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Faber–Laurent series in variable Smirnov classes

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Abstract: In this work, the maximal convergence properties of partial sums of Faber–Laurent series in the variable exponent Smirnov classes of analytic functions defined on a doubly connected domain of the complex plane are investigated.

Key words: Faber–Laurent series, variable spaces, maximal convergence

1. Introduction and main results

Let K be a bounded continuum of the complex plane $\mathbb{C}$ with the complementary $\mathbb{C}\backslash K$, consisting of two simple connected domains G and B. We assume that B is bounded and G is unbounded component of this complementary. Without loss of generality, we assume that $0 \in B$. Moreover, let $D := \{w \in \mathbb{C} : |w| < 1\}$, $D^{-} := \{w \in \mathbb{C} : |w| > 1\}$, and $\mathbb{T} := \partial D$.

We denote by $w = \varphi_1(z)$ the conformal mapping of G onto D^{-} normalized by the conditions

$$\varphi_1(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi_1(z)}{z} > 0$$

and by ψ_1 the inverse mapping of φ_1.

We also denote by $w = \varphi_2(z)$ the conformal mapping of B onto D^{-} normalized by the conditions

$$\varphi_2(0) = \infty, \quad \lim_{z\to 0} z\varphi_2(z) > 0$$

and by ψ_2 the inverse mapping of φ_2.

Since the Laurent expansion of φ_1 in some neighborhood of the infinity has the form

$$\varphi_1(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + ...,$$

we have

$$[\varphi_1(z)]^n = \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k + \sum_{k<0} \gamma_{n,k} z^k.$$

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The polynomial

$$\Phi_n^1(z) := \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k$$

is called the Faber polynomial of order n for continuum K. Let $E_n^1(z) := -\sum_{k<0} \gamma_{n,k} z^k$.

The function φ_2 has an expansion in some neighborhood of the origin:

$$\varphi_2(z) = \frac{\beta}{z} + \beta_0 + \beta_1 z + ... + \beta_k z^k....$$

Raising this function to the power n, we obtain

$$[\varphi_2(z)]^n = \Phi_n^2(1/z) - E_n^2(z), \quad z \in B \tag{1.1}$$

where $\Phi_n^2(1/z)$ denotes the polynomial of negative powers of z and the term $E_n^2(z)$ contains nonnegative powers of z; hence, this is an analytic function in the domain B.

Note that the polynomials Φ_n^1 and Φ_n^2 can be found also as the Taylor coefficients of the series representations

$$\frac{\psi_1'(t)}{\psi_1(t) - z} = \sum_{k=0}^{\infty} \Phi_k^1(z) \frac{1}{t^{k+1}}, \quad |t| > R_1 > 1,\ z \in K \tag{1.2}$$

and

$$\frac{\psi_2'(t)}{\psi_2(t) - z} = -\sum_{k=0}^{\infty} \Phi_k^2(1/z) \frac{1}{t^{k+1}}, \quad |t| > R_2 > 1,\ z \in K \tag{1.3}$$

respectively.

Let $p(\cdot) : \Gamma \to \mathbb{R}^+ := [0, \infty)$ be a Lebesgue measurable function defined on the Jordan rectifiable curve $\Gamma \subset \mathbb{C}$, such that

$$1 < p_- := \operatorname*{ess\,inf}_{z\in\Gamma} p(z) \leq \operatorname*{ess\,sup}_{z\in\Gamma} p(z) := p_+ < \infty. \tag{1.4}$$

Definition 1.1 *We say that $p(\cdot) \in P_0(\Gamma)$, if $p(\cdot)$ satisfies the conditions (1.4) and for some constant $c_0 > 0$ the inequality*

$$|p(z_1) - p(z_2)| \leq \frac{c_0}{\log(|\Gamma| / |z_1 - z_2|)}, \ \forall z_1, z_2 \in \Gamma$$

holds, where $|\Gamma|$ is the Lebesgue measure of Γ.

