An integral formula for bi-slant submanifolds in complex space forms

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Abstract

The present paper deals with bi-slant submanifolds in complex space forms. We obtain an integral formula of Simons' type for bi-slant submanifolds of a complex space form with positive constant holomorphic sectional curvature. Then, we apply it to prove our main result. We also discuss the same result in different submanifolds such as hemi-slant and semi-slant submanifolds.

1. Introduction

As a generalization of totally real submanifolds and holomorphic submanifolds, *CR*-submanifolds of Kaehler manifolds were introduced by Bejancu in [2]. Since then, such submanifolds have attracted many geometers (see for example [3,8,11,16,22]). As another interesting subject, the notion of a slant submanifold in an almost Hermitian manifold was introduced by B.-Y. Chen in [9,10]. A. Lotta introduced the notion of slant immersions of Riemannian manifolds into an almost contact metric manifolds. Afterwards, these submanifolds were studied by J.L. Cabrerizo et al. in the setting of Sasakian manifolds. Now, it has become a very active research area in both complex geometry and contact geometry (see for example [5, 6, 20]). Further, N. Papaghiuc [17] defined semi-slant submanifolds as a generalization of slant and *CR*-submanifolds. Slant and semi-slant submanifolds are a generalization of invariant, anti-invariant, semi-invariant and totally real submanifolds. In [4], J.L. Cabrerizo et al. generalized these notions by introducing bi-slant submanifolds in almost Hermitian manifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds which are studied by J.L. Cabrerizo et al. B. Şahin [20] named these submanifolds as hemi-slant submanifolds and also called pseudo-slant submanifolds. Since then, many geometers have studied these types of submanifolds in different ambient spaces.

J. Simons [19] studied the basic exposition of immersed minimal varieties in a Riemannian manifold. The principal result of this general investigation became the derivation of

the linear elliptic second order equation, which was satisfied by the second fundamental form of any minimal variety in any ambient manifold. In 1981, A. Bejancu et al. [3] derived Simons' type formula for CR-submanifolds of a complex space form. Since then, many authors derived Simons' type formula for different submanifolds in ambient spaces (see for example [14, 15, 21]). Recently, A.N. Siddiqui et al. [18] studied Simons' type formula for Kaehlerian slant submanifolds in complex space forms. The purpose of the present paper is to continue the study of bi-slant submanifolds and to construct Simons' type formula for bi-slant, hemi-slant, and semi-slant submanifolds of complex space forms.

The paper has been organized as follows: Section 2 is preliminary in nature. Some examples of semi-slant, hemi-slant, and bi-slant submanifolds in complex space forms are also discussed. In Section 3, some lemmas are quoted for later use. We derive Simons' type formula for a minimal bi-slant submanifold in a complex space form with positive constant holomorphic sectional curvature and also construct for hemi-slant and semi-slant submanifolds in the same ambient space. In Section 4, we prove the main result by using an integral formula of Simons' type and Hopf's Lemma. Some immediate consequences are also obtained.

2. Preliminaries

Let M be an n-dimensional submanifold of a Riemannian manifold $(\overline{M}, g)$ and ∇, $\overline{\nabla}$ denote the induced Riemannian and Riemannian connections on M and $\overline{M}$, respectively. The Gauss and Weingarten formulas are given by $\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)$ and $\overline{\nabla}_X V = -A_V(X) + \nabla_X^\perp V$, for any vector fields X, Y tangent to M and a vector field V normal to M, where $\nabla^\perp$ denotes the normal connection, B the second fundamental form and A_V the shape operator in the direction of V. If $B = 0$, then M is called *totally geodesic* [8]. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_p M$ and $\{e_{n+1}, \ldots, e_{2n}\}$ an orthonormal basis of $T_p^\perp M$. The mean curvature vector H of a submanifold M at p is defined by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} B(e_i, e_i).$$

If $H = 0$, i.e., trace $B \equiv 0$, then M is called *minimal* [8].

