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Degree of approximation by means of hexagonal Fourier series

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Abstract: Let f be a continuous function which is periodic with respect to the hexagon lattice, and let A be a lower triangular infinite matrix of nonnegative real numbers with nonincreasing rows. The degree of approximation of the function f by matrix means $T_n^{(A)}(f)$ of its hexagonal Fourier series is estimated in terms of the modulus of continuity of f.

Key words: Hexagonal domain, hexagonal Fourier series, Hölder class, matrix mean

1. Introduction

Estimation of the degree of approximation is one of the most important problems in approximation theory. Especially, mathematicians are interested in the degree of approximation of periodic functions. Fourier series and their summation methods are most useful tools in study of approximation problems of such functions. The degree of approximation by Cesàro, Nörlund, Riesz, and more general matrix means of trigonometric Fourier series of continuous $2\pi-$periodic functions was investigated by many authors in recent decades (see, for example [1, 2, 9, 10, 13, 14, 18, 19]).

Investigation of the degree of approximation of functions of several real variables is also important. Summation methods of multiple trigonometric Fourier series are used for studying approximation problems of such functions (see, for example [15–17]), [20, Sections 5.3 and 6.3], [23, Vol II, Chapter XVII], [22, Part 2]. In all of these studies it was assumed that the functions are $2\pi-$periodic in each of their variables.

Approximation problems on nontensor product domains, for example on hexagonal domains of $\mathbb{R}^2$, are studied by using another kind of periodicity. The periodicity defined by lattices allows us to study approximation problems on such domains. In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the simplest lattice is the hexagon lattice and the simplest spectral set is the regular hexagon. The hexagon lattice has importance, since it offers the densest packing of the plane with unit circles. Now, we give basic information about hexagonal lattice and hexagonal Fourier series. More detailed information can be found in [11] and [21].

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

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and

$$\Omega_H = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \le x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1\right\}.$$

It is more convenient to use the homogeneous coordinates (t_1, t_2, t_3) that satisfies $t_1 + t_2 + t_3 = 0$. As it is pointed out in [21], using homogeneous coordinates reveals symmetry in various formulas. If we set

$$t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2},$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \le t_1, t_2, -t_3 < 1,\ t_1 + t_2 + t_3 = 0\right\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we set

$$\mathbb{R}^3_H := \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\right\}$$

and

$$\mathbb{Z}^3_H := \mathbb{Z}^3 \cap \mathbb{R}^3_H.$$

A function $f : \mathbb{R}^2 \to \mathbb{C}$ is called $H-$periodic (or periodic with respect to the hexagon lattice) if

$$f(x + Hk) = f(x)$$

for all $k \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$. If we define $\mathbf{t} \equiv \mathbf{s}\ (mod 3)$ as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3\ (mod 3)$$

for $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3_H$, it follows that the function f is $H-$periodic if and only if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0}\ (mod 3)$, and

$$\int_\Omega f(\mathbf{t} + \mathbf{s})\, d\mathbf{t} = \int_\Omega f(\mathbf{t})\, d\mathbf{t}\ \ \left(\mathbf{s} \in \mathbb{R}^3_H\right)$$

for $H-$periodic integrable function f [21].

$L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g\rangle_H := \frac{1}{|\Omega|}\int_\Omega f(\mathbf{t})\,\overline{g(\mathbf{t})}d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω. The functions

$$\varphi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle}\ \ \left(\mathbf{t} \in \mathbb{R}^3_H\right),$$

where $\langle \mathbf{j}, \mathbf{t} \rangle$ is the usual Euclidean inner product of $\mathbf{j}$ and $\mathbf{t}$, are $H-$periodic, and by a theorem of B. Fuglede, the set

$$\left\{ \varphi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_H^3 \right\}$$

becomes an orthonormal basis of $L^2(\Omega)$ [3] (see also [11]).

For every natural number n, we define a subset of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \leq j_1, j_2, j_3 \leq n \right\}.$$

The subspace

$$\mathcal{H}_n := \operatorname{span} \left\{ \varphi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n \right\} \quad (n \in \mathbb{N})$$

has dimension $\#\mathbb{H}_n = 3n^2 + 3n + 1$, and its members are called hexagonal trigonometric polynomials of degree n.

The hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \varphi_{\mathbf{j}}(\mathbf{t}), \tag{1.1}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}) \overline{\varphi_{\mathbf{j}}(\mathbf{t})} d\mathbf{t} \quad \left(\mathbf{j} \in \mathbb{Z}_H^3 \right).$$

The nth hexagonal partial sum of the series (1.1) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \varphi_{\mathbf{j}}(\mathbf{t}) \quad (n \in \mathbb{N}).$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t} - \mathbf{u}) D_n(\mathbf{u}) d\mathbf{u},$$

where

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \varphi_{\mathbf{j}}(\mathbf{t})$$

is the Dirichlet kernel of order n.

