, $LM(\varphi_1) = x_2^2$, $LM(\psi_0) = x_1y^a$, $LM(\psi_1) = x_2y^a$ and $LM(\theta) = y^{a+1}$. We begin with ξ_{11} and φ_0. $LM(\xi_{11}) = x_1^2$ and $LM(\varphi_0) = x_1x_2$ and $\text{spoly}(\xi_{11}, \varphi_0) = x_0x_1y - x_0x_2^2$. $LM(\text{spoly}(\xi_{11}, \varphi_0)) = x_0x_2^2$. Among the leading monomials of the elements of G, only $LM(\varphi_1)$ divides $LM(\text{spoly}(\xi_{11}, \varphi_0))$. $\text{ecart}(\varphi_1) = \text{ecart}(\text{spoly}(\xi_{11}, \varphi_0)) = 0$. $\text{spoly}(\varphi_1, \text{spoly}(\xi_{11}, \varphi_0)) = 0$ implies $NF(\text{spoly}(\xi_{11}, \varphi_0) \mid G) = 0$. Choose ξ_{11} and φ_1. $\text{lcm}(LM(\xi_{11}), LM(\varphi_1)) = LM(\xi_{11}).LM(\varphi_1)$, thus $NF(\text{spoly}(\xi_{11}, \varphi_1) \mid \{\xi_{11}, \varphi_1\}) = 0$. This implies that $NF(\text{spoly}(\xi_{11}, \varphi_1) \mid G) = 0$. $NF(\text{spoly}(\xi_{11}, \psi_1) \mid G) = 0$, $NF(\text{spoly}(\xi_{11}, \theta) \mid G) =$

