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On Cosymplectic-Like Statistical Submersions

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Abstract. We study cosymplectic-like statistical submersions. It is shown that for a cosymplectic-like statistical submersion, the base space is a Kähler-like statistical manifold and each fiber is a cosymplectic-like statistical manifold. We find the characterizations of the total and the base spaces under certain conditions. Examples of cosymplectic-like statistical manifolds and their submersions are also given.

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1. Introduction and Preliminaries

Let M and N be two Riemannian manifolds. A Riemannian submersion $F : M \to N$ is a mapping such that $rankF_* = boyN$ and F_* preserves lengths of horizontal vectors (see [3,5,7,9,14]). Recently, Abe and Hasegawa [1] studied an affine submersion with horizontal distribution when the total space is a statistical manifold.

Statistical manifolds with almost complex structure and its statistical submersions, statistical submersion of the space of the multivariate normal distribution, statistical manifolds with almost contact structures and its statistical submersions were studied in [10–12], respectively, by Takano.

Motivated by the above studies, in the present study, we consider cosymplectic-like statistical submersions. The paper is organized as follows. In Sect. 2, we give a brief introduction about statistical submersions. In Sect. 3, we study cosymplectic-like statistical submersions. We prove that for a cosymplectic-like statistical submersion, the base space is a Kähler-like statistical manifold and each fiber is a cosymplectic-like statistical manifold. We characterize the total and the base spaces under certain conditions. Examples of cosymplectic-like statistical manifolds and their submersions are also given.

Birkhäuser

Let M be a Riemannian manifold. Define a torsion-free affine connection by ∇. The triple (M, ∇, g) is called a *statistical manifold* if ∇g is symmetric [2]. For a statistical manifold (M, ∇, g), we define another affine connection ∇^* by

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y) \tag{1.1}$$

for vector fields X, Y, Z on (M, g) [13]. The affine connection ∇^* is called *conjugate* (or *dual*) of the connection ∇ w.r.t. g. The affine connection ∇^* is torsion free, $\nabla^* g$ is symmetric and satisfies $(\nabla^*)^* = \nabla$. Clearly, (M, ∇^*, g) is a statistical manifold. Every Riemannian manifold (M, ∇, g) with its Riemannian connection ∇ is a *trivial statistical manifold*. We denote R and R^* the curvature tensors on M with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Then we have

$$g(R(X,Y)Z, W) = -g(Z, R^*(X,Y)W) \tag{1.2}$$

for vector fields X, Y, Z and W on (M, g) [4].

In [10], Takano considered a semi-Riemannian manifold (M, g) with almost complex structure J which has another tensor field J^* of type $(1,1)$ satisfying

$$g(JX, Y) + g(X, J^*Y) = 0 \tag{1.3}$$

for vector fields X and Y on (M, g). Then (M, g, J) is called an *almost Hermite-like manifold* [10]. It is easy to see that $(J^*)^* = J$, $(J^*)^2 = -I$ and $g(JX, J^*Y) = g(X, Y)$. Since $J^2 = -I$, the tensor field J is not symmetric to g [10].

In [10], Takano considered statistical manifolds on almost Hermite-like manifolds. If J is parallel with respect to ∇, then (M, ∇, g, J) is called a *Kähler-like statistical manifold*.

By virtue of (1.3), we get

$$g((\nabla_X J)Y, Z) + g(Y, (\nabla^*_X J^*)Z) = 0$$

(see [10]).

On a Kähler-like statistical manifold (M, ∇, g, J), Takano [10] considered the curvature tensor R w.r.t. ∇ such that

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y - g(Y,JZ)JX + g(X,JZ)JY \\ &+ [g(X,JY) - g(Y,JX)]JZ\}. \end{aligned} \tag{1.4}$$

2. Statistical Submersions

Let (M^m, g) and $(N^n, \widehat{g})$ be Riemannian manifolds and $F : M \to N$ a Riemannian submersion. For $x \in N$, Riemannian submanifold $F^{-1}(x)$ with the induced metric $\overline{g}$ is called a *fiber* and denoted by $\overline{M}$. The dimension of each fiber is always $(m - n) = s$. In the tangent bundle TM of M, the vertical and horizontal distributions are denoted by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. We call a vector field X on M *projectable* if there exists a vector field X_* on N such that $F_*(X_p) = X_{*F(p)}$ for each $p \in M$, in this case X and X_* are

F-related. Also, a vector field X on $\mathcal{H}(M)$ is called *basic* if it is projectable (see [7,8]).

The fundamental tensors of a submersion were introduced in [7]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A are defined for vector fields E, F on M by

$$T_E F = h\nabla_{\upsilon E}\upsilon F + \upsilon\nabla_{\upsilon E} hF \tag{2.1}$$

and

$$A_E F = h\nabla_{hE}\upsilon F + \upsilon\nabla_{hE} hF.$$

Let (M, ∇, g) be a statistical manifold and $F : M \to N$ a Riemannian submersion. Let $\overline{\nabla}$ and $\overline{\nabla}^*$ denote the affine connections on $\overline{M}$. It is clear that $\overline{\nabla}_U V = \upsilon\nabla_U V$ and $\overline{\nabla}^*_U V = \upsilon\nabla^*_U V$. It can be easily seen that $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion free and conjugate to each other w.r.t. $\overline{g}$.