It is known that the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}) \quad (n \geq 1), \tag{1.2}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin \frac{(n+1)(t_1 - t_2)\pi}{3} \sin \frac{(n+1)(t_2 - t_3)\pi}{3} \sin \frac{(n+1)(t_3 - t_1)\pi}{3}}{\sin \frac{(t_1 - t_2)\pi}{3} \sin \frac{(t_2 - t_3)\pi}{3} \sin \frac{(t_3 - t_1)\pi}{3}} \tag{1.3}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3$ [11].

The degree of approximation of $H-$periodic continuous functions by Cesàro, Riesz, and Nörlund means of their hexagonal Fourier series was investigated by the author in [4–8]. In the present paper, approximation properties of more general means of hexagonal Fourier series are studied and generalizations of previous results are obtained.

2. Main results

Let $C_H\left(\overline{\Omega}\right)$ be the Banach space of complex valued $H-$periodic continuous functions defined on $\mathbb{R}_H^3$, whose norm is the uniform norm:

$$\|f\|_{C_H\left(\overline{\Omega}\right)} := \sup\left\{|f(\mathbf{t})| : \mathbf{t}\in\overline{\Omega}\right\}.$$

The modulus of continuity of the function $f\in C_H\left(\overline{\Omega}\right)$ is defined by

$$\omega_H(f,\delta) := \sup_{0<\|\mathbf{t}\|\leq\delta}\|f - f(\cdot+\mathbf{t})\|_{C_H\left(\overline{\Omega}\right)},$$

where

$$\|\mathbf{t}\| := \max\{|t_1|,|t_2|,|t_3|\}$$

for $\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}_H^3$. $\omega_H(f,\cdot)$ is a nonnegative and nondecreasing function, and satisfies

$$\omega_H(f,\lambda\delta)\leq(1+\lambda)\,\omega_H(f,\delta) \tag{2.1}$$

for $\lambda>0$ [21].

A function $f\in C_H\left(\overline{\Omega}\right)$ is said to belong to the Hölder space $H^\alpha\left(\overline{\Omega}\right)$ $(0<\alpha\leq1)$ if

$$\Lambda^\alpha(f) := \sup_{\mathbf{t}\neq\mathbf{s}}\frac{|f(\mathbf{t})-f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha}<\infty.$$

$H^\alpha\left(\overline{\Omega}\right)$ becomes a Banach space with respect to the Hölder norm

$$\|f\|_{H^\alpha\left(\overline{\Omega}\right)} := \|f\|_{C_H\left(\overline{\Omega}\right)} + \Lambda^\alpha(f).$$

Let $A=(a_{n,k})$ $(n,k=0,1,...)$ be a lower triangular infinite matrix of real numbers. The $A-$transform of the sequence $(S_n(f))$ of partial sums the series (1.1) is defined by

$$T_n^{(A)}(f)(\mathbf{t}) := \sum_{k=0}^{n}a_{n,k}S_k(f)(\mathbf{t}) \quad (n\in\mathbb{N}).$$

We shall assume that the lower triangular matrix $A=(a_{n,k})$ satisfies the conditions

$$a_{n,k}\geq0 \quad (n=0,1,...,0\leq k\leq n), \tag{2.2}$$

$$a_{n,k}\geq a_{n,k+1} \quad (n=0,1,...,0\leq k\leq n-1), \tag{2.3}$$

and

$$\sum_{k=0}^{n}a_{n,k}=1 \quad (n=0,1,...). \tag{2.4}$$

Further, we use the notations

$$A_{n,k} := \sum_{\nu=0}^{k}a_{n,\nu} \quad (0\leq k\leq n),\; A_n(u) := A_{n,[u]},\, a_n(u) := a_{n,[u]} \quad (u>0),$$

where $[u]$ denotes the integer part of u.

In the rest of the paper, the relation $x \lesssim y$ will mean that there exists an absolute constant $c>0$ such that $x \leq cy$ holds for quantities x and y.

Main results of this paper are the following.

Theorem 2.1 *Let* $f \in C_H\left(\overline{\Omega}\right)$ *and let* $A=(a_{n,k})$ $(n,k=0,1,...)$ *be a lower triangular infinite matrix of real numbers which satisfies* (2.2), (2.3), *and* (2.4). *Then the estimate*

$$\left\| f - T_n^{(A)}(f) \right\|_{C_H\left(\overline{\Omega}\right)} \lesssim \log(n+1) \sum_{k=1}^{n} \frac{\omega_H(f,1/k)}{k} A_{n,k} \quad (n \in \mathbb{N}) \tag{2.5}$$

holds.