Denote by $\overline{R}$ and R the curvature tensor fields of $\overline{M}$ and M, respectively. Then the Gauss equation is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(A_{B(Y,Z)}X, W) + g(A_{B(X,Z)}Y, W), \qquad (2.1)$$

for all vector fields X, Y, Z, W tangent to M [8].

For the Laplacian of the second fundamental form A of an n-dimensional minimal submanifold M in an m-dimensional Riemannian manifold $\overline{M}$, the following Simons' type formula is well known ([19], pg - 81):

$$\nabla^2 A = -A \circ \widetilde{A} - \underset{\sim}{A} \circ A + \overline{R}(A) + \overline{R}', \qquad (2.2)$$

where the operators $\widetilde{A}$ and $\underset{\sim}{A}$ are defined as follows:

$$\widetilde{A} = A^t \circ A \text{ and } \underset{\sim}{A} = \sum_{a=1}^{m-n} adA_a adA_a$$

for a normal frame $\{e_a\}, a = 1, \ldots, m - n$, and $A_a = A_{e_a}$, where $\overline{R}(A)$ and $\overline{R}'$ are given by the following relations:

$$g(\overline{R}(A)^N(X), Y) = \sum_{i=1}^{n} \Big[2g(\overline{R}(e_i, Y)B(X, e_i), N) + 2g(\overline{R}(e_i, X)B(Y, e_i), N)$$

$$-g(A_N(X), \overline{R}(e_i, Y)e_i) - g(A_N(Y), \overline{R}(e_i, X)e_i)$$

$$+g(\overline{R}(e_i, B(X,Y))e_i, N) - 2g(A_N(e_i), \overline{R}(e_i, X)Y)\Big] \tag{2.3}$$

and

$$g(\overline{R'}^N(X), Y) = \sum_{i=1}^{n} \Big(g((\overline{\nabla}_X \overline{R})(e_i, Y)e_i, N) + g((\overline{\nabla}_{e_i}\overline{R})(e_i, X)Y, N) \Big) \tag{2.4}$$

for all vector fields X, Y tangent to M and N normal to M.

An almost Hermitian manifold $\overline{M}$ with an almost complex structure J is called a *Kaehler manifold* if J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of g, i.e., $\overline{\nabla}J = 0$.

A Kaehler manifold of constant holomorphic sectional curvature c is called a *complex space form* and it is denoted by $\overline{M}(c)$ [23]. The Riemannian curvature tensor field of $\overline{M}(c)$ is given by

$$\begin{aligned}
\overline{R}(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(X,JZ)J\,Y \\
&\quad -g(Y,JZ)JX + 2g(X,JY)JZ\},
\end{aligned} \tag{2.5}$$

for any vector fields X, Y, Z tangent to $\overline{M}(c)$ [23].

Now, let $\overline{M}$ be a real $2n$-dimensional Kaehler manifold with complex structure J and M be an isometrically immersed submanifold in $\overline{M}$. For any vector field X tangent to M, we put

$$JX = PX + FX, \tag{2.6}$$

where PX and FX denote the tangential and normal components of JX, respectively. Then P is an endomorphism of TM and F is the normal bundle valued 1-form on TM.

For any vector field V normal to M, we put

$$JV = tV + fV, \tag{2.7}$$

where tV and fV denote tangential and normal components of JV, respectively.

By the use of Gauss equation (2.1) and (2.5), we can write the Riemannian curvature tensor field R of M as

$$\begin{aligned}
R(X,Y,Z,W) &= \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
&\quad +g(PY,Z)g(PX,W) - g(PX,Z)g(PY,W) \\
&\quad +2g(X,PY)g(PZ,W)\} + g(A_{B(Y,Z)}X, W) \\
&\quad -g(A_{B(X,Z)}Y, W)
\end{aligned} \tag{2.8}$$

for any vector fields X, Y, Z tangent to TM.

A submanifold M of an almost Hermitian manifold $\overline{M}$ is called *slant* [9], if for each non-zero vector X tangent to M, the angle $\theta(X)$ between JX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$.