For a given exponent $p(\cdot)$ we define the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ as the set of Lebesgue measurable functions f defined on Γ such that $\int_\Gamma |f(z)|^{p(z)} |dz| < \infty$. Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf\left\{\lambda \geq 0 : \int_\Gamma |f(z)/\lambda|^{p(z)} |dz| \leq 1\right\} < \infty,$$

it becomes a Banach space, which in the case of $[0, 2\pi]$ is the variable exponent Lebesgue space $L^{p(\cdot)}([0, 2\pi])$, investigated in [3, 5, 19].

Let us take the level lines

$$\Gamma_{R_1} := \{z : |\varphi_1(z)| = R_1 > 1\},$$

$$L_{R_2} := \{z : |\varphi_2(z)| = R_2 > 1\}$$

and let $G_{R_1} := int\Gamma_{R_1}$, $G^{-}_{R_1} := ext\Gamma_{R_1}$, $B_{R_2} := intL_{R_2}$, $B^{-}_{R_2} := extL_{R_2}$. Moreover, let G_{R_1,R_2} be a doubly-connected domain bounded by the curves Γ_{R_1} and L_{R_2}.

Let $E^p(G_{R_1,R_2})$, $p \geq 1$, be a classical Smirnov class of analytic functions in the doubly-connected domain G_{R_1,R_2}. We mention [6] that $f \in E^p(G_{R_1,R_2})$, iff f is analytic in G_{R_1,R_2} and there exists a sequence $(\Delta_\nu)_{\nu=1}^{\infty}$, $\Delta_\nu \subset G_{R_1,R_2}$ of domains Δ_ν whose boundaries $(\Gamma_\nu)_{\nu=1}^{\infty}$ consist of two rectifiable Jordan curves, such that the domain Δ_ν contains each compact subset G^* of G_{R_1,R_2} for every $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and

$$\underset{\nu\to\infty}{limsup} \int_{\Gamma_\nu} |f(z)|^p |dz| < \infty.$$

Definition 1.2 *Let $p_1(\cdot)$ and $p_2(\cdot)$ be the Lebesgue measurable functions defined on Γ_{R_1} and L_{R_2}, respectively. The set*

$$E^{p_1(\cdot),p_2(\cdot)}(G_{R_1,R_2}) := \left\{f \in E^1(G_{R_1,R_2}) : f \in L^{p_1(\cdot)}(\Gamma_{R_1}) \cap L^{p_2(\cdot)}(L_{R_2})\right\}$$

is called the variable exponent Smirnov class of analytic functions in G_{R_1,R_2}.

Since $L^{p(\cdot)}(\mathbb{T})$ is noninvariant with respect to the usual shift operator, we consider the mean value operator: $\sigma_h : f \to \sigma_h f := \frac{1}{h}\int_0^h f(we^{it})\,dt$, $w \in \mathbb{T}$, $0 < h < \pi$, which is bounded[4] in $L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. Using this operator we define the modulus of smoothness as following.

Definition 1.3 *Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in P_0(\mathbb{T})$. The function $\Omega(f,\cdot)_{p(\cdot),\mathbb{T}} : [0,\infty) \to [0,\infty)$ defined by*

$$\Omega(f,\cdot)_{p(\cdot),\mathbb{T}} := \sup_{0<h\leq\delta} \|f - \sigma_h f\|_{L^{p(\cdot)}(\mathbb{T})}$$

is called the modulus of smoothness of f in $L^{p(\cdot)}(\mathbb{T})$.

If $f(z) \in E^1(G_{R_1,R_2})$ is an analytic function in the doubly-connected domain G_{R_1,R_2}, then it has the integral representation[20, pp. 256]:

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_{R_1}} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{L_{R_2}} \frac{f(\zeta)}{\zeta - z}d\zeta, \quad z \in K.$$

Let

$$\begin{aligned} f^{+}_{R_1}(z) \quad : \quad &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}} \frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_{|w|=R_1} \frac{\psi_1'(w)}{\psi_1(w) - z} f(\psi_1(w))\,dw\,, \qquad (1.5) \end{aligned}$$

$$\begin{aligned} f_{R_2}^{-}(z) \quad : &= \frac{-1}{2\pi i}\int_{L_{R_2}} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{-1}{2\pi i}\int_{|w|=R_2} \frac{\psi_2'(w)}{\psi_2(w) - z} f(\psi_2(w))\, dw. \end{aligned} \tag{1.6}$$