Corollary 2.2 *Let* $f \in H^{\alpha}\left(\overline{\Omega}\right)$ $(0<\alpha\leq 1)$ *and let the matrix* $A=(a_{n,k})$ $(n,k=0,1,...)$ *satisfies conditions of Theorem 1. Then we have*

$$\left\| f - T_n^{(A)}(f) \right\|_{C_H\left(\overline{\Omega}\right)} \lesssim \log(n+1) \sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \quad (n \in \mathbb{N}). \tag{2.6}$$

Theorem 2.3 *Let* $0 \leq \beta < \alpha \leq 1$, $f \in H^{\alpha}\left(\overline{\Omega}\right)$ *and let* $A=(a_{n,k})$ $(n,k=0,1,...)$ *be a lower triangular infinite matrix of real numbers which satisfies* (2.2), (2.3), *and* (2.4). *Then,*

$$\left\| f - T_n^{(A)}(f) \right\|_{H^{\beta}\left(\overline{\Omega}\right)} \lesssim \log(n+1) \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k} \right)^{\beta/\alpha} \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right)^{1-\beta/\alpha} \quad (n \in \mathbb{N}). \tag{2.7}$$

For means of trigonometric Fourier series of continuous $2\pi-$periodic functions, analogue of Theorem 1 was proved in [2] and analogue of Theorem 2 was proved in [14]. In these theorems, analogues of estimates (2.5) and (2.7) do not contain the multiplier $\log(n+1)$.

3. Proofs of main results

Proof [Proof of Theorem 2.1] It is clear that

$$\begin{aligned} \left| f(\mathbf{t}) - T_n^{(A)}(f)(\mathbf{t}) \right| &\leq \frac{1}{|\Omega|} \int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u})| \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \\ &\lesssim \frac{1}{|\Omega|} \int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u}. \end{aligned}$$

If we set $\Theta_{-1}(\mathbf{u}) := 0$, by (1.2) we get

$$\int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} = \int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} \left(\Theta_k(\mathbf{u}) - \Theta_{k-1}(\mathbf{u})\right) \right| d\mathbf{u}.$$

The function

$$\mathbf{t} \to \omega_H\left(f, \|\mathbf{t}\|\right)\left|\sum_{k=0}^{n} a_{n,k}\left(\Theta_k\left(\mathbf{t}\right) - \Theta_{k-1}\left(\mathbf{t}\right)\right)\right|$$

is symmetric with respect to variables t_1, t_2, and t_3, where $\mathbf{t} = (t_1, t_2, t_3) \in \Omega$. Hence it is sufficient to estimate the integral over the triangle

$$\begin{aligned} \Delta \quad &: \quad = \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3 : 0 \le t_1, t_2, -t_3 \le 1\right\} \\ &= \quad \left\{(t_1, t_2) : t_1 \ge 0,\ t_2 \ge 0,\ t_1 + t_2 \le 1\right\}, \end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By considering the formula (1.3), we obtain

$$\begin{aligned} &\int_{\Delta} \omega_H\left(f, \|\mathbf{t}\|\right)\left|\sum_{k=0}^{n} a_{n,k}\left(\Theta_k\left(\mathbf{t}\right) - \Theta_{k-1}\left(\mathbf{t}\right)\right)\right| d\mathbf{t} \\ = \quad &\int_{\Delta} \omega_H\left(f, t_1 + t_2\right)\left|\sum_{k=0}^{n} a_{n,k}\left(\begin{array}{c} \frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \\ -\frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \end{array}\right)\right| d\mathbf{t}. \end{aligned}$$

If we use the change of variables

$$s_1 := \frac{t_1 - t_3}{3} = \frac{2t_1 + t_2}{3},\ \ s_2 := \frac{t_2 - t_3}{3} = \frac{t_1 + 2t_2}{3}, \tag{3.1}$$

the integral becomes

$$3\int_{\widetilde{\Delta}} \omega_H\left(f, s_1 + s_2\right)\left|\sum_{k=0}^{n} a_{n,k}\left(\begin{array}{c} \frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \\ -\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \end{array}\right)\right| ds_1 ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta} := \left\{(s_1, s_2) : 0 \le s_1 \le 2s_2,\ 0 \le s_2 \le 2s_1,\ s_1 + s_2 \le 1\right\}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, estimating the integral over the triangle