0, $NF(\text{spoly}(\varphi_0,\theta)\,|\,G) = 0$, $NF(\text{spoly}(\varphi_1,\psi_0)\,|\,G) = 0$ and $NF(\text{spoly}(\varphi_1,\theta)\,|\,G) = 0$. Now, we compute $\text{spoly}(\xi_{11},\psi_0) = x_0^{a+d+1}x_1 - x_0x_2y^a$. Only $LM(\psi_1)$ divides $LM(\text{spoly}(\xi_{11},\psi_0)) = x_0x_2y^a$. $\text{ecart}(\psi_1)=\text{ecart}(\text{spoly}(\xi_{11},\psi_0)) = d$. Since $\text{spoly}(\psi_1,\text{spoly}(\xi_{11},\psi_0)) =0$, $NF(\text{spoly}\ (\xi_{11},\psi_0)\,|\,G) = 0$. $\text{spoly}(\varphi_0,\varphi_1) = x_1^2y - x_0x_2y$. $LM(\xi_{11})$ divides $LM(\text{spoly}(\varphi_0,\varphi_1)) = x_1^2y$. $\text{ecart}(\xi_{11}) =\text{ecart}(\text{spoly}(\varphi_0,\varphi_1)) = 0$. $\text{spoly}(\xi_{11},\text{spoly}(\varphi_0,\varphi_1)) = 0$ implies $NF(\text{spoly}(\varphi_0,\varphi_1)\,|\,G) = 0$. Choose φ_0 and ψ_0. $\text{spoly}(\varphi_0,\psi_0) = x_0^{a+d+1}x_2 - x_0y^{a+1}$. $LM(\theta)$ divides $LM(\text{spoly}(\varphi_0,\psi_0)) = x_0y^{a+1}$. $\text{ecart}(\theta) = \text{ecart}(\text{spoly}(\varphi_0,\psi_0)) = d$. $\text{spoly}(\theta,\text{spoly}(\varphi_0,\psi_0)) = 0$ implies $NF(\text{spoly}(\varphi_0,\psi_0)\,|\,G) = 0$. $\text{spoly}(\varphi_0,\psi_1) = x_0^{a+d}x_1^2 - x_0y^{a+1}$. $LM(\theta)$ divides $LM(\text{spoly}(\varphi_0,\psi_1)) = x_0y^{a+1}$. $\text{ecart}(\theta) = \text{ecart}(\text{spoly}(\varphi_0,\psi_1)) = d$. $\text{spoly}(\theta,\text{spoly}(\varphi_0,\psi_1)) = x_0^{a+d}x_1^2 - x_0^{a+d+1}x_2$. Among the leading monomials, $LM(\xi_{11}) = x_1^2$ divides $LM(\text{spoly}(\text{spoly}(\theta,\text{spoly}(\varphi_0,\psi_1)))) = x_0^{a+d}x_1^2$. $\text{ecart}(\xi_{11}) = \text{ecart}(\text{spoly}(\theta,\text{spoly}(\varphi_0,\psi_1))) = 0$. $\text{spoly}(\xi_{11},\text{spoly}(\theta,\text{spoly}(\varphi_0,\psi_1))) = 0$ implies $NF(\text{spoly}(\varphi_0,\psi_1)\,|\,G) = 0$. We compute $\text{spoly}(\varphi_1,\psi_1) = x_0^{a+d}x_1x_2 - x_1y^{a+1}$. $LM(\psi_0)$ and $LM(\theta)$ divides $LM(\text{spoly}(\varphi_1,\psi_1)) = x_1y^{a+1}$. Note that $\text{ecart}(\psi_0) = \text{ecart}(\theta) = d$. Firstly, beginning with ψ_0, $\text{spoly}(\psi_0,\text{spoly}(\varphi_1,\psi_1)) = x_0^{a+d}x_1x_2 - x_0^{a+d+1}y$. $LM(\varphi_1) = x_1x_2$ divides $LM(\text{spoly}(\psi_0,\text{spoly}(\varphi_1,\psi_1)))$. $\text{ecart}(\varphi_1) = \text{ecart}(\text{spoly}(\psi_0,\text{spoly}(\varphi_1,\psi_1))) = 0$. $\text{spoly}(\varphi_1,\text{spoly}(\psi_0,\text{spoly}(\varphi_1,\psi_1))) = 0$. Secondly, $\text{spoly}(\theta,\text{spoly}(\varphi_1,\psi_1)) = 0$. Thus, $NF(\text{spoly}(\varphi_1,\psi_1)\,|\,G) = 0$. We continue by $\text{spoly}(\psi_0,\psi_1) = x_0^{a+d}x_1^2 - x_0^{a+d+1}x_2$. $LM(\xi_{11}) = x_1^2$ divides $LM(\text{spoly}(\psi_0,\psi_1)) = x_0^{a+d}x_1^2$. $\text{ecart}(\xi_{11}) = \text{ecart}(\text{spoly}(\psi_0,\psi_1)) = 0$. $\text{spoly}(\xi_{11},\text{spoly}\ (\psi_0,\psi_1) = 0$ implies $NF(\text{spoly}(\psi_0,\psi_1)\,|\,G) = 0$. In the same manner, $\text{spoly}(\psi_0,\theta) = x_0^{a+d}x_1x_2 - x_0^{a+d+1}y$. $LM(\varphi_0) = x_1x_2$ divides $LM(\text{spoly}(\psi_0,\theta)) = x_0^{a+d}x_1x_2$. Also $\text{ecart}(\varphi_0) = \text{ecart}(\text{spoly}(\psi_0,\theta)) = 0$. $\text{spoly}(\varphi_0,\text{spoly}(\psi_0,\theta)) = 0$ implies $NF(\text{spoly}(\psi_0,\theta)\,|\,G) = 0$. Finally, we compute $\text{spoly}(\psi_1,\theta) = x_0^{a+d}x_2^2 - x_0^{a+d}x_1y$. $LM(\varphi_1) = x_2^2$ divides $LM(\text{spoly}(\psi_1,\theta)) = x_0^{a+d}x_2^2$. Also $\text{ecart}(\varphi_1) = \text{ecart}(\text{spoly}(\psi_1,\theta)) = 0$. $\text{spoly}(\varphi_1,\text{spoly}(\psi_1,\theta)) = 0$ implies $NF(\text{spoly}(\psi_0,\theta)\,|\,G) = 0$.

Case $b = 2$.

As in the previous case, we obtain by the minimal generating set G in Theorem 1,

$$G = \{\xi_{11} = x_1^2 - x_0x_2,\ \ \varphi_0 = x_1x_2 - x_0y,\ \ \varphi_1 = x_2^2 - x_1y,$$