Let $\widehat{\nabla}$ be an affine connection on N. We call $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ a *statistical submersion* if $F : M \to N$ satisfies $F_*(\nabla_X Y)_p = (\widehat{\nabla}_{X_*} Y_*)_{F(p)}$ for basic vector fields X, Y and $p \in M$ [10]. Changing ∇ for ∇^* in the above equations, we define T^* and A^*, respectively [10]. A and A^* are equal to zero if and only if $\mathcal{H}(M)$ is integrable with respect to ∇ and ∇^*, respectively. For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$, we obtain

$$g(T_U V, X) = -g(V, T^*_U X), \; g(A_X Y, U) = -g(Y, A^*_X U). \tag{2.2}$$

Takano gave the following two lemmas in [10].

Lemma 2.1. *For $X, Y \in \mathcal{H}(M)$, we have $A_X Y = -A^*_Y X$.*

Lemma 2.2. *For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$, we have*

$$\begin{aligned}
\nabla_U V &= T_U V + \overline{\nabla}_U V, & \nabla^*_U V &= T^*_U V + \overline{\nabla}^*_U V,\\
\nabla_U X &= h\nabla_U X + T_U X, & \nabla^*_U X &= h\nabla^*_U X + T^*_U X,\\
\nabla_X U &= A_X U + \upsilon\nabla_X U, & \nabla^*_X U &= A^*_X U + \upsilon\nabla^*_X U,\\
\nabla_X Y &= h\nabla_X Y + A_X Y, & \nabla^*_X Y &= h\nabla^*_X Y + A^*_X Y.
\end{aligned}$$

Furthermore, if X is basic, then $h\nabla_U X = A_X U$ and $h\nabla^*_U X = A^*_X U$.

Let $\overline{R}$ be the curvature tensor w.r.t. the induced affine connection $\overline{\nabla}$ of each fiber. Moreover, let $\widehat{R}(X, Y)Z$ be horizontal vector field such that $F_*(\widehat{R}(X, Y)Z) = \widehat{R}(F_*X, F_*Y)F_*Z$ at each $p \in M$, where $\widehat{R}$ is the curvature tensor on N of the affine connection $\widehat{\nabla}$.

Theorem 2.1 [10]. *If $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ is a statistical submersion, then for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$*

(i) $g\left(R(U,V)W,W^{'}\right)=g\left(\overline{R}(U,V)W,W^{'}\right)+g\left(T_UW,T_V^*W^{'}\right)-g\left(T_VW,T_U^*W^{'}\right)$,
(ii) $g(R(X,U)V,Y)=g\left((\nabla_XT)_UV,Y\right)-g\left((\nabla_UA)_XV,Y\right)$
$+g(A_XU,A_Y^*V)-g(T_UX,T_V^*Y)$,
(iii) $g(R(X,U)Y,V)=g\left((\nabla_XT)_UY,V\right)-g\left((\nabla_UA)_XY,V\right)$
$+g(T_UX,T_VY)-g(A_XU,A_YV)$,
(iv) $g\left(R(X,Y)Z,Z^{'}\right)=g\left(\widehat{R}(X,Y)Z,Z^{'}\right)-g\left(A_YZ,A_X^*Z^{'}\right)$
$+g\left(A_XZ,A_Y^*Z^{'}\right)+g\left((A_X+A_X^*)Y,A_Z^*Z^{'}\right)$.

We define by $\{E_1,\ldots,E_m\}$, $\{X_1,\ldots,X_n\}$ and $\{U_1,\ldots,U_\alpha\}$ the orthonormal basis of $\chi(M)$, $\mathcal{H}(M)$ and $\mathcal{V}(M)$, respectively, such that $E_i=X_i$, $(1\leq i\leq n)$ and $E_{n+\alpha}=U_\alpha$ $(1\leq\alpha\leq s)$. Denote, respectively, by ω_b^a and ω_a^{*b} the connection forms in terms of local coordinates w.r.t. $\{E_1,\ldots,E_m\}$ of the affine connection ∇ and its conjugate ∇^*, where $1\leq a,b\leq m$. Using (1.1), we get

$$\omega_a^{*b}=-\omega_b^a, \tag{2.3}$$

(see [10]). From [12], we have

$$g(TX,TY)=\sum_{\alpha=1}^{s}g\left(T_{U_\alpha}X,T_{U_\alpha}Y\right)$$

for $X,Y\in\mathcal{H}(M)$. The mean curvature vector fields of the fiber w.r.t. the affine connection ∇ and its conjugate connection ∇^* are given by the horizontal vector fields, respectively,

$$H=\sum_{\alpha=1}^{s}T_{U_\alpha}U_\alpha$$

and

$$H^*=\sum_{\alpha=1}^{s}T_{U_\alpha}^*U_\alpha.$$

3. Cosymplectic-Like Statistical Submersions with Certain Conditions

Let (M,g) be an odd dimensional semi-Riemannian manifold with the almost contact structure (φ,ξ,η) which has an another tensor field φ^* of type $(1,1)$ satisfying