Combining (1.5) and (1.6) with the series representations (1.2) and (1.3), respectively we can write

$$f_{R_1}^{+}(z) \sim \sum_{k=0}^{\infty} a_k\left(f_{R_1}^{+}\right)\Phi_k^1(z), \quad z \in G_{R_1}, \tag{1.7}$$

$$f_{R_2}^{-}(z) \sim \sum_{k=0}^{\infty} b_k\left(f_{R_2}^{-}\right)\Phi_k^2(1/z), \quad z \in B_{R_2}^{-}, \tag{1.8}$$

where

$$\begin{aligned} a_k\left(f_{R_1}^{+}\right) &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}} \frac{f_{R_1}^{+}(z)\,\varphi_1'(z)}{[\varphi_1(z)]^{k+1}} dz \\ &= \frac{1}{2\pi i}\int_{|w|=R_1} \frac{f_{R_1}^{+}(\psi_1(w))}{w^{k+1}} dw, \end{aligned} \tag{1.9}$$

$$\begin{aligned} b_k\left(f_{R_2}^{-}\right) &= \frac{1}{2\pi i}\int_{L_{R_2}} \frac{f_{R_2}^{-}(z)\,\varphi_2'(z)}{[\varphi_2(z)]^{k+1}} dz \\ &= \frac{1}{2\pi i}\int_{|w|=R_2} \frac{f_{R_2}^{-}(\psi_2(w))}{w^{k+1}} dw. \end{aligned} \tag{1.10}$$

Let us introduce the value

$$\begin{aligned} R_n(z,f) \quad : &= f(z) - \left\{\sum_{k=0}^{n} a_k\left(f_{R_1}^{+}\right)\Phi_k^1(z)\right. \\ &\quad \left. + \sum_{k=1}^{n} b_k\left(f_{R_2}^{-}\right)\Phi_k^2(1/z)\right\} \end{aligned} \tag{1.11}$$

and the best approximation numbers

$$E_n\left(f, G_{R_1}\right)_{p_1(\cdot)} := \inf \|f - p_n\|_{L^{p_1(\cdot)}\left(\Gamma_{R_1}\right)} \quad \text{for } f \in E^{p_1(\cdot)}\left(G_{R_1}\right), \tag{1.12}$$

$$E_n\left(f, B_{R_2}^{-}\right)_{p_2(\cdot)} := \inf \|f - q_n\|_{L^{p_2(\cdot)}\left(L_{R_2}\right)} \quad \text{for } f \in E^{p_2(\cdot)}\left(B_{R_2}^{-}\right), \tag{1.13}$$

where inf is taken over the polynomials $p_n(z)$ and $q_n(1/z)$, respectively.

Since Γ_{R_1} and L_{R_2} are analytic curves, the following lemmas are true[14]:

Lemma 1.4 *If* $f \in L^{p_1(\cdot)}(\Gamma_{R_1})$, $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, *then* $f_{R_1}^{+}(z) \in E^{p_1(\cdot)}(G_{R_1})$.

Lemma 1.5 *If* $f \in L^{p_2(\cdot)}(L_{R_2})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$, *then* $f_{R_2}^{-}(z) \in E^{p_2(\cdot)}(B_{R_2}^{-})$.

By $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot), \ldots$, we denote the constants depending in general of parameters given in the brackets.