$$\Delta^* := \left\{(s_1, s_2) \in \widetilde{\Delta} : s_1 \le s_2\right\} = \left\{(s_1, s_2) : s_1 \le s_2 \le 2s_1,\ s_1 + s_2 \le 1\right\},$$

which is the half of $\widetilde{\Delta}$, will be sufficient. The change of variables

$$s_1 := \frac{u_1 - u_2}{2},\ \ s_2 := \frac{u_1 + u_2}{2} \tag{3.2}$$

transforms the triangle Δ^* to the triangle

$$\Gamma := \left\{(u_1, u_2) : 0 \le u_2 \le \frac{u_1}{3},\ 0 \le u_1 \le 1\right\}.$$

Thus, we have to estimate the integral

$$I_n := \int_{\Gamma} \omega_H (f, u_1) \left| \sum_{k=0}^{n} a_{n,k} D_k^* (u_1, u_2) \right| du_1 du_2,$$

where

$$\begin{aligned} D_k^* (u_1, u_2) &= \frac{\sin ((k+1)(u_2)\pi) \sin \left((k+1) \frac{u_1+u_2}{2}\pi\right) \sin \left((k+1)\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin ((u_2)\pi) \sin \left(\frac{u_1+u_2}{2}\pi\right) \sin \left(\frac{u_1-u_2}{2}\pi\right)} \\ &\quad - \frac{\sin (k(u_2)\pi) \sin \left(k\frac{u_1+u_2}{2}\pi\right) \sin \left(k\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin ((u_2)\pi) \sin \left(\frac{u_1+u_2}{2}\pi\right) \sin \left(\frac{u_1-u_2}{2}\pi\right)}. \end{aligned}$$

By elementary trigonometric identities, we obtain

$$D_k^* (u_1, u_2) = D_{k,1}^* (u_1, u_2) + D_{k,2}^* (u_1, u_2) + D_{k,3}^* (u_1, u_2), \tag{3.3}$$

where

$$\begin{aligned} D_{k,1}^* (u_1, u_2) &: = 2\cos\left(\left(k+\frac{1}{2}\right) u_2 \pi\right) \\ &\quad \times \frac{\sin\left(\frac{1}{2}u_2\pi\right) \sin\left((k+1)\frac{u_1+u_2}{2}\pi\right) \sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}, \\ D_{k,2}^* (u_1, u_2) &: = 2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1+u_2}{2}\pi\right) \\ &\quad \times \frac{\sin(ku_2\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}, \end{aligned}$$

and

$$\begin{aligned} D_{k,3}^* (u_1, u_2) &: = 2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1-u_2}{2}\pi\right) \\ &\quad \times \frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}. \end{aligned}$$

We partition the triangle Γ as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &: = \left\{ (u_1, u_2) \in \Gamma : u_1 \le \frac{1}{n+1} \right\}, \\ \Gamma_2 &: = \left\{ (u_1, u_2) \in \Gamma : u_1 \ge \frac{1}{n+1},\ u_2 \le \frac{1}{3(n+1)} \right\}, \\ \Gamma_3 &: = \left\{ (u_1, u_2) \in \Gamma : u_1 \ge \frac{1}{n+1},\ u_2 \ge \frac{1}{3(n+1)} \right\}. \end{aligned}$$

Hence, $I_n = I_{n,1} + I_{n,2} + I_{n,3}$, where

$$I_{n,j} := \int\limits_{\Gamma_j} \omega_H\left(f, u_1\right) \left| \sum_{k=0}^{n} a_{n,k} D_k^*\left(u_1, u_2\right) \right| du_1 du_2 \ \ (j = 1, 2, 3) .$$

We shall use the inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \leq n, \ \ \left(n \in \mathbb{N}\right), \tag{3.4}$$

and

$$\sin t \geq \frac{2}{\pi} t, \ \ \left(0 \leq t \leq \frac{\pi}{2}\right) \tag{3.5}$$

to estimate integrals $I_{n,1}, I_{n,2}$, and $I_{n,3}$. By (3.4),

$$\begin{aligned} I_{n,1} &= \int\limits_{\Gamma_1} \omega_H\left(f, u_1\right) \left| \sum_{k=0}^{n} a_{n,k} D_k^*\left(u_1, u_2\right) \right| du_1 du_2 \\ &\lesssim \int\limits_{\Gamma_1} \omega_H\left(f, u_1\right) \left(\sum_{k=0}^{n} \left(k+1\right)^2 a_{n,k} \right) du_1 du_2 \\ &\leq \left(n+1\right)^2 \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{3u_2}^{\frac{1}{n+1}} \omega_H\left(f, u_1\right) du_1 du_2 \leq \omega_H\left(f, \frac{1}{n+1}\right). \end{aligned}$$