$$g(\varphi E,F)+g(E,\varphi^*F)=0,$$

for vector fields E and F on M. Then (M,g,φ,ξ,η) is called an *almost contact metric manifold of certain kind* [12]. It is easy to see that $\varphi^{*2}E=-E+\eta(E)\xi$ and

$$g(\varphi E,\varphi^*F)=g(E,F)-\eta(E)\eta(F).$$

So φ is not symmetric. Equations $\varphi\xi = 0$ and $\eta(\varphi E) = 0$ hold on the almost contact manifold. We obtain $\varphi^*\xi = 0$ and $\eta(\varphi^* E) = 0$ on the almost contact metric manifold of certain kind [12].

Moreover, for $E \in \chi(M)$, if

$$\nabla_E \xi = 0, \ \nabla_E \varphi = 0, \tag{3.1}$$

then $(M, \nabla, g, \varphi, \xi, \eta)$ is called a *cosymplectic-like statistical manifold* [6].

Lemma 3.1 [6]. *$(M, \nabla, g, \varphi, \xi, \eta)$ is a cosymplectic-like statistical manifold if and only if $(M, \nabla^*, g, \varphi^*, \xi, \eta)$ is a cosymplectic-like statistical manifold.*

On a cosymplectic-like statistical manifold, we consider the curvature tensor R w.r.t. ∇ such that

$$\begin{aligned} R(E,F)G = \frac{c}{4}\{&g(F,G)E - g(E,G)F + g(E,\varphi G)\varphi F - g(F,\varphi G)\varphi E \\ &+ [g(E,\varphi F) - g(\varphi E, F)]\varphi G + \eta(E)\eta(G)F - \eta(F)\eta(G)E \\ &+ g(E,G)\eta(F)\xi - g(F,G)\eta(E)\xi\}, \end{aligned} \tag{3.2}$$

where c is a constant. Changing φ for φ^* in (3.2), we get the expression of the curvature tensor R^*. Now we give the following examples.

Example 3.1. The Euclidean space $\mathbb{R}^4$ with local coordinate system $\{x_1, x_2, y_1, y_2\}$, which admits the following almost complex structure J:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

the metric $g_{\mathbb{R}^4} = 2\mathrm{d}x_1^2 + 2\mathrm{d}x_2^2 - \mathrm{d}y_1^2 - \mathrm{d}y_2^2$ and the flat affine connection $\nabla^{\mathbb{R}^4}$ is a Kähler-like statistical manifold (see [12]). If $(\mathbb{R}, \nabla^{\mathbb{R}}, \mathrm{d}t^2)$ is a trivial statistical manifold, it is known from [6] that the product manifold $(\mathbb{R} \times \mathbb{R}^4, \widetilde{\nabla}, \widetilde{g} = \mathrm{d}t^2 + g_{\mathbb{R}^4})$ is a cosymplectic-like statistical manifold. The curvature tensor of $(\mathbb{R} \times \mathbb{R}^4, \widetilde{\nabla}, \widetilde{g} = \mathrm{d}t^2 + g_{\mathbb{R}^4}, \varphi, \xi, \eta)$ satisfies Eq. (3.2) with $c = 0$.

We define φ, ξ and η by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \ \xi = \mathrm{d}t = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $\eta = (1, 0, -y_1, 0, -y_2)$. We also find

$$\varphi^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

This manifold is not cosymplectic with respect to the Levi-Civita connection.

Example 3.2. The Euclidean space $\mathbb{R}^2$ with local coordinate system $\{x, y\}$, which admits the following almost complex structure J:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the metric $g_{\mathbb{R}^2} = \frac{2}{y^2}\mathrm{d}x^2 + \frac{1}{y^2}\mathrm{d}y^2$ and $\nabla^{\mathbb{R}^2}$ defined by

$$\nabla_{\partial_x}\partial_x = -\nabla_{\partial_y}\partial y = \frac{4}{3y}\partial y,$$

and

$$\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial x = -\frac{4}{3y}\partial x$$

is a Kähler-like statistical manifold (see [10]) and $\left(\mathbb{R}, \nabla^{\mathbb{R}}, \mathrm{d}t^2\right)$ is a trivial statistical manifold. So similar to the previous example, $(\mathbb{R} \times \mathbb{R}^2, \widetilde{\nabla}, \widetilde{g} = \mathrm{d}t^2 + g_{\mathbb{R}^2})$ is a cosymplectic-like statistical manifold. We define φ and ξ by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \xi = \mathrm{d}t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We also find

$$\varphi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -2 & 0 \end{pmatrix}.$$

Furthermore, it is easy to see that $(\mathbb{R} \times \mathbb{R}^2, \widetilde{\nabla}, \widetilde{g} = \mathrm{d}t^2 + g_{\mathbb{R}^2}, \varphi, \xi, \eta)$ satisfies Eq. (3.2) with $c = -\frac{8}{9}$.