Our main results are as follows:

Theorem 1.6 *Let* $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$. *If* $f \in E^{p_1(\cdot),p_2(\cdot)}(G_{R_1,R_2})$, $R_1, R_2 > 1$, *then there is a constant* $c(p_1,p_2) > 0$ *such that for* $\forall z \in K$

$$|R_n(z,f)| \leq c(p_1,p_2)\sqrt{n \ln n}\left[\frac{E_n\left(f_{R_1}^{+}, G_{R_1}\right)_{p_1(\cdot)}}{R_1^{n+1}(R_1-1)} + \frac{E_n\left(f_{R_2}^{-}, B_{R_2}^{-}\right)_{p_2(\cdot)}}{R_2^{n+1}(R_2-1)}\right].$$

Theorem 1.6 is new also for the simple connected domains. When G is a simple connected domain, we have

Corollary 1.7 *Let* $p(\cdot) \in \mathcal{P}_0(\Gamma_R)$. *If* $f \in E^{p(\cdot)}(G_R)$, $R > 1$, *then there is a constant* $c(p) > 0$ *such that*

$$|R_n(z,f)| \leq c(p)\sqrt{n \ln n}\frac{E_n\left(f_R^{+}, G_R\right)_{p(\cdot)}}{R_1^{n+1}(R_1-1)}\ , \ \ z \in K,$$

where

$$R_n(z,f) := f(z) - \sum_{k=0}^{n} a_k\left(f_R^{+}\right)\Phi_k^1(z).$$

We denote by $w = \varphi_1(z)$ the conformal mapping of $G_{R_1}^{-}$ onto D^{-} and by φ_1^{-1} the inverse mapping of φ_1. Moreover, we denote by $w = \varphi_2(z)$ the conformal mapping of B_{R_2} onto D^{-} and by φ_2^{-1} the inverse mapping of φ_2. It is easy to see that

$$\varphi_1(z) = \frac{\varphi_1(z)}{R_1} \quad \text{and} \quad \varphi_1^{-1}(w) = \psi_1(R_1 w),$$

$$\varphi_2(z) = \frac{\varphi_2(z)}{R_2} \quad \text{and} \quad \varphi_2^{-1}(w) = \psi_2(R_2 w).$$

Let

$$f_1(w) := f\left(\varphi_1^{-1}(w)\right) \text{ and } p_{1,1}(w) := p_1\left(\varphi_1^{-1}(w)\right),$$

$$f_2(w) := f\left(\varphi_2^{-1}(w)\right) \text{ and } p_{2,2}(w) := p_2\left(\varphi_2^{-1}(w)\right).$$

The following theorem gives a qualitative estimation for the error of maximal convergence:

Theorem 1.8 *Let* $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$. *If* $f \in E^{p_1(\cdot),p_2(\cdot)}(G_{R_1,R_2})$, $R_1, R_2 > 1$, *then there exists a constant* $c(p_1,p_2) > 0$ *such that*

$$|R_n(z,f)| \leq c(p_1,p_2)\sqrt{n \ln n}\left[\frac{\Omega(f_1,1/n)_{p_{1,1}(\cdot),\mathbb{T}}}{R_1^{n+1}(R_1-1)} + \frac{\Omega(f_2,1/n)_{p_{2,2}(\cdot),\mathbb{T}}}{R_2^{n+1}(R_2-1)}\right].$$

When $p(\cdot) = const > 1$, different versions of Theorems 1.6 and 1.8 in the classical Smirnov, Smirnov–Orlicz classes of analytic functions defined on the simple connected domains can be found in the monograph [20, chapter X] and also in [8–12]. In the case of variable exponent $p(\cdot)$ similar problems were investigated in [1, 2, 7, 13–17, 22].

2. Auxiliary results

For fixed $R_1, R_2 > 1$ the level curves Γ_{R_1} and L_{R_2} are analytic curves and hence by [21] there exist the positive constants $c_i > 0$, $i = 1, 2, ..., 8$, such that

$$0 < c_1 \leq |\varphi_1'(z)| \leq c_2 < \infty, \quad z \in \Gamma_{R_1},$$

$$0 < c_3 \leq |\psi_1'(w)| \leq c_4 < \infty, \quad |w| = R_1 \tag{2.1}$$

$$0 < c_5 \leq |\varphi_2'(z)| \leq c_6 < \infty, \quad z \in L_{R_2},$$

$$0 < c_7 \leq |\psi_2'(w)| \leq c_8 < \infty, \quad |w| = R_2 \tag{2.2}$$

The proof of the following Lemma goes by a similar way to the proof of Lemma 1 in [14].