$$\psi_0 = x_2y^a - x_0^{a+d+1},\ \ \theta = y^{a+1} - x_0^{a+d}x_1\}.$$

$LM(\xi_{11}) = x_1^2$, $LM(\varphi_0) = x_1x_2$, $LM(\varphi_1) = x_2^2$, $LM(\psi_0) = x_2y^a$ and $LM(\theta) = y^{a+1}$ with respect to the negative degree reverse lexicographical ordering. We begin with ξ_{11} and φ_0. This case is exactly the same as in $b = 1$. Next, we choose ξ_{11} and φ_1. As in the first case, $\text{lcm}(LM(\xi_{11}),LM(\varphi_1)) = LM(\xi_{11}).LM(\varphi_1)$, then $NF(\text{spoly}(\xi_{11},\varphi_1)\,|\,\{\xi_{11},\varphi_1\}) = 0$. Thus, this implies that $NF(\text{spoly}(\xi_{11},\varphi_1)\,|\,G) = 0$. In the same manner, $NF(\text{spoly}(\xi_{11},\psi_0)\,|\,G) = 0$, $NF(\text{spoly}(\xi_{11},\theta)\,|\,G) = 0$, $NF(\text{spoly}(\varphi_0,\theta)\,|\,G) = 0$ and $NF(\text{spoly}(\varphi_1,\theta)\,|\,G) =0$. Again, we compute S-polynomial of φ_0 and φ_1. This one is also the same as in the previous case. Now choose φ_0 and ψ_0. Then, $\text{spoly}(\varphi_0,\psi_0) = x_0^{a+d+1}x_1 - x_0y^{a+1}$. Once again, only

$LM(\theta) = y^{a+1}$ divides $LM(\text{spoly}(\varphi_0, \psi_0)) = x_0 y^{a+1}$ among the leading monomials. Also, $\text{ecart}(\theta) = \text{ecart}(\text{spoly}(\varphi_0, \psi_0)) = d$. $\text{spoly}(\theta, \text{spoly}(\varphi_0, \psi_0)) = 0$ implies $NF(\text{spoly}(\varphi_0, \psi_0) \,|\, G) = 0$. Similarly, we compute $\text{spoly}(\varphi_1, \psi_0) = x_0^{a+d+1} x_2 - x_1 y^{a+1}$. Among the leading monomials , only $LM(\theta)$ divides $LM(\text{spoly}(\varphi_1, \psi_0)) = x_1 y^{a+1}$. $\text{ecart}(\theta) = \text{ecart}(\text{spoly}(\varphi_1, \psi_0)) = d$. $\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0)) = x_0^{a+d+1} x_2 - x_0^{a+d} x_1^2$. Since $\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0))$ is not zero, again among the leading monomials, $LM(\xi_{11}) = x_1^2$ divides $LM(\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0))) = x_0^{a+d} x_1^2$. $\text{ecart}(\xi_{11}) = \text{ecart}(\text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0))) = 0$. $\text{spoly}(\xi_{11}, \text{spoly}(\theta, \text{spoly}(\varphi_1, \psi_0))) = 0$. Thus, $NF(\text{spoly}(\varphi_1, \psi_0) \,|\, G) = 0$. Finally, we compute $\text{spoly}(\psi_0, \theta) = x_0^{a+d} x_1 x_2 - x_0^{a+d+1} y$. $LM(\varphi_0) = x_1 x_2$ divides $LM(\text{spoly}(\psi_0, \theta)) = x_0^{a+d} x_1 x_2$. Also, $\text{ecart}(\varphi_0) = \text{ecart}(\text{spoly}(\psi_0, \theta)) = 0$. $\text{spoly}(\varphi_0, \text{spoly}(\psi_0, \theta)) = 0$ implies $NF(\text{spoly}(\psi_0, \theta) \,|\, G) = 0$.

Case $b = 3$.

Finally, by writing 3 instead of b in the minimal generating set G in Theorem 1, we obtain

$$G = \{\xi_{11} = x_1^2 - x_0 x_2,\ \ \varphi_0 = x_1 x_2 - x_0 y,\ \ \varphi_1 = x_2^2 - x_1 y,\ \ \theta = y^{a+1} - x_0^{a+d+1}\}.$$

In the same manner, $LM(\xi_{11}) = x_1^2$, $LM(\varphi_0) = x_1 x_2$, $LM(\varphi_1) = x_2^2$ and $LM(\theta) = y^{a+1}$ with respect to the negative degree reverse lexicographical ordering $>_{ds}$. As in the previous cases, we begin with ξ_{11} and φ_0 and this case is exactly the same as in $b = 1$. In the same manner, $NF(\text{spoly}(\{\xi_{11}, \varphi_1) \,|\, G) = 0$, $NF(\text{spoly}(\xi_{11}, \theta) \,|\, G) = 0$, $NF(\text{spoly}(\varphi_0, \theta) \,|\, G) = 0$ and $NF(\text{spoly}(\varphi_1, \theta) \,|\, G) = 0$. Finally, the computation of the S-polynomial of φ_0 and φ_1 also results as in the case $b = 1$.