Let $(M, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. If $F : M \to N$ is a Riemannian submersion, each fiber is a φ-invariant Riemannian submanifold of M and tangent to the vector field ξ, then F is said to be an *almost contact metric submersion.* If X is basic on M, which is F-related to X_* on N, then φX (resp. $\varphi^* X$) is basic and F-related to φX_* (resp. $\varphi^* X_*$) [10].

Similar to the Takano's definition for Sasaki-like statistical submersion (see [12]), we define cosymplectic-like statistical submersion as follows: a statistical submersion $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ is called a *cosymplectic-like statistical submersion* if $(M, \nabla, g, \varphi, \xi, \eta)$ is a cosymplectic-like statistical manifold, each fiber is a φ-invariant Riemannian submanifold of M and tangent to ξ.

So we have the following lemmas.

Lemma 3.2. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion. Then*

$$A_X\xi = 0,$$
$$T_U\xi = 0,$$
$$\upsilon\nabla_X\xi = 0$$

and

$$\overline{\nabla}_U\xi = 0,$$

where $X \in \mathcal{H}(M)$ *and* $U \in \mathcal{V}(M)$.

Proof. Since each fiber is a φ-invariant Riemannian submanifold of M such that tangent to ξ and M is a cosymplectic-like statistical manifold, from Lemma 2.2, we obtain the above equations. □

Lemma 3.3. *If* $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ *is a cosymplectic-like statistical submersion, then*

$$\begin{aligned} (h\nabla_X \varphi) Y &= 0, \\ A_X \varphi Y &= \overline{\varphi} A_X Y, \\ A_{\varphi X} U &= \varphi A_X U, \ \textit{if } X \textit{ is basic}, \\ T_U \varphi X &= \overline{\varphi} T_U X, \\ A_X \overline{\varphi} U &= \varphi A_X U, \\ (\upsilon \nabla_X \overline{\varphi}) U &= 0 \end{aligned}$$

and

$$\left(\overline{\nabla}_U \overline{\varphi}\right) V = 0,$$

where $X, Y \in \mathcal{H}(M)$ *and* $U, V \in \mathcal{V}(M)$.

Proof. Since horizontal and vertical distributions are φ-invariant, for $X, Y \in \mathcal{H}(M)$, from (3.1) and Lemma 2.2, we have

$$A_X \varphi Y + h\nabla_X \varphi Y - \overline{\varphi} A_X Y - \varphi h \nabla_X Y = 0.$$

So we obtain the first two equations. Similarly, for $U \in \mathcal{V}(M)$ and $X \in \mathcal{H}(M)$, we have

$$T_U \varphi X + h\nabla_U \varphi X - \varphi T_U X - \varphi h \nabla_U X = 0. \tag{3.3}$$

If we take X as basic, from Lemma 2.2, we find the third and the fourth equations. Similarly, we obtain the fifth and the sixth equations.

Finally, for $U, V \in \mathcal{V}(M)$, from (3.1) and Lemma 2.2, we have

$$T_U \overline{\varphi} V + \upsilon \nabla_U \overline{\varphi} V - \varphi T_U V - \overline{\varphi} \upsilon \nabla_U V = 0.$$

This gives us $\left(\overline{\nabla}_U \overline{\varphi}\right) V = 0$. □

Using Lemmas 3.2 and 3.3, we can state the following theorem.

Theorem 3.1. *Let* $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ *be a cosymplectic-like statistical submersion. Then* $(N, \widehat{\nabla}, \widehat{g})$ *is a Kähler-like statistical manifold and* $\left(\overline{M}, \overline{\nabla}, \overline{g}, \overline{\varphi}, \xi, \eta\right)$ *a cosymplectic-like statistical manifold.*

Proof. From Lemmas 3.2 and 3.3, it is clear that each fiber is a cosymplectic-like statistical manifold.

Now we shall show that $(N, \widehat{\nabla}, \widehat{g})$ is a Kähler-like statistical manifold. Let X, Y, Z be basic vector fields and F related to $X_*, Y_*, Z_* \in N$. Since

$$\widehat{g}\left(\left(\widehat{\nabla}_{X_*} J\right) Y_*, Z_*\right) = \widehat{g}\left(\widehat{\nabla}_{X_*} J Y_* - J \widehat{\nabla}_{X_*} Y_*, Z_*\right),$$

and F is a cosymplectic-like statistical submersion, we find

$$\widehat{g}\left(\widehat{\nabla}_{X_*} J Y_* - J \widehat{\nabla}_{X_*} Y_*, Z_*\right) = \widehat{g}\left(\widehat{\nabla}_{X_*} F_*(\varphi Y) - J \widehat{\nabla}_{X_*} F_* Y, F_* Z\right)$$

$$= \widehat{g}\left(F_*\left(\nabla_X \varphi Y\right) - F_*\left(\varphi \nabla_X Y\right), F_* Z\right)$$
$$= g\left(\nabla_X \varphi Y - \varphi \nabla_X Y, Z\right) = g\left(\left(\nabla_X \varphi\right) Y, Z\right).$$

Since (M, ∇, g) is a cosymplectic-like statistical manifold, $(\nabla_X \varphi)Y = 0$. Hence, from the above equation, we obtain $(\widehat{\nabla}_{X_*} J)Y_* = 0$, which shows that the base space is a Kähler-like statistical manifold. $\square$