Lemma 2.1 *The following equivalences are true*

$$p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1}) \Leftrightarrow p_{1,1}(\cdot) \in \mathcal{P}_0(\mathbb{T}), \quad p_2(\cdot) \in \mathcal{P}_0(L_{R_2}) \Leftrightarrow p_{2,2}(\cdot) \in \mathcal{P}_0(\mathbb{T}),$$

$$f \in L^{p_1(\cdot)}(\Gamma_{R_1}) \Leftrightarrow f_1 \in L^{p_{1,1}(\cdot)}(\mathbb{T}), \quad f \in L^{p_2(\cdot)}(L_{R_2}) \Leftrightarrow f_2 \in L^{p_{2,2}(\cdot)}(\mathbb{T}).$$

We mention that if Γ is a rectifiable Jordan curve and $f \in L^1(\Gamma)$, then the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta := \frac{1}{2\pi i} (P.V) \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\Gamma(z, \varepsilon) := \{\zeta \in \Gamma : |\zeta - z| < \varepsilon\}$, existing for almost all $z \in \Gamma$ is called the Cauchy singular integral of f at $z \in \Gamma$.

For a given $f \in L^1(\Gamma)$ we associate the singular integral $S_\Gamma(f)$ taking the value $S_\Gamma(f)(z)$ *a.e.* on Γ. The linear operator S_Γ defined in such way is called the *Cauchy singular operator*. By [18] it is a bounded linear operator from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$.

If $f \in L^{p(\cdot)}(\Gamma)$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$, then the functions

$$f^+(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in int\Gamma,$$

$$f^-(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in ext\Gamma$$

are analytic in $int\Gamma$ and $ext\Gamma$, respectively. According to Privalov's theorem they have nontangential limits *a.e.* on Γ and the relations

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z) \quad \text{and} \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z) \tag{2.3}$$

are valid $a.e.$ on Γ. Hence,

$$f(z) = f^{+}(z) - f^{-}(z)$$

holds $a.e.$ on Γ.

Lemma 2.2 *If* $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, *then* $\Omega(S_{\mathbb{T}}(f), \cdot)_{p(\cdot)} \leq c(p)\, \Omega(f, \cdot)_{p(\cdot)}$.

Proof Let $\delta \in (0, \pi)$, $h < \delta$ and $w \in \mathbb{T}$. By Fubini's theorem

$$\begin{aligned}
\sigma_h(S_{\mathbb{T}}(f)(w)) &= (1/h) \int_0^h S_{\mathbb{T}}(f)\left(we^{it}\right) dt \\
&= (1/h) \int_0^h \frac{1}{2\pi i} (P.V.) \left(\int_{\mathbb{T}} \frac{f\left(\tau e^{it}\right) d\tau}{\tau - w} \right) dt \\
&= \frac{1}{2\pi i} (P.V.) \int_{\mathbb{T}} \frac{(1/h) \int_0^h f\left(\tau e^{it}\right) dt}{\tau - w} d\tau \\
&= \frac{1}{2\pi i} (P.V.) \int_{\mathbb{T}} \frac{\sigma_h(f)(\tau)}{\tau - w} d\tau \\
&= S_{\mathbb{T}}(\sigma_h(f)(w))
\end{aligned}$$

and hence using the boundedness of singular operator $S_{\mathbb{T}}$ in $L^{p(\cdot)}(\mathbb{T})$, we have that

$$\begin{aligned}
\|S_{\mathbb{T}}(f) - \sigma_h(S_{\mathbb{T}}(f))\|_{L^{p(\cdot)}(\mathbb{T})} &= \|S_{\mathbb{T}}(f - \sigma_h(f))\|_{L^{p(\cdot)}(\mathbb{T})} \\
&\leq c(p) \|f - \sigma_h(f)\|_{L^{p(\cdot)}(\mathbb{T})},
\end{aligned}$$

which implies the desired relation $\Omega(S_{\mathbb{T}}(f), \cdot)_{p(\cdot)} \leq c(p)\, \Omega(f, \cdot)_{p(\cdot)}$. □

In [14] were proved some direct theorems of approximation theory in the variable exponent Smirnov classes of analytic functions which in our terms can be formulated as follows:

Theorem 2.3 *If* $f \in E^{p_1(\cdot)}(G_{R_1})$, $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, *then for* $\forall n \in \mathbb{N}$ *there is an algebraic polynomial* $p_n(z, f)$ *such that for some constant* $c(p_1) > 0$ *the inequality*

$$\|f - p_n(\cdot, f)\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \leq c(p_1)\, \Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}}$$

holds.