If we divide Γ_2 into two parts as

$$\begin{aligned} \Gamma_2' &:= \left\{ \left(u_1, u_2\right) \in \Gamma_2 : u_2 \leq \frac{a_{n,0}}{3\left(n+1\right)} \right\}, \\ \Gamma_2'' &:= \left\{ \left(u_1, u_2\right) \in \Gamma_2 : u_2 \geq \frac{a_{n,0}}{3\left(n+1\right)} \right\}, \end{aligned}$$

we have $I_{n,2} = I_{n,2}' + I_{n,2}''$, where

$$I_{n,2}' := \int\limits_{\Gamma_2'} \omega_H\left(f, u_1\right) \left| \sum_{k=0}^{n} a_{n,k} D_k^*\left(u_1, u_2\right) \right| du_1 du_2$$

and

$$I_{n,2}'' := \int\limits_{\Gamma_2''} \omega_H\left(f, u_1\right) \left| \sum_{k=0}^{n} a_{n,k} D_k^*\left(u_1, u_2\right) \right| du_1 du_2.$$

We also need the inequality

$$\frac{\omega_H\left(f, \delta_2\right)}{\delta_2} \leq 2 \frac{\omega_H\left(f, \delta_1\right)}{\delta_1} \ \ \left(\delta_1 < \delta_2\right), \tag{3.6}$$

which is obtained from (2.1). By (3.5) and (3.6),

$$\begin{aligned} &\int\limits_{\Gamma_2'} \omega_H\left(f,u_1\right)\left|\sum_{k=0}^{n} a_{n,k} D_{k,1}^{*}\left(u_1,u_2\right)\right| du_1 du_2 \\ \lesssim\ & \int\limits_{0}^{\frac{a_{n,0}}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H\left(f,u_1\right)}{u_1^2} du_1 du_2 = \frac{a_{n,0}}{3\left(n+1\right)} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H\left(f,u_1\right)}{u_1^2} du_1 \\ \leq\ & 2\frac{a_{n,0}}{3\left(n+1\right)}\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right) \int\limits_{\frac{1}{n+1}}^{1} \frac{du_1}{u_1} \lesssim \log\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right). \end{aligned}$$

By (3.4), (3.5), and (3.6) we obtain

$$\begin{aligned} &\int\limits_{\Gamma_2'} \omega_H\left(f,u_1\right)\left|\sum_{k=0}^{n} a_{n,k} D_{k,j}^{*}\left(u_1,u_2\right)\right| du_1 du_2 \\ \lesssim\ & n \int\limits_{0}^{\frac{a_{n,0}}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H\left(f,u_1\right)}{u_1} du_1 du_2 = n\frac{a_{n,0}}{3\left(n+1\right)} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H\left(f,u_1\right)}{u_1} du_1 \\ \leq\ & \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H\left(f,u_1\right)}{u_1^2} du_1 \lesssim \log\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right), \end{aligned}$$

for $j=2,3$. These last two estimates yield

$$I_{n,2}' \lesssim \log\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right).$$

Since

$$\sin 2x + \sin 2y + \sin 2z = -4\sin x \sin y \sin z$$

for $x+y+z=0$, we also get the expression

$$D_k^{*}\left(u_1,u_2\right) = H_{k,1}\left(u_1,u_2\right) + H_{k,2}\left(u_1,u_2\right) + H_{k,3}\left(u_1,u_2\right), \tag{3.7}$$

where

$$\begin{aligned} H_{k,1}\left(u_1,u_2\right) &: = \frac{1}{2}\frac{\cos\left(\left(2k+1\right)u_2\pi\right)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}, \\ H_{k,2}\left(u_1,u_2\right) &: = -\frac{1}{2}\frac{\cos\left(\left(2k+1\right)\frac{u_1+u_2}{2}\pi\right)}{\sin\left(u_2\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}, \\ H_{k,3}\left(u_1,u_2\right) &: = \frac{1}{2}\frac{\cos\left(\left(2k+1\right)\frac{u_1-u_2}{2}\pi\right)}{\sin\left(u_2\pi\right)\sin\left(\frac{u_1+u_2}{2}\pi\right)}. \end{aligned}$$