Therefore, if b =1,2 and 3, we conclude that the set G is a standard basis with respect to the negative degree reverse lexicographical ordering $>_{ds}$.

□

We can now find the minimal generating set of the tangent cone by using the above lemma.

Proposition 4. *Let C be a certain monomial curve having parametrization*

$$x_0 = t^{m_0},\ x_1 = t^{m_1},\ x_2 = t^{m_2},\ y = t^{m_3}$$

$m_0 = 3a + b$ *for positive integers* $a \geq 1$ *and* $b \in [1, 3]$ *and* $0 < m_0 < m_1 < m_2 < m_3$ *form an arithmetic progression with common difference d and let the generators of the defining ideal $I(C)$ be given by the set G in Theorem 1. Then the defining ideal $I(C)^*$ of the tangent cone is generated by the set G^* consisting of the least homogeneous summands of the binomials in G.*

Proof. By the Lemma 3,

$$G := \begin{cases} \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0, \psi_1\} \cup \{\theta\} & if \quad b = 1, \\ \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\psi_0\} \cup \{\theta\} & if \quad b = 2, \\ \{\xi_{11}\} \cup \{\varphi_0, \varphi_1\} \cup \{\theta\} & if \quad b = 3. \end{cases}$$

as in Theorem 1, is a standard basis of $I(C)$ with respect to a local degree ordering $>_{ds}$ with respect to $x_0 > x_1 > x_2 > y$. Then, from [5, Lemma 5.5.11], $I(C)^*$

is generated by the least homogeneous summands of the elements in the standard basis. Thus, $I(C)^*$ is generated by
if $b=1$

$$G^* = \{\xi_{11}^* = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \psi_0^* = x_1y^a,$$

$$\psi_1^* = x_2y^a,\ \ \theta^* = y^{a+1}\},$$

if $b=2$

$$G^* = \{\xi_{11} = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \psi_0^* = x_2y^a, \theta^* = y^{a+1}\},$$

and if $b=3$

$$G^* = \{\xi_{11}^* = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \theta^* = y^{a+1}\}.$$

□

Theorem 5. *Let C be a certain monomial curve having parametrization*

$$x_0 = t^{m_0},\ x_1 = t^{m_1},\ x_2 = t^{m_2},\ y = t^{m_3}$$

$m_0 = 3a+b$ for positive integers $a \geq 1$ and $b \in [1,3]$ and $0 < m_0 < m_1 < m_2 < m_3$ form an arithmetic progression with common difference d. The certain monomial curve C with the defining ideal $I(C)$ as in Theorem 1 has Cohen-Macaulay tangent cone at the origin.

Proof. We can apply the Theorem 2.1 in [1] to the generators of the tangent cone which are given by the set
if $b=1$

$$G^* = \{\xi_{11}^* = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \psi_0^* = x_1y^a,$$

$$\psi_1^* = x_2y^a,\ \ \theta^* = y^{a+1}\},$$

if $b=2$

$$G^* = \{\xi_{11} = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \psi_0^* = x_2y^a, \theta^* = y^{a+1}\},$$

and if $b=3$

$$G^* = \{\xi_{11}^* = x_1^2 - x_0x_2,\ \ \varphi_0^* = x_1x_2 - x_0y,\ \ \varphi_1^* = x_2^2 - x_1y,\ \ \theta^* = y^{a+1}\}.$$

All of these sets are Gröbner bases with respect to the reverse lexicographic order with $x_0 > y > x_1 > x_2$. Since x_0 does not divide the leading monomial of any element in G^* in all three cases, the ring $k[x_0, x_1, x_2, y]/I(C)^*$ is Cohen-Macaulay from Theorem 2.1 in [1]. Thus, $R = \mathrm{gr}_m(k[[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}]]) \cong k[x_0, x_1, x_2, y]/I(C)^*$ is Cohen-Macaulay. □

3. Minimal free resolution of the associated graded ring

Here, we study the minimal free resolution of $\mathrm{gr}_m(k[[t^{m_0}, t^{m_1}, t^{m_2}, t^{m_3}]])$ of the certain monomial curve C in affine 4-space.