Furthermore,

(i) A submanifold M of $\overline{M}$ is called a *semi-slant submanifold* [17] of $\overline{M}$, if there exists a pair of orthogonal distributions D and D_θ such that D is holomorphic and D_θ is proper slant.

(ii) A submanifold M of $\overline{M}$ is called a *hemi-slant submanifold* (or pseudo-slant) [7] of $\overline{M}$, if there exists a pair of orthogonal distributions $D_\perp$ and D_θ such that $D_\perp$ is totally real and D_θ is proper slant.

(iii) A submanifold M of $\overline{M}$ is called a *bi-slant submanifold* [4] of $\overline{M}$, if there exists a pair of orthogonal distributions D_{θ_1} and D_{θ_2} such that D_{θ_1} is proper slant with slant angle θ_1 and D_{θ_2} is proper slant with slant angle θ_2.

Now we give some examples of bi-slant, semi-slant and hemi-slant submanifolds of complex space forms.

Example 2.1. [13] Let $(\mathbb{R}^{2m}, J)$ be the Euclidean $2m-$space and endowed with the Euclidean metric g, where J is an almost complex structure on $\mathbb{R}^{2m}$. The Euclidean metric g is defined by

$$g((x_1, \ldots, x_m, y_1, \ldots, y_m), (z_1, \ldots, z_m, w_1, \ldots, w_m))$$
$$= x_1 z_1 + \cdots + x_m z_m + y_1 w_1 + \cdots + y_m w_m.$$

Now let $\varphi : M \to \mathbb{R}^6$ defined by

$$\varphi(u, v, w, t) = \left(u\sqrt{2}, v\sqrt{2}, u + v, u - v, w - t, t - w\right).$$

Then we have an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ of $T_p M$, $p \in M$ such that

$$e_1 = \frac{1}{2}(\sqrt{2}, 0, 1, 1, 0, 0),$$

$$e_2 = \frac{1}{2}(0, \sqrt{2}, 1, -1, 0, 0),$$

$$e_3 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, -1),$$

$$e_4 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, -1, 1).$$

Thus, M is a totally geodesic bi-slant submanifold of $\mathbb{R}^6$ such that $D_{\theta_1} = \text{span}\{e_1, e_2\}$ with slant angle α and $D_{\theta_2} = \text{span}\{e_3, e_4\}$ with slant angle β, where

$$J(a, b, c, d, e, f) = (b, -a, d, -c, f, -e).$$

Using [12] (see page 10), we can state the following four examples.

Example 2.2. Let $\overline{M}(c)$ be a complex space form with two orthogonal submanifolds M_θ, a slant submanifold with slant angle $\theta \neq \frac{\pi}{2}$, and M_T a holomorphic submanifold. Then the Riemannian product $M = M_\theta \times M_T$ is a semi-slant submanifold of $\overline{M}(c)$.

Example 2.3. Let $\overline{M}(c)$ be a complex space form with two orthogonal submanifolds M_θ, a slant submanifold with slant angle $\theta \neq \frac{\pi}{2}$, and $M_\perp$ a totally real submanifold. Then the Riemannian product $M = M_\perp \times M_\theta$ is a hemi-slant submanifold of $\overline{M}(c)$.

Example 2.4. Let $\overline{M}(c)$ be a complex space form with two orthogonal submanifolds M_θ, a slant submanifold with slant angle θ, and M_T a holomorphic submanifold. Then the Riemannian product $M = M_\theta \times M_T$ is a bi-slant submanifold of $\overline{M}(c)$.

Example 2.5. Let $\overline{M}(c)$ be a complex space form with two orthogonal submanifolds M_{θ_1} and M_{θ_2} such that M_{θ_2} and M_{θ_2} are slant submanifolds of $\overline{M}(c)$ with slant angles θ_1 and θ_2, respectively. Then the Riemannian product $M = M_{\theta_1} \times M_{\theta_2}$ is a bi-slant submanifold of $\overline{M}(c)$.