Theorem 2.4 *If* $f \in E^{p_2(\cdot)}\left(B_{R_2}^{-}\right)$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$, *then for* $\forall n \in \mathbb{N}$ *there is an algebraic polynomial* $p_n(1/z, f)$ *such that for some constant* $c(p_2) > 0$ *the inequality*

$$\|f - p_n(\cdot, f)\|_{L^{p_2(\cdot)}(L_{R_2})} \leq c(p_2)\, \Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}$$

holds.

The following theorem also will be used.

Theorem 2.5 *If $r_i > 1$ and $|w| \geq r > 1$, $i = 1, 2$, then the inequality*

$$\frac{1}{2\pi} \int_{|t|=ri} \left| \frac{\psi_i'(t)}{\psi_i(t) - \psi_i(w)} - \frac{1}{t-w} \right| |dt| \leq \sqrt{\frac{r_i^2}{r_i^4 - 1} \ln \frac{r^2}{r^2 - 1}} \tag{2.4}$$

holds.

Note that Theorem 2.5 in the case of $i = 1$ was proved in [20, pp. 174], and in the case of $i = 2$ the proof goes by a similar way.

Using (1.7) and (1.8) in (1.11), for $z \in K$ we obtain

$$|R_n(z, f)| = \left| f(z) - \sum_{k=0}^{n} a_k \left(f_{R_1}^+\right) \Phi_k^1(z) - \sum_{k=1}^{n} b_k \left(f_{R_2}^-\right) \Phi_k^2(1/z) \right|$$

$$= \left| f_{R_1}^+(z) + f_{R_2}^-(z) - \sum_{k=0}^{n} a_k \left(f_{R_1}^+\right) \Phi_k^1(z) - \sum_{k=1}^{n} b_k \left(f_{R_2}^-\right) \Phi_k^2(1/z) \right|$$

$$\begin{aligned}
&\leq |f_{R_1}^+(z) - \sum_{k=0}^{n} a_k(f_{R_1}^+)\Phi_k^1(z)| + |f_{R_2}^-(z) - \sum_{k=1}^{n} b_k(f_{R_2}^-)\Phi_k^2(1/z)| \\
&= \left| \sum_{k=n+1}^{\infty} a_k \left(f_{R_1}^+\right) \Phi_k^1(z) \right| + \left| \sum_{k=n+1}^{\infty} b_k \left(f_{R_2}^-\right) \Phi_k^2(1/z) \right| \\
&=: \left| R_n^1\left(z, f_{R_1}^+\right) \right| + \left| R_n^2\left(z, f_{R_2}^-\right) \right|. \qquad (2.5)
\end{aligned}$$

It is clear that

$$\begin{aligned}
R_n^1\left(z, f_{R_1}^+\right) &= \sum_{k=n+1}^{\infty} a_k \left(f_{R_1}^+\right) \Phi_k^1(z) \\
&= \frac{1}{2\pi i} \int_{|t|=R_1} f_{R_1}^+(\psi(t)) \left[\sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} \right] dt \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
R_n^2\left(z, f_{R_2}^-\right) &= \sum_{k=n+1}^{\infty} b_k \left(f_{R_2}^-\right) \Phi_k^2(1/z) \\
&= \frac{1}{2\pi i} \int_{|t|=R_2} f_{R_2}^-(\psi_2(t)) \sum_{k=n+1}^{\infty} \frac{\Phi_k^2(1/z)}{t^{k+1}} dt. \qquad (2.7)
\end{aligned}$$

If p_n is a polynomial of degree at most n, then

$$R_n^1\left(z, f_{R_1}^+\right) = \frac{1}{2\pi i} \int_{|t|=R_1} \left[f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))\right] \sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} dt. \tag{2.8}$$