Theorem 6. *Let C be a certain affine monomial curve in $\mathbb{A}^4$ having parametrization*

$$x_0 = t^{m_0},\ x_1 = t^{m_1},\ x_2 = t^{m_2},\ y = t^{m_3}$$

$m_0 = 3a + b$ *for positive integers* $a \geq 1$ *and* $b \in [1,3]$ *and* $0 < m_0 < m_1 < m_2 < m_3$ *form an arithmetic progression with common difference d. Then the sequence of R-modules*

$$0 \longrightarrow R^{\beta_3(b)} \xrightarrow{\phi_3(b)} R^{\beta_2(b)} \xrightarrow{\phi_2(b)} R^{\beta_1(b)} \xrightarrow{\phi_1(b)} R \xrightarrow{\phi} G \longrightarrow 0$$

is a minimal free resolution for the tangent cone of C, where

$$\beta_1(b) = \begin{cases} 6 & if\ \ b=1, \\ 5 & if\ \ b=2, \\ 4 & if\ \ b=3, \end{cases} \quad , \quad \beta_2(b) = \begin{cases} 8 & if\ \ b=1, \\ 5 & if\ \ b=2, \\ 5 & if\ \ b=3, \end{cases} \quad , \quad \beta_3(b) = \begin{cases} 3 & if\ \ b=1, \\ 1 & if\ \ b=2, \\ 2 & if\ \ b=3. \end{cases}$$

and ϕ's denote the canonical surjections and the maps between R-modules depend on b

$$\phi_1(b=1) = \begin{pmatrix} g_1 = x_1^2 - x_0x_2 & g_2 = x_1x_2 - x_0y & g_3 = x_2^2 - x_1y & g_4 = x_1y^a & g_5 = x_2y^a & g_6 = y^{a+1} \end{pmatrix}$$

$$\phi_2(b=1) = \begin{pmatrix} x_2 & y^a & -y & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & x_2 & y^a & 0 & 0 & 0 & 0 \\ x_0 & 0 & -x_1 & 0 & y^a & 0 & 0 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_2 & y & 0 \\ 0 & x_0 & 0 & 0 & -x_2 & -x_1 & 0 & y \\ 0 & 0 & 0 & x_0 & x_1 & 0 & -x_1 & -x_2 \end{pmatrix},$$

$$\phi_3(b=1) = \begin{pmatrix} y^a & 0 & 0 \\ -x_2 & y & 0 \\ 0 & y^a & 0 \\ x_1 & -x_2 & 0 \\ -x_0 & x_1 & 0 \\ 0 & -x_2 & y \\ 0 & x_1 & -x_2 \\ 0 & -x_0 & x_1 \end{pmatrix},$$

$$\phi_1(b=2) = \begin{pmatrix} g_1 = x_1^2 - x_0x_2 & g_2 = x_1x_2 - x_0y & g_3 = x_2^2 - x_1y & g_4 = x_2y^a & g_5 = y^{a+1} \end{pmatrix}$$

$$\phi_2(b=2) = \begin{pmatrix} x_2 & -y & 0 & 0 & 0 \\ -x_1 & x_2 & y^a & 0 & 0 \\ x_0 & -x_1 & 0 & y^a & 0 \\ 0 & 0 & -x_1 & -x_2 & y \\ 0 & 0 & x_0 & x_1 & -x_2 \end{pmatrix},$$

$$\phi_3(b=2) = \begin{pmatrix} g_5 = y^{a+1} \\ g_4 = x_2 y^a \\ -g_3 = -x_2^2 + x_1 y \\ g_2 = x_1x_2 - x_0 y \\ g_1 = x_1^2 - x_0x_2 \end{pmatrix},$$

$$\phi_1(b=3) = \begin{pmatrix} g_1 = x_1^2 - x_0x_2 & g_2 = x_1x_2 - x_0y & g_3 = x_2^2 - x_1y & g_4 = y^{a+1} \end{pmatrix}$$

$$\phi_2(b=3) = \begin{pmatrix} x_2 & y^{a+1} & -y & 0 & 0 \\ -x_1 & 0 & x_2 & y^{a+1} & 0 \\ x_0 & 0 & -x_1 & 0 & y^{a+1} \\ 0 & -x_1^2 + x_0x_2 & 0 & -x_1x_2 + x_0y & -x_2^2 + x_1y \end{pmatrix},$$

$$\phi_3(b=3) = \begin{pmatrix} y^{a+1} & 0 \\ -x_2 & y \\ 0 & y^{a+1} \\ x_1 & -x_2 \\ -x_0 & x_1 \end{pmatrix},$$

Proof. We will prove the theorem for the three cases, $b = 1$, 2, and 3.