Remark 2.6. It is easy to see that if M_1 and M_2 are minimal submanifolds, then the Riemannian product $M = M_1 \times M_2$ is also a minimal submanifold. Hence if we take M_1 and M_2 as minimal submanifolds in the above four examples, we will have minimal submanifolds.

3. Simons' type formula

This section is devoted to some preparatory results while our main result of this paper is given in the next section.

Lemma 3.1. [9] *Let M be any submanifold of a Kaehler manifold $\overline{M}$. Then $\overline{\nabla}P = 0$ if and only if*

$$A_{FX}(Y) = A_{FY}(X),$$

for any vector fields X, Y tangent to M.

Lemma 3.2. [9] *Let M be any submanifold of a Kaehler manifold $\overline{M}$. Then $\overline{\nabla}F = 0$ if and only if*

$$A_{fN}(X) = -A_N(PX),$$

for any vector fields X tangent to M and N normal to M.

Lemma 3.3. [8] *Let M be any submanifold of a Kaehler manifold $\overline{M}$. If $\overline{\nabla}P = 0$, then*

$$A_N P + P A_N = 0,$$

for any vector field N normal to M.

For a bi-slant submanifold M of dimension $n = 2n_1 + 2n_2$ in a complex space form $\overline{M}(c)$, we have

$$TM = D_{\theta_1} \oplus D_{\theta_2},$$

then

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j) = 2(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2),$$

where $\dim(D_{\theta_1}) = 2n_1$ and $\dim(D_{\theta_2}) = 2n_2$.

Using above Lemmas and equations (2.2), (2.3) and (2.4), we obtain the following result to get the required Simons' type formula for an n-dimensional minimal bi-slant submanifold in a complex space form $\overline{M}(c)$ of dimension $2n$, also for hemi-slant and semi-slant submanifolds, which play crucial role in working out the main results of this paper.

Theorem 3.4. *Let M be an n-dimensional minimal bi-slant submanifold of a complex space form $\overline{M}(c)$ of dimension $2n$. If $\overline{\nabla}P = 0$, then*

$$g(\nabla^2 A, A) = -g\left(A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A\right)$$

$$+ \frac{c}{4}\left[n + 1 + 6(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2)\right]||A||^2. \tag{3.1}$$

Proof. Since $\overline{M}(c)$ is locally symmetric, i.e., $\overline{\nabla}\,\overline{R} = 0$, then (2.4) implies

$$g(\overline{R'}^N(X), Y) = 0. \tag{3.2}$$

Now, we compute $g(\overline{R}(A)^N(X), Y)$ by putting (2.5) into (2.3) and then from Lemmas 3.1-3.3, we get the following easily,

$$g(\overline{R}(A)^N(X), Y) = \frac{c}{4}\Big[ng(A_N(X), Y) - 2g(A_{FY}(X), tN) - 2g(A_{FX}(Y), tN)$$

$$-4g(A_{fN}(X), PY) - 4g(A_{fN}(Y), PX) + 3g(PX, PA_N(Y))$$

$$+3g(PY, PA_N(X)) - 6g(A_N(PX), PY) - 2\sum_{i=1}^{n}\Big(g(A_{Fe_i}(e_i), X)g(FY, N)$$

$$+g(A_{Fe_i}(e_i), Y)g(FX, N) + \frac{3}{2}g(A_{Fe_i}(X), Y)g(Fe_i, N)\Big)\Big]$$

$$= \frac{c}{4}\left[n - 3 + 14(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2)\right]g(A_N(X), Y)$$

$$- cg(A_{FY}(X), tN). \tag{3.3}$$

With the help of (2.2), (3.2), and (3.3), we derive

$$g(\nabla^2 A(X), Y) = -g\left(A \circ \widetilde{A}(X) + \underset{\sim}{A} \circ A(X), Y\right)$$