By the method used in [12, p.179], we get

$$\left|\sum_{k=0}^{n} a_{n,k}\cos\left(2k+1\right)t\right| \lesssim A_n\left(\frac{1}{t}\right)+a_n\left(\frac{1}{t}\right)\frac{1}{\sin t}\ \ (0<t<\pi) \tag{3.8}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k}\cos\left(2k+1\right)t\right| \lesssim A_n\left(\frac{1}{t}\right)\ \ \left(0<t\leq\frac{\pi}{2}\right). \tag{3.9}$$

By (3.9) we obtain

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,1}\left(u_1,u_2\right)\right| \lesssim \frac{1}{u_1^2}A_n\left(\frac{1}{\pi u_2}\right) \tag{3.10}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,3}\left(u_1,u_2\right)\right| \lesssim \frac{1}{u_1u_2}A_n\left(\frac{3}{\pi u_1}\right) \tag{3.11}$$

for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$. Also, for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$, the relation (3.8) and the fact

$$\sin\left(\frac{u_1\pi}{2}\right)\lesssim\sin\left(\frac{\left(u_1+u_2\right)\pi}{2}\right)$$

yield

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,2}\left(u_1,u_2\right)\right| \lesssim \frac{1}{u_1u_2}A_n\left(\frac{3}{\pi u_1}\right). \tag{3.12}$$

If we consider (3.5) and (3.6), we get

$$\begin{aligned}
&\int_{\Gamma_2''}\omega_H\left(f,u_1\right)\left|\sum_{k=0}^{n} a_{n,k}H_{k,1}\left(u_1,u_2\right)\right|du_1du_2\\
\lesssim\ & \int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{\omega_H\left(f,u_1\right)}{u_1^2}du_1du_2\leq 2\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right)\int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{du_1du_2}{u_1}\\
\leq\ & \log\left(n+1\right)\omega_H\left(f,\frac{1}{n+1}\right).
\end{aligned}$$

(3.11) and (3.12) give

$$
\begin{aligned}
&\int\limits_{\Gamma_2''} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2)\right| du_1 du_2 \lesssim \int\limits_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H(f,u_1)}{u_1 u_2} A_n\left(\frac{3}{\pi u_1}\right) du_1 du_2 \\
= \;& \log\left(\frac{1}{a_{n,0}}\right) \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H(f,u_1)}{u_1} A_n\left(\frac{3}{\pi u_1}\right) du_1 = \log\left(\frac{1}{a_{n,0}}\right) \int\limits_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n(t)\, dt \\
= \;& \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \left(\int\limits_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n(t)\, dt \right) \leq \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n\left(\frac{3}{\pi}(k+1)\right) \\
\leq \;& \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_{n,k+1} \lesssim \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_{n,k}
\end{aligned}
$$

for $j = 2, 3$. Hence, we get

$$
I_{n,2}'' \lesssim \log(n+1) \left\{ \omega_H\left(f, \frac{1}{n+1}\right) + \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_{n,k} \right\}.
$$

By considering (3.10) and (3.6),

$$
\begin{aligned}
&\int\limits_{\Gamma_3} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2)\right| du_1 du_2 \lesssim \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int\limits_{3u_2}^{1} \frac{\omega_H(f,u_1)}{u_1^2} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2 \\
\leq \;& \frac{2}{3} \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int\limits_{3u_2}^{1} \frac{\omega_H(f,3u_2)}{u_1 u_2} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2 = \frac{2}{3} \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_H(f,3u_2)}{u_2} \log\left(\frac{1}{3u_2}\right) A_n\left(\frac{1}{\pi u_2}\right) du_2 \\
\leq \;& \log(n+1) \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_H(f,3u_2)}{u_2} A_n\left(\frac{1}{\pi u_2}\right) du_2 = \log(n+1) \int\limits_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n(t)\, dt \\
= \;& \log(n+1) \sum_{k=1}^{n} \left(\int\limits_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n(t)\, dt \right) \leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n\left(\frac{3}{\pi}(k+1)\right) \\
\lesssim \;& \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_{n,k}.
\end{aligned}
$$

For $j=2,3$ have

$$\int\limits_{\Gamma_3}\omega_H\left(f,u_1\right)\left|\sum_{k=0}^{n}a_{n,k}H_{k,j}\left(u_1,u_2\right)\right|du_1du_2\lesssim\int\limits_{\frac{1}{n+1}}^{1}\int\limits_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}}\frac{\omega_H\left(f,u_1\right)}{u_1u_2}A_n\left(\frac{3}{\pi u_1}\right)du_2du_1$$

$$=\int\limits_{\frac{1}{n+1}}^{1}\frac{\omega_H\left(f,u_1\right)}{u_1}\log\left(\left(n+1\right)u_1\right)A_n\left(\frac{3}{\pi u_1}\right)du_1\leq\log\left(n+1\right)\int\limits_{\frac{1}{n+1}}^{1}\frac{\omega_H\left(f,u_1\right)}{u_1}A_n\left(\frac{3}{\pi u_1}\right)du_1$$

$$\lesssim\quad\log\left(n+1\right)\sum_{k=1}^{n}\frac{\omega_H\left(f,\frac{1}{k}\right)}{k}A_{n,k}$$

by (3.11) and (3.12). Thus, we get

$$I_{n,3}\lesssim\log\left(n+1\right)\sum_{k=1}^{n}\frac{\omega_H\left(f,\frac{1}{k}\right)}{k}A_{n,k}.$$