Case $b = 1$.
It is easy to show that $\phi_1(1)\phi_2(1) = \phi_2(1)\phi_3(1) = 0$ which proves that the above sequence is a complex. To prove the exactness, we use Corollary 2 of Buchsbaum-Eisenbud theorem in [3]. We have to show that rank $\phi_1(1) = 1$, rank $\phi_2(1) = 5$ and rank $\phi_3(1) = 3$, and also that $I(\phi_i(1))$ contains a regular sequence of length i for all $1 \leq i \leq 3$. rank $\phi_1(1) = 1$ is trivial. We want to show that rank $\phi_2(1) = 5$. Since the columns of the matrix $\phi_2(1)$ are related by the generators of the defining ideal $I(C)$, note that all 6×6 minors of $\phi_2(1)$ are zero. $\phi_2(1)$ has a non zero divisor in the kernel. By McCoy's theorem rank $\phi_2(1) \leq 5$. The determinants of 5×5 minors of $\phi_2(1)$ are $x_0g_6^2$ when the 6th row and the columns 3, 5 and 6 are deleted, and $x_1g_2^2$ when the 2nd row and the columns 2, 5 and 8 are deleted. Since $\{x_0g_6^2, x_1g_2^2\}$ are relatively prime, $I(\phi_2(1))$ contains a regular sequence of length 2. Also, among the 3×3 minors of $\phi_3(1)$, we have $\{-x_0g_1, -x_1g_2, -x_2g_3\}$. They are relatively prime, so $I(\phi_3(1))$ contains a regular sequence of length 3.

Case $b = 2$.
Clearly $\phi_1(2)\phi_2(2) = \phi_2(2)\phi_3(2) = 0$ and rank $\phi_1(2) = 1$ and rank $\phi_3(2) = 1$. We have to show that rank $\phi_2(2) = 4$ and $I(\phi_i(2))$ contains a regular sequence of length i for all $1 \leq i \leq 3$. Among the 4×4 minors of $\phi_2(2)$, $I(\phi_2(2))$ contains $\{-g_1^2, -g_2^2\}$. These two determinants constitute a regular sequence of length 2, since they are relatively prime.

Case $b = 3$.
As in the previous cases, we have to show that rank $\phi_1(3) = 1$, rank $\phi_2(3) = 3$ and rank $\phi_3(3) = 2$, and also that $I(\phi_i(3))$ contains a regular sequence of length i for all $1 \leq i \leq 3$. rank $\phi_1(3) = 1$ is trivial. We have to show that rank $\phi_2(3) = 3$. $\phi_2(3)$ has a non zero divisor in the kernel. By McCoy's theorem rank $\phi_2(3) \leq 3$. Among the 3×3 minors of $\phi_2(3)$, $I(\phi_2(3))$ contains $\{g_1^2, g_2^2\}$ which is a regular sequence of length 2, since they are relatively prime. Also, among the 2×2 minors of $\phi_3(3)$, we have $\{g_1, -g_2, g_3\}$. They are relatively prime, thus $I(\phi_3(3))$ contains a regular sequence of length 3. $\square$

Corollary 7. *Under the hypothesis of Theorem 6., the minimal graded free resolution of the associated graded ring G is given by*

if b =1

$$0 \longrightarrow R(-(a+3))^3 \xrightarrow{\phi_3(b)} R(-3)^2 \bigoplus R(-(a+2))^6 \xrightarrow{\phi_2(b)} R(-2)^3 \bigoplus R(-(a+1))^3 \xrightarrow{\phi_1(b)} R$$

if b =2

$$0 \longrightarrow R(-(a+4)) \xrightarrow{\phi_3(b)} R(-3)^2 \bigoplus R(-(a+2))^3 \xrightarrow{\phi_2(b)} R(-2)^3 \bigoplus R(-(a+1))^2 \xrightarrow{\phi_1(b)} R$$

if b =3

$$0 \longrightarrow R(-(a+4))^2 \xrightarrow{\phi_3(b)} R(-3)^2 \bigoplus R(-(a+3))^3 \xrightarrow{\phi_2(b)} R(-2)^3 \bigoplus R(-(a+1)) \xrightarrow{\phi_1(b)} R$$