Lemma 3.4. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion. If* $\dim \overline{M} = 1$*, then $\mathcal{H}(M)$ is integrable.*

Proof. Assume that $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ is a cosymplectic-like statistical submersion. Then

$$\left(\nabla_X \varphi\right) Y = \nabla_X \varphi Y - \varphi \nabla_X Y = 0.$$

Changing Y with φY in the above equation, we write

$$\nabla_X \varphi^2 Y - \varphi \nabla_X \varphi Y = 0.$$

Since $\varphi^2 Y = -Y + \eta(Y)\xi$, we get

$$-\nabla_X Y + g\left(\nabla_X Y, \xi\right)\xi + g\left(Y, \nabla^*_X \xi\right)\xi + \eta(Y)\nabla_X \xi - \varphi \nabla_X \varphi Y = 0.$$

Using Lemma 2.2, we obtain

$$-A_X Y - h\nabla_X Y + g\left(A_X Y, \xi\right)\xi - \varphi A_X \varphi Y - \varphi h \nabla_X \varphi Y = 0. \tag{3.4}$$

Hence, the vertical part of (3.4) satisfies

$$-A_X Y + g\left(A_X Y, \xi\right)\xi - \overline{\varphi} A_X \varphi Y = 0.$$

Since $g\left(A_X Y, \xi\right) = 0$, we have $A_X Y = -\overline{\varphi} A_X \varphi Y$. Because of $\dim \overline{M} = 1$, we find $A_X \varphi Y = 0$. So we get $A = 0$ on $\mathcal{H}(M)$. Thus, $\mathcal{H}(M)$ is integrable. $\square$

From (1.3), if we take $E = F = A_X Y$, then we have

$$\left(\overline{\varphi} + \overline{\varphi}^*\right) A_X Y = 0. \tag{3.5}$$

Then we can state the following theorem.

Theorem 3.2. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion. If rank* $(\overline{\varphi} + \overline{\varphi}^*) = \dim \overline{M} - 1$*, then $\mathcal{H}(M)$ is integrable.*

Proof. From Lemma 3.3, we find $A_X Y = -\overline{\varphi} A_X \varphi Y$. Let $\{U_1, U_2, \ldots, U_{s-1}, \xi\}$ be orthonormal frame field. Since *rank* $(\overline{\varphi} + \overline{\varphi}^*) = \dim \overline{M} - 1$, the vector fields $(\overline{\varphi} + \overline{\varphi}^*)U_1$, $(\overline{\varphi} + \overline{\varphi}^*)U_2, \ldots,$ $(\overline{\varphi} + \overline{\varphi}^*)U_{s-1}$ are linearly independent. So we obtain

$$A_X \varphi Y = g\left(A_X \varphi Y, \xi\right)\xi$$

and

$$\varphi A_X \varphi Y = 0.$$

Hence, we have $A = 0$ on $\mathcal{H}(M)$. Then $\mathcal{H}(M)$ is integrable. $\square$

So in view of Lemma 3.3 and Eq. (3.5), we have the following corollary.

Corollary 3.1. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion. If $\overline{\varphi} = \overline{\varphi}^*$, then $\mathcal{H}(M)$ is integrable.*

Let $F:(M,\nabla,g)\to(N,\widehat{\nabla},\widehat{g})$ be a cosymplectic-like statistical submersion. So from Theorem 2.1 and Eq. (3.2), we have

$$\begin{aligned}g\left(R(U,V)W,W^{'}\right)&=\frac{c}{4}\Big\{g(V,W)\,g\left(U,W^{'}\right)\\&-g(U,W)\,g\left(V,W^{'}\right)+\eta(U)\,\eta(W)\,g\left(V,W^{'}\right)-\eta(V)\,\eta(W)\,g\left(U,W^{'}\right)\\&+\eta(V)\,\eta\left(W^{'}\right)g(U,W)-\eta(U)\,\eta\left(W^{'}\right)g(V,W)-g(V,\overline{\varphi}W)\,g\left(W^{'},\overline{\varphi}U\right)\\&+g(U,\overline{\varphi}W)\,g\left(\overline{\varphi}V,W^{'}\right)+[g(U,\overline{\varphi}V)-g(\overline{\varphi}U,V)]\,g\left(\overline{\varphi}W,W^{'}\right)\Big\},\end{aligned}\tag{3.6}$$

$$g(R(X,U)V,Y)=\frac{c}{4}\{[g(U,V)-\eta(U)\,\eta(V)]\,g(X,Y)-g(U,\overline{\varphi}V)\,g(\varphi X,Y)\},\tag{3.7}$$

$$g(R(X,U)Y,V)=-\frac{c}{4}\{[g(U,V)-\eta(U)\,\eta(V)]\,g(X,Y)-g(\overline{\varphi}U,V)\,g(X,\varphi Y)\},\tag{3.8}$$

$$\begin{aligned}g\left(R(X,Y)Z,Z^{'}\right)&=\frac{c}{4}\Big\{g(Y,Z)\,g\left(X,Z^{'}\right)-g(X,Z)\,g\left(Y,Z^{'}\right)-g(Y,\varphi Z)\,g\left(\varphi X,Z^{'}\right)\\&+g\left(\varphi Y,Z^{'}\right)g(X,\varphi Z)+[g(X,\varphi Y)-g(\varphi X,Y)]\,g\left(\varphi Z,Z^{'}\right)\Big\}.\end{aligned}\tag{3.9}$$

Hence, we can state the following theorem.