$$+ \frac{c}{4}\left(n - 3 + 14(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2)\right)g(A_N(X), Y)$$

$$- cg(A_{FY}(X), tN). \tag{3.4}$$

We calculate the preceding equation (3.4) for $X = e_i$ and $Y = A_N(e_i)$, we get

$$g(\nabla^2 A(e_i), A_N(e_i)) = -g\left(A \circ \widetilde{A}(e_i) + \underset{\sim}{A} \circ A(e_i), A_N(e_i)\right)$$

$$+\frac{c}{4}\left(n - 3 + 14(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right) g(A_N(e_i), A_N(e_i))$$

$$- cg(A_{FA_N(e_i)}(e_i), tN). \tag{3.5}$$

Again applying Lemmas 3.1-3.3 and by summation for $i = 1, \ldots, n$, we find

$$\sum_{i=1}^{n} g(A_{FA_N(e_i)}(e_i), tN) = -(1 - 2(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2))\|A\|^2. \tag{3.6}$$

Thus, our assertion (3.1) follows on combining equations (3.5) and (3.6). $\qquad\square$

The following is Simons' type formula for a hemi-slant submanifold in $\overline{M}(c)$.

Corollary 3.5. *Let M be an n-dimensional minimal hemi-slant submanifold in a complex space form $\overline{M}(c)$ of dimension $2n$. If $\overline{\nabla}P = 0$, then*

$$g(\nabla^2 A, A) = -g\left(A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A\right) + \frac{c}{4}\left[n + 1 + 6n_2 \cos^2\theta\right]\|A\|^2. \tag{3.7}$$

For a semi-slant submanifold in $\overline{M}(c)$, we have the following Simons' type formula.

Corollary 3.6. *Let M be an n-dimensional minimal semi-slant submanifold in a complex space form $\overline{M}(c)$ of dimension $2n$. If $\overline{\nabla}P = 0$, then*

$$g(\nabla^2 A, A) = -g\left(A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A\right) + \frac{c}{4}\left[n + 1 + 6(n_1 + n_2 \cos^2\theta)\right]\|A\|^2. \tag{3.8}$$

4. Main results

This section gives the applications of Simons' type formula which has been derived in the last section for different submanifolds in the same ambient space. At this moment, we need to construct an inequality for minimal bi-slant submanifold in $\overline{M}(c)$, which shall be used in proving our main theorem.

Lemma 4.1. *Let M be an n-dimensional minimal bi-slant submanifold of a complex space form $\overline{M}(c)$ of dimension $2n$ such that $c > 0$. If $\overline{\nabla}P = 0$, then*

$$g(\nabla^2 A, A) \geq \left\{\left(-2 + \frac{1}{n}\right)\|A\|^2 + \frac{c}{4}\left[n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right]\right\}\|A\|^2. \tag{4.1}$$

Proof. Using Lemma 5.3.1 of J. Simons ([19], pg - 93), i.e.,

$$g\left(A \circ \widetilde{A} + \underset{\sim}{A} \circ A, A\right) \leq \left(2 - \frac{1}{n}\right)\|A\|^4$$

in Theorem 3.4, we find

$$g(\nabla^2 A, A) \geq \left(-2 + \frac{1}{n}\right)\|A\|^4 + \frac{c}{4}\left[n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right]\|A\|^2.$$

Thus, our assertion follows. $\qquad\square$

At this stage, we are ready to prove our main result by using the developed inequality (4.1) and well known Hopf's Lemma.