Since

$$\Phi_k^1(z) = [\varphi_1(z)]^k + E_k^1(z), \quad z \in K, \tag{2.9}$$

we have

$$\sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty} \frac{[\varphi_1(z)]^k}{t^{k+1}} + \sum_{k=n+1}^{\infty} \frac{E_k^1(z)}{t^{k+1}}. \tag{2.10}$$

Hence, from (2.8), taking into account (2.10), for $z = \psi_1(w)$ we get

$$\left|R_n^1\left(z, f_{R_1}^+\right)\right| \leq \frac{1}{2\pi} \int_{|t|=R_1} \left|f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| |dt|$$

$$+\frac{1}{2\pi} \int_{|t|=R_1} \left|f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty} E_k^1(\psi_1(w)) \frac{1}{t^{k+1}}\right| |dt|. \tag{2.11}$$

Similarly, using (1.1) we obtain

$$\left|R_n^2\left(z, f_{R_2}^-\right)\right| \leq \frac{1}{2\pi} \int_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| dt$$

$$+\frac{1}{2\pi} \int_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}}\right| dt. \tag{2.12}$$

We shall also use the relations [20, pp. 63]:

$$E_k^i(\psi_i(w)) = \frac{1}{2\pi i} \int_{|\tau|=r_i} \tau^k F_i(\tau, w)\, d\tau, \quad |w| \geq r_i > 1, \quad i = 1, 2 \tag{2.13}$$

where

$$F_i(\tau, w) := \frac{\psi_i'(\tau)}{\psi_i(\tau) - \psi_i(w)} - \frac{1}{\tau - w}, \quad |\tau| > 1,\ |w| > 1.$$

3. Proof of main results

Proof [Proof of Theorem 1.6] Let $z \in \Gamma_{r_1}$, $1 < r_1 < R_1$ and p_n be the best approximating polynomial of degree at most n to $f_{R_1}^+ \in E^{p_1(\cdot)}(G_{R_1})$. Denoting

$$I_1 := \frac{1}{2\pi} \int_{|t|=R_1} \left|f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| |dt|,$$

$$I_2 := \frac{1}{2\pi} \int_{|t|=R_1} \left|f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty} E_k^1(\psi_1(w)) \frac{1}{t^{k+1}}\right| |dt|$$

by virtue of (2.11), we see that

$$\left|R_n^1\left(z, f_{R_1}^+\right)\right| \leq I_1 + I_2. \tag{3.1}$$

Using relations (2.1), (1.12) and applying Hölder's inequality [3, pp. 27], we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{\Gamma_{R_1}} \left|f_{R_1}^{+}(\zeta)-p_n(\zeta)\right| \left|\sum_{k=n+1}^{\infty}\frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}}\right| |\varphi_1'(\zeta)|\,|d\zeta| \\ &\le c_1\int_{\Gamma_{R_1}} \left|f_{R_1}^{+}(\zeta)-p_n(\zeta)\right| \left|\sum_{k=n+1}^{\infty}\frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}}\right| |d\zeta| \\ &\le c_2(p_1)\left\|f_{R_1}^{+}(\zeta)-p_n(\zeta)\right\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \left\|\sum_{k=n+1}^{\infty}\frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}}\right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})} \\ &\le c_3(p_1)\,E_n\left(f_{R_1}^{+},G_{R_1}\right)_{p_1(\cdot)} \left\|\frac{|\varphi_1(z)|^{n+1}|\varphi_1(\cdot)|^{-n-1}}{|\varphi_1(\cdot)|-|\varphi_1(z)|}\right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})} \\ &\le \frac{c_4(p_1)\,r_1^{n+1}}{R_1^{n+1}(R_1-r_1)}E_n\left(f_{R_1}^{+},G_{R_1}\right)_{p_1(\cdot)}. \end{aligned} \tag{3.2}$$