Theorem 3.3. *Let $F:(M,\nabla,g)\to(N,\widehat{\nabla},\widehat{g})$ be a cosymplectic-like statistical submersion. If $\mathcal{H}(M)$ is integrable and the curvature tensor of M is of the form (3.2), then the curvature tensor of N is of the form (1.4).*

Proof. Assume that $\mathcal{H}(M)$ is integrable. Then $A=0$. Since the curvature tensor of the total space satisfies Eq. (3.2), we have (3.9). So if we take the vector fields X,Y,Z as basic and F-related to X_*,Y_*,Z_*, then from (3.9), we obtain

$$\begin{aligned}F_*\left(\widehat{R}(X,Y)Z\right)=\widehat{R}(F_*X,F_*Y)F_*Z&=\frac{c}{4}\{\widehat{g}(Y_*,Z_*)X_*-\widehat{g}(X_*,Z_*)Y_*\\&-\widehat{g}(Y_*,JZ_*)JX_*+\widehat{g}(X_*,JZ_*)JX_*+[g(X_*,JY_*)-g(JX_*,Y_*)]JZ_*\}.\end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let $F:(M,\nabla,g)\to(N,\widehat{\nabla},\widehat{g})$ be a cosymplectic-like statistical submersion. If $\dim\overline{M}=1$ or rank $(\overline{\varphi}+\overline{\varphi}^*)=\dim\overline{M}-1$ and the curvature tensor of M is of the form (3.2), then the curvature tensor of N is of the form (1.4).*

If $\mathcal{H}(M)$ is integrable, from Eq. (3.7), we find

$$\begin{aligned}g((\nabla_XT)_U V,Y)-g(T_UX,T_V^*Y)=\frac{c}{4}&\{[g(U,V)-\eta(U)\,\eta(V)]\,g(X,Y)\\&-g(U,\overline{\varphi}V)\,g(\varphi X,Y)\}.\end{aligned}$$

By a contraction from the last equation over U and V, we get

$$\begin{aligned}&\sum_{\alpha=1}^{s}g\left((\nabla_XT)_{U_\alpha}U_\alpha,Y\right)-\sum_{\alpha=1}^{s}g\left(T_{U_\alpha}X,T_{U_\alpha}^*Y\right)\\&\quad=\frac{c}{4}\{(s-1)\,g(X,Y)-(tr\overline{\varphi})\,g(\varphi X,Y)\}.\end{aligned}\tag{3.10}$$

Since T is symmetric on $\overline{M}$, from (2.2), we obtain

$$\sum_{\alpha=1}^{s} g\left((\nabla_X T)_{U_\alpha} U_\alpha, Y\right) = g(\nabla_X H, Y) + \sum_{\alpha=1}^{s} \left\{ g\left(T^*_{U_\alpha} Y, \nabla_X U_\alpha\right) + g\left(T^*_{U_\alpha} Y, \nabla_X U_\alpha\right) \right\}. \tag{3.11}$$

Using Eq. (2.3), we find

$$\sum_{\alpha=1}^{s} g\left(T^*_{U_\alpha} Y, \nabla^*_X U_\alpha\right) = -\sum_{\alpha=1}^{s} g\left(T^*_{U_\alpha} Y, \nabla_X U_\alpha\right).$$

By the use of the last equation in (3.11), from (2.1), we get

$$\sum_{\alpha=1}^{s} g\left((\nabla_X T)_{U_\alpha} U_\alpha, Y\right) = g(\nabla_X H, Y) + \sum_{\alpha=1}^{s} g\left(T^*_{U_\alpha} Y, T_{U_\alpha} X - T^*_{U_\alpha} X\right). \tag{3.12}$$

In view of (3.10) and (3.12), we have

$$g(\nabla_X H, Y) - g(T^* Y, T^* X) = \frac{c}{4}\left\{(s-1)\, g(X,Y) - (tr\overline{\varphi})\, g(\varphi X, Y)\right\}. \tag{3.13}$$

If $h\nabla_X H = 0$, then we find

$$-g(T^* Y, T^* X) = \frac{c}{4}\left\{(s-1)\, g(X,Y) - (tr\overline{\varphi})\, g(\varphi X, Y)\right\}. \tag{3.14}$$

Thus, using (3.14), we obtain the following theorem and corollary.