Since the sequence $\left(\frac{A_{n,k}}{k}\right)$ is nonincreasing with respect to k we have

$$\begin{aligned}\omega_H\left(f,\frac{1}{n+1}\right)&\leq\omega_H\left(f,\frac{1}{n}\right)=\frac{n\omega_H\left(f,\frac{1}{n}\right)}{n}=\sum_{k=1}^{n}\frac{\omega_H\left(f,\frac{1}{n}\right)}{n}\\&=\sum_{k=1}^{n}\omega_H\left(f,\frac{1}{n}\right)\frac{A_{n,n}}{n}\leq\sum_{k=1}^{n}\omega_H\left(f,\frac{1}{k}\right)\frac{A_{n,k}}{k}.\end{aligned}$$

(2.5) follows from estimates of $I_{n,j}$ $(j=1,2,3)$ and from the last estimate. □

Proof [Proof of Theorem 2.3] By the same method used in proof of Theorem 1, we obtain

$$\int\limits_{\Omega}\left|\sum_{k=0}^{n}a_{n,k}D_k\left(\mathbf{u}\right)\right|d\mathbf{u}\lesssim\log\left(n+1\right)\sum_{k=1}^{n}\frac{A_{n,k}}{k}\tag{3.13}$$

and

$$\int\limits_{\Omega}\left\|\mathbf{u}\right\|^{\alpha}\left|\sum_{k=0}^{n}a_{n,k}D_k\left(\mathbf{u}\right)\right|d\mathbf{u}\lesssim\log\left(n+1\right)\sum_{k=1}^{n}\frac{A_{n,k}}{k^{1+\alpha}}\quad\left(0<\alpha\leq1\right).\tag{3.14}$$

We set $e_n\left(\mathbf{t}\right):=f\left(\mathbf{t}\right)-T_n^{(A)}\left(f\right)\left(\mathbf{t}\right)$. Hence,

$$\left\|f-T_n^{(A)}\left(f\right)\right\|_{H^\beta\left(\overline{\Omega}\right)}=\left\|f-T_n^{(A)}\left(f\right)\right\|_{C_H\left(\overline{\Omega}\right)}+\Lambda^\beta\left(e_n\right).\tag{3.15}$$

Since

$$\left|e_n\left(\mathbf{t}\right)-e_n\left(\mathbf{s}\right)\right|\leq\frac{1}{\left|\Omega\right|}\int\limits_{\Omega}\left|f\left(\mathbf{t}\right)-f\left(\mathbf{t}-\mathbf{u}\right)-f\left(\mathbf{s}\right)+f\left(\mathbf{s}-\mathbf{u}\right)\right|\left|\sum_{k=0}^{n}a_{n,k}D_k\left(\mathbf{u}\right)\right|d\mathbf{u},$$

we have to estimate the integral

$$J_n := \int\limits_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u}.$$

Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$ we have

$$|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{t}-\mathbf{s}\|^{\alpha} \tag{3.16}$$

and

$$|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{u}\|^{\alpha}. \tag{3.17}$$

Hence, by (3.16) and (3.13) we get

$$\begin{aligned} (J_n)^{\frac{\beta}{\alpha}} &= \left(\int\limits_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}} \\ &\lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta} \left(\int\limits_{\Omega} \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}} \\ &\lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta} \left(\log(n+1) \sum_{k=1}^{n} \frac{A_{n,k}}{k} \right)^{\frac{\beta}{\alpha}}. \end{aligned}$$

Also, by (3.17) and (3.14) we obtain

$$(J_n)^{1-\frac{\beta}{\alpha}} \lesssim \left(\int\limits_{\Omega} \|\mathbf{u}\|^{\alpha} \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{1-\frac{\beta}{\alpha}} \lesssim \left(\log(n+1) \sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right)^{1-\frac{\beta}{\alpha}}.$$