Theorem 4.2. *Let M be an n-dimensional compact minimal bi-slant submanifold of a complex space form $\overline{M}(c)$ of dimension $2n$ such that $c > 0$ and $\overline{\nabla}P = 0$. If the second fundamental form A satisfies*

$$||A||^2 \leq \frac{c}{4\left(2 - \frac{1}{n}\right)}\left(n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right),$$

then either M is totally geodesic or

$$||A||^2 = \frac{c}{4\left(2 - \frac{1}{n}\right)}\left(n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right).$$

Proof. Since M is compact, we have

$$\int_M g(\nabla^2 A, A) = -\int_M g(\nabla A, \nabla A). \tag{4.2}$$

Using Lemma 4.1 and (4.2), it follows that

$$0 \leq \int_M g(\nabla A, \nabla A)$$

$$\leq \int_M \left\{\left[\left(2 - \frac{1}{n}\right)||A||^2 - \frac{c}{4}\left(n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right)\right]||A||^2\right\}. \tag{4.3}$$

So, (4.3) is known as *an integral formula of Simons' type.* Further, by considering the hypothesis of the theorem, we find $\nabla A = 0$ and hence $\nabla ||A||^2$ is a constant, i.e., either $||A||^2 = 0$ or

$$||A||^2 = \frac{c}{4\left(2 - \frac{1}{n}\right)}\left[n + 1 + 6(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2)\right].$$

From $A = 0$, we conclude that M is totally geodesic. Hence, our assertions follow. $\square$

Since we know that bi-slant submanifolds generalize the hemi-slant and semi-slant submanifolds, therefore, we can easily state the following theorem by using Corollary 3.5.

Theorem 4.3. *Let M be an n-dimensional compact minimal hemi-slant submanifold of a complex space form $\overline{M}(c)$ of dimension $2n$ such that $c > 0$ and $\overline{\nabla}P = 0$. If the second fundamental form A satisfies*

$$||A||^2 \leq \frac{c}{4\left(2 - \frac{1}{n}\right)}\left[n + 1 + 6n_2 \cos^2\theta\right],$$

then either M is totally geodesic or

$$||A||^2 = \frac{c}{4\left(2 - \frac{1}{n}\right)}\left[n + 1 + 6n_2 \cos^2\theta\right].$$

The following theorem can be stated by using Corollary 3.6.

Theorem 4.4. *Let M be an n-dimensional compact minimal semi-slant submanifold of a complex space form $\overline{M}(c)$ of dimension $2n$ such that $c > 0$ and $\overline{\nabla}P = 0$. If the second fundamental form A satisfies*

$$||A||^2 \leq \frac{c}{4\left(2 - \frac{1}{n}\right)}\left(n + 1 + 6(n_1 + n_2 \cos^2\theta)\right),$$

then either M is totally geodesic or

$$||A||^2 = \frac{c}{4\left(2 - \frac{1}{n}\right)}\left(n + 1 + 6(n_1 + n_2 \cos^2\theta)\right).$$

As the ambient manifold is a complex space form $\overline{M}(c)$, we can take a complex projective space CP^{2n} with constant holomorphic sectional curvature 4 of real dimension $2n$. Then the following corollary follows from Theorem 4.2 and Lemma 4 of K. Abe [1].

Corollary 4.5. *Let M be an n-dimensional complete, compact minimal bi-slant submanifold of CP^{2n}. If the second fundamental form A satisfies*

$$||A||^2 < \left(\frac{n + 1 + 6(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2)}{2 - \frac{1}{n}} \right),$$

then M is either a

(1) *complex projective space, or*
(2) *real projective space.*

In the same manner, we can have the following corollaries based on Theorems 4.3 and 4.4, respectively.

Corollary 4.6. *Let M be an n-dimensional complete, compact minimal hemi-slant submanifold of CP^{2n}. If the second fundamental form A satisfies*

$$||A||^2 < \left(\frac{n + 1 + 6n_2 \cos^2 \theta}{2 - \frac{1}{n}} \right),$$

then M is either a

(1) *complex projective space, or*
(2) *real projective space.*

Corollary 4.7. *Let M be an n-dimensional complete, compact minimal semi-slant submanifold of CP^{2n}. If the second fundamental form A satisfies*

$$||A||^2 < \left(\frac{n + 1 + 6(n_1 + n_2 \cos^2 \theta)}{2 - \frac{1}{n}} \right),$$

then M is either a

(1) *complex projective space, or*
(2) *real projective space.*

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