Theorem 3.4. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion such that the curvature tensor of M is of the form (3.2). Suppose that $\mathcal{H}(M)$ is integrable and $h\nabla_X H = 0$ for $X \in \mathcal{H}(M)$.*

(i) *If $c = 0$, then M and N are flat, each fiber is a totally geodesic submanifold of M.*
(ii) *In the cases of $tr\overline{\varphi} = 0$ and $c < 0$, we find $\dim \overline{M} > 1$.*

Corollary 3.3. *Let $F : (M, \nabla, g) \to (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion such that the curvature tensor of M is of the form (3.2). If $\mathcal{H}(M)$ is integrable and H is a constant vector field, then we have similar results to Theorem 3.4.*

Similarly, from Eqs. (3.8) and (1.2), we have

$$g\left((\nabla^*_X T^*)_U V, Y\right) - g\left((\nabla^*_U A^*)_X V, Y\right) + g\left(A^*_X U, A_Y V\right) - g\left(T^*_U X, T_V Y\right) = \frac{c}{4}\left\{\left[g(U,V) - \eta(U)\eta(V)\right] g(X,Y) - g(\overline{\varphi} U, V)\, g(X, \varphi Y)\right\}.$$

If $\mathcal{H}(M)$ is integrable, then the last equation can be written as

$$g\left((\nabla^*_X T^*)_U V, Y\right) - g\left(T^*_U X, T_V Y\right) = \frac{c}{4}\left\{\left[g(U,V) - \eta(U)\eta(V)\right] g(X,Y) - g(\overline{\varphi} U, V)\, g(X, \varphi Y)\right\}.$$

By a contraction from the last equation over U and V, we get

$$\sum_{\alpha=1}^{s}\left\{g\left(\left(\nabla_X^* T^*\right)_{U_\alpha} U_\alpha, Y\right)-g\left(T_{U_\alpha}^* X, T_{U_\alpha} Y\right)\right\}$$
$$=\frac{c}{4}\left\{(s-1)\, g(X, Y)-(tr\overline{\varphi})\, g(X, \varphi Y)\right\}. \tag{3.15}$$

Since T is symmetric on $\overline{M}$, from (2.2), we obtain

$$\sum_{\alpha=1}^{s} g\left(\left(\nabla_X^* T^*\right)_{U_\alpha} U_\alpha, Y\right)=g\left(\nabla_X^* H^*, Y\right)+\sum_{\alpha=1}^{s}\left\{g\left(T_{U_\alpha} Y, \nabla_X^* U_\alpha\right)\right.$$
$$\left.+g\left(T_{U_\alpha} Y, \nabla_X^* U_\alpha\right)\right\}. \tag{3.16}$$

Similarly, from Eq. (2.3), we find

$$\sum_{\alpha=1}^{s} g\left(T_{U_\alpha} Y, \nabla_X^* U_\alpha\right)=-\sum_{\alpha=1}^{s} g\left(T_{U_\alpha} Y, \nabla_X U_\alpha\right). \tag{3.17}$$

By the use of (2.1), (3.16) and (3.17), Eq. (3.15) gives

$$g\left(\nabla_X^* H^*, Y\right)-g(T Y, T X)=\frac{c}{4}\left\{(s-1)\, g(X, Y)-(tr\overline{\varphi})\, g(X, \varphi Y)\right\}.$$

So using the above equation, we give the following theorem and corollary.

Theorem 3.5. *Let $F:(M, \nabla, g) \rightarrow (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion such that the curvature tensor of M is of the form (3.2). Suppose that $\mathcal{H}(M)$ is integrable and $h\nabla_X^* H^*=0$ for $X \in \mathcal{H}(M)$.*

- (i) *If $c=0$, then M and N are flat, each fiber is a totally geodesic submanifold of M.*
- (ii) *In the cases of $tr\overline{\varphi}=0$ and $c<0$, we find $\dim \overline{M}>1$.*

Corollary 3.4. *Let $F:(M, \nabla, g) \rightarrow (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion such that the curvature tensor of M is of the form (3.2). If $\dim \overline{M}=1$ or rank $(\overline{\varphi}+\overline{\varphi}^*)=\dim \overline{M}-1$ and H^* is a constant vector field, then we have similar results to Theorem 3.5.*

Takano [10] considered F as a statistical submersion with conformal fibers. For $U, V \in \mathcal{V}(M)$ if $T_U V=0$ (resp. $T_U V=\frac{1}{s} g(U, V) H$) holds, then F is called a *statistical submersion with isometric fibers* (resp. *conformal fibers*). Hence, we get the following Proposition.

Proposition 3.1. *If $F:(M, \nabla, g) \rightarrow (N, \widehat{\nabla}, \widehat{g})$ is a cosymplectic-like statistical submersion with conformal fibers then F has isometric fibers.*

Proof. Let F be a cosymplectic-like statistical submersion with conformal fibers. So we have

$$T_U V=\frac{1}{s} g(U, V) H.$$

If we take $V=\xi$, from Lemma 3.2, $\frac{1}{s} g(U, \xi) H=0$. Since $U, \xi \in \mathcal{V}(M)$, we find $H=0$. Thus, the proof of the proposition is completed. □

Theorem 3.6. *Let $F:(M, \nabla, g) \rightarrow (N, \widehat{\nabla}, \widehat{g})$ be a cosymplectic-like statistical submersion with isometric fibers such that the curvature tensor of M is of the form (3.2). Then each fiber is a totally geodesic submanifold of M such that the curvature tensor is of the form (3.2).*