Since

$$\begin{aligned} |e_n(\mathbf{t}) - e_n(\mathbf{s})| &\leq J_n = (J_n)^{\frac{\beta}{\alpha}} (J_n)^{1-\frac{\beta}{\alpha}} \\ &\lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta} \log(n+1) \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k} \right)^{\frac{\beta}{\alpha}} \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right)^{1-\frac{\beta}{\alpha}}, \end{aligned}$$

we get

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \lesssim \log(n+1) \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k} \right)^{\frac{\beta}{\alpha}} \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right)^{1-\frac{\beta}{\alpha}} \quad (\mathbf{t} \neq \mathbf{s}),$$

which implies

$$\Lambda^{\beta}(e_n) \lesssim \log(n+1) \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k} \right)^{\frac{\beta}{\alpha}} \left(\sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right)^{1-\frac{\beta}{\alpha}}.$$

The proof is finished by combining (2.6) and (3.15). □

4. Remarks

Remark 4.1 *Let* $p=(p_k)$ *be a nonincreasing sequence of positive real numbers. If we take*

$$a_{n,k}:=\left\{\begin{array}{cc}\frac{p_k}{P_n}, & 0\le k\le n\\ 0, & k>n\end{array}\right. ,$$

where $P_n:=\sum\limits_{k=0}^{n}p_k$, *then the matrix* $A=(a_{n,k})$ *satisfies* (2.2), (2.3), *and* (2.4). *In this case* $T_n^{(A)}$ *becomes the Riesz mean*

$$R_n(p;f)=\frac{1}{P_n}\sum_{k=0}^{n}p_kS_k(f).$$

Theorem 1 gives

$$\|f-R_n(p;f)\|_{C_H(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k\omega_H(f,1/k)}{k}\tag{4.1}$$

for $f\in C_H\left(\overline{\Omega}\right)$, *and Theorem 2 yields*

$$\|f-R_n(p;f)\|_{H^\beta(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\left(\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\tag{4.2}$$

for $f\in H^\alpha\left(\overline{\Omega}\right)$, $(0\le\beta<\alpha\le1)$.

Remark 4.2 *Let* $p=(p_k)$ *be a nondecreasing sequence of positive real numbers. In this case the matrix* $A=(a_{n,k})$ *with entries*

$$a_{n,k}:=\left\{\begin{array}{cc}\frac{p_{n-k}}{P_n}, & 0\le k\le n\\ 0, & k>n\end{array}\right. ,$$

satisfies (2.2), (2.3), *and* (2.4), *and* $T_n^{(A)}$ *becomes the Nörlund mean*

$$N_n(p;f)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}S_k(f).$$

If we set $Q_{n,k}:=\sum\limits_{\nu=n-k}^{n}p_\nu$, *we conclude from Theorem 1*

$$\|f-N_n(p;f)\|_{C_H(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_0}\right)\sum_{k=1}^{n}\frac{Q_{n,k}\omega_H(f,1/k)}{k}\tag{4.3}$$

for $f\in C_H\left(\overline{\Omega}\right)$, *and by Theorem 2 we get*

$$\|f-N_n(p;f)\|_{H^\beta(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_0}\right)\left(\sum_{k=1}^{n}\frac{Q_{n,k}}{k}\right)^{\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n}\frac{Q_{n,k}}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\tag{4.4}$$

for $f\in H^\alpha\left(\overline{\Omega}\right)$ $(0\le\beta<\alpha\le1)$.

Remark 4.3 *If we take* $p_k = 1 \ (k = 0, 1, ...)$, $R_n(p; f)$ *and* $N_n(p; f)$ *become* $(C, 1)$ *means* $S_n^{(1)}(f)$, *and both of* (4.1) *and* (4.3) *reduce to*

$$\left\| f - S_n^{(1)}(f) \right\|_{C_H(\overline{\Omega})} \lesssim \frac{\log(n+1)}{n+1} \sum_{k=1}^{n} \omega_H\left(f, \frac{1}{k}\right)$$

for $f \in C_H\left(\overline{\Omega}\right)$. *Furthermore,* (4.2) *and* (4.4) *give the estimate*

$$\left\| f - S_n^{(1)}(f) \right\|_{H^{\beta}(\overline{\Omega})} \lesssim \begin{cases} \frac{\log(n+1)}{n^{\alpha-\beta}}, & \alpha < 1 \\ \frac{(\log(n+1))^{2-\beta}}{n^{1-\beta}}, & \alpha = 1 \end{cases}$$

for $(C, 1)$ *means of* $f \in H^{\alpha}\left(\overline{\Omega}\right)$ $(0 \leq \beta < \alpha \leq 1)$.

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