$\square$

If $H_G(i) = dim_k(m^i/m^{i+1})$ is the Hilbert function of G, then

Corollary 8. *Under the hypothesis of Theorem 6., the Hilbert function of the associated graded ring G is given by*

if b = 1

$$H_G(i) = \binom{i+3}{3} - 3\binom{i+1}{3} - 3\binom{i-a+2}{3} + 2\binom{i}{3} + 6\binom{i-a+1}{3} - 3\binom{i-a}{3}$$

if b = 2

$$H_G(i)=\binom{i+3}{3}-3\binom{i+1}{3}-2\binom{i-a+2}{3}+2\binom{i}{3}+3\binom{i-a+1}{3}-\binom{i-a-1}{3}$$

if b=3

$$H_G(i)=\binom{i+3}{3}-3\binom{i+1}{3}-\binom{i-a+2}{3}+2\binom{i}{3}+3\binom{i-a}{3}-2\binom{i-a-1}{3}$$

Acknowledgments. Authors acknowledge partial financial support from Balikesir University under the project number BAP 2017/108.

References

[1] Arslan, S.F., Cohen-Macaulayness of tangent cones, *Proc. Amer. Math. Soc.* 128 (2000) 2243-2251.

[2] Barucci, V.,Fröberg, R. and Şahin, M., On free resolutions of some semigroup rings, *J. Pure and Appl. Algebra* 218 (6) (2014) 1107-1116.

[3] Buchsbaum, D. and Eisenbud, D., What makes a complex exact?, *Journal of Algebra* 25 (1973) 259-268.

[4] Gimenez, P., Sengupta, I. andSrinivasan, H., Minimal free resolutions for certain affine monomial curves, *Contemporary Mathematics* 555 (2011) 87-95.

[5] Greuel, G-M, Pfister, G., A Singular Introduction to Commutative Algebra, Springer-Verlag, 2002.

[6] Decker, W., Greuel, G-M., Pfister, G., and Schönemann, H., SINGULAR 4-1-0 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2016).

[7] Molinelli, S. and Tamone, G., On the Hilbert function of certain rings of monomial curves, *J. Pure and Appl. Algebra* 101 (2) (1995) 191-206.

[8] Molinelli, S. and Tamone, G., On the Hilbert function of certain non-Cohen-Macaulay one dimensional rings, *Rocky Mountain J. Math.* 29 (1) (1999) 271-300.

[9] Molinelli, S., Patil, D.P. and Tamone, G., On the Cohen-Macaulayness of the associated graded ring of certain monomial curves. *Beitrage Algebra Geom.* 39 (2) (1998) 433-446.

[10] Oneto, A. and Tamone, G., Explicit minimal resolution for certain monomial curves, arXiv:1312.0789 [math.AC].

[11] Patil, D.P., Minimal sets of generators for the relation ideals of certain monomial curves, *Manuscripta Math.* 80 (1993) 239-248.

[12] Rossi, M., Hilbert functions of Cohen-Macaulay local rings, Commutative Algebra and its Connections to Geometry, *Contemporary Math* 555 (2011) 173-200.

[13] Rossi, M.E. and Sharifan, L., Minimal free resolution of a finitely generated module over a regular local ring, *Journal of Algebra* 322 (10) (2009) 3693-3712.

[14] Rossi, M.E. and Valla, G., Hilbert functions of filtered modules, Lecture Notes the Unione Matematica Italiana 9, Springer, 2010.

[15] Sengupta, I., A minimal free resolution for certain monomial curves in A^4, *Comm. in Algebra* 31 (6) (2003) 2791-2809.

[16] Sengupta, I., A Gröbner basis for certain affine monomial curves, *Comm. in Algebra* 31 (3) (2003) 1113-1129.

[17] Sharifan, L. and Zaare-Nahandi, R., Minimal free resolution of the associated graded ring of monomial curves of generalized arithmetic sequences, *J. of Pure and Appl. Algebra* 213 (2009) 360-369.

Current address: Department of Mathematics, Balıkesir University, Balıkesir , 10145 Turkey
E-mail address: pinarm@balikesir.edu.tr
ORCID Address: http://orcid.org/0000-0002-3369-2838
Current address: Department of Mathematics, Balıkesir University, Balıkesir , 10145 Turkey
E-mail address: esrazengin103@gmail.com
ORCID Address: http://orcid.org/0000-0002-5195-9364