Proof. Assume that F has isometric fibers. Then $T=0$. Since the curvature tensor of the total space is of the form (3.2), we get Eq. (3.6). So we obtain the result. □

Theorem 3.7. *Let $F:(M,\nabla,g)\to(N,\widehat{\nabla},\widehat{g})$ be a cosymplectic-like statistical submersion with isometric fibers such that the curvature tensor of (M,∇,g) is of the form (3.2). If $\mathcal{H}(M)$ is integrable, then M and N are flat.*

Proof. From Theorem F in [12], we have

$$g\left(\left(\nabla_X T\right)_U Y,V\right)-g\left(\left(\nabla_U A\right)_X Y,V\right)+g\left(T_U X,T_V Y\right)-g\left(A_X U,A_Y V\right)$$
$$=-\frac{c}{4}\left\{g(X,Y)\,g(U,V)-g(X,Y)\,\eta(U)\,\eta(V)-g(X,\varphi Y)\,g(\overline{\varphi}U,V)\right\}$$

and

$$g\left(\left(\nabla_Y T\right)_U X,V\right)-g\left(\left(\nabla_U A\right)_Y X,V\right)+g\left(T_U Y,T_V X\right)-g\left(A_Y U,A_X V\right)$$
$$=-\frac{c}{4}\left\{g(Y,X)\,g(U,V)-g(Y,X)\,\eta(U)\,\eta(V)-g(Y,\varphi X)\,g(\overline{\varphi}U,V)\right\}.$$

Assume that $\mathcal{H}(M)$ is integrable and F has isometric fibers. Then the above equations are reduced to

$$0=\frac{c}{4}\left\{g(X,Y)\,g(U,V)-g(X,Y)\,\eta(U)\,\eta(V)-g(X,\varphi Y)\,g(\overline{\varphi}U,V)\right\} \quad (3.18)$$

and

$$0=\frac{c}{4}\left\{g(X,Y)\,g(U,V)-g(X,Y)\,\eta(U)\,\eta(V)-g(\varphi X,Y)\,g(\overline{\varphi}U,V)\right\}. \quad (3.19)$$

Subtracting Eq. (3.18) from (3.19), we find

$$0=\frac{c}{4}g(\overline{\varphi}U,V)\left\{g(\varphi X,Y)-g(X,\varphi Y)\right\}.$$

Hence, contracting the last equation with respect to U and V, we get

$$0=\frac{c}{4}\left(tr\overline{\varphi}\right)\left\{g(\varphi X,Y)-g(X,\varphi Y)\right\}.$$

Since $g(\varphi X,Y)\neq g(X,\varphi Y)$, we obtain $c=0$ or $tr\overline{\varphi}=0$.

Furthermore, from Eq. (3.14), we have

$$0=\frac{c}{4}\left\{(s-1)\,g(X,Y)-\left(tr\overline{\varphi}\right)g(\varphi X,Y)\right\}.$$

Now assume that $tr\overline{\varphi}=0$. So from the above equation

$$0=\frac{c}{4}(s-1)\,g(X,Y).$$

Since $s>1$, we find $c=0$ again. Hence, (M,∇,g) and $\left(N,\widehat{\nabla},\widehat{g}\right)$ are flat. □

Example 3.3. Let $(\mathbb{R}\times\mathbb{R}^4,\widehat{\nabla},\widehat{g}=dt^2+g_{\mathbb{R}^4})$ be the cosymplectic-like statistical manifold given in Example 3.1. Now we define the cosymplectic-like statistical submersion $F:(\mathbb{R}\times\mathbb{R}^4,\widehat{\nabla},\widehat{g})\to(\mathbb{R}^4,\nabla^{\mathbb{R}^4},g_{\mathbb{R}^4})$ as the projection mapping

$$F(t,x_1,x_2,y_1,y_2)=(x_1,x_2,y_1,y_2).$$

Then we find $\mathcal{V}(M) = span\left\{\frac{\partial}{\partial t}\right\}$ and $\mathcal{H}(M) = span\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\}$. It is trivial that $\dim \overline{M} = 1$. Since $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \in \mathcal{H}(M)$, we obtain $A = 0$.

Example 3.4. Let $(\mathbb{R} \times \mathbb{R}^2, \widehat{\nabla}, \widehat{g} = \mathrm{d}t^2 + g_{\mathbb{R}2})$ be the cosymplectic-like statistical manifold given in Example 3.2. Now we define the cosymplectic-like statistical submersion $F : (\mathbb{R} \times \mathbb{R}^2, \widehat{\nabla}, \widehat{g}) \to (\mathbb{R}^2, \nabla^{\mathbb{R}^2}, g_{\mathbb{R}^2})$ as the projection mapping

$$F(t, x, y) = (x, y).$$

Then we find $\mathcal{V}(M) = span\left\{\frac{\partial}{\partial t}\right\}$ and $\mathcal{H}(M) = span\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. It is trivial that $\dim \overline{M} = 1$. Since $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in \mathcal{H}(M)$, we obtain $A = 0$.

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