Now, we estimate the integral I_2. By (2.13) we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_{|t|=R_1} \left|f_{R_1}^{+}(\psi_1(t))-p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty}\frac{E_k^1(\psi_1(w))}{t^{k+1}}\right| |dt| \\ &= \frac{1}{2\pi}\int_{|t|=R_1} \left\{\left|f_{R_1}^{+}(\psi_1(t))-p_n(\psi_1(t))\right| \left|\sum_{k=n+1}^{\infty}[\frac{1}{2\pi i}\int_{|\tau|=r_1}\tau^k F_1(\tau,w)\,d\tau]\frac{1}{t^{k+1}}\right|\right\} |dt| \\ &\le \frac{1}{2\pi}\int_{|t|=R_1} \left\{\left|f_{R_1}^{+}(\psi_1(t))-p_n(\psi_1(t))\right| \frac{1}{2\pi}\int_{|\tau|=r_1}\left|\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}\right| |F_1(\tau,w)|\,|d\tau|\right\} |dt| \\ &= \frac{1}{2\pi}\int_{|t|=R_1} \left\{\left|f_{R_1}^{+}(\psi_1(t))-p_n(\psi_1(t))\right| \frac{1}{2\pi}\int_{|\tau|=r_1}\left|\frac{\tau^{n+1}}{t^{n+1}(t-\tau)}\right| |F_1(\tau,w)|\,|d\tau|\right\} |dt|. \end{aligned}$$

By Fubini's theorem

$$I_2 \le \frac{r_1^{n+1}}{2\pi R_1^{n+1}}\int_{|\tau|=r_1}\left\{|F_1(\tau,w)|\frac{1}{2\pi}\int_{|t|=R_1}\left|f_{R_1}^{+}(\psi_1(t))-p_n(\psi_1(t))\right|\frac{|dt|}{|t-\tau|}\right\}|d\tau|$$

and then changing the variables and using Hölder's inequality we obtain

$$\begin{aligned} I_2 &\le \frac{r_1^{n+1}}{2\pi R_1^{n+1}}\int_{|\tau|=r_1}\left\{|F_1(\tau,w)|\frac{1}{2\pi}\int_{\Gamma_{R_1}}\left|f_{R_1}^{+}(\zeta)-p_n(\zeta)\right|\frac{|\varphi_1'(\zeta)|\,|d\zeta|}{|\varphi_1(\zeta)-\varphi_1(z)|}\right\}|d\tau| \\ &\le \frac{c_5(p_1)\,r_1^{n+1}}{R_1^{n+1}}\int_{|\tau|=r_1}|F_1(\tau,w)|\left\{\left\|f_{R_1}^{+}(\zeta)-p_n(\zeta)\right\|_{L^{p_1(\cdot)}(\Gamma_{R_1})}\left\|\frac{\varphi_1'(\zeta)}{\varphi_1(\zeta)-\varphi_1(z)}\right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})}\right\}|d\tau| \\ &\le \frac{c_6(p_1)\,r_1^{n+1}}{R_1^{n+1}(R_1-r_1)}\int_{|\tau|=r_1}|F_1(\tau,w)|\,E_n\left(f_{R_1}^{+},G_{R_1}\right)_{p_1(\cdot)}|d\tau|. \end{aligned}$$

From this, by using (2.4), we have that

$$I_2 \leq \frac{c_7(p_1) r_1^{n+1}}{R_1^{n+1}(R_1 - r_1)} E_n\left(f_{R_1}^+, G_{R_1}\right)_{p_1(\cdot)} \sqrt{\frac{r_1^2}{r_1^4 - 1} \ln \frac{r_1^2}{r_1^2 - 1}}. \tag{3.3}$$

Now, the inequalities (3.1)–(3.3) imply that

$$\left|R_n^1\left(z, f_{R_1}^+\right)\right| \leq \frac{c_8(p_1) r_1^{n+1} E_n\left(f_{R_1}^+, G_{R_1}\right)_{p_1(\cdot)}}{R_1^{n+1}(R_1 - r_1)} \sqrt{\frac{r_1^2}{r_1^4 - 1} \ln \frac{r_1^2}{r_1^2 - 1}}.$$

Consequently, setting $z \in K$ and $r_1 = 1 + \frac{1}{n}$ in this estimate, we obtain the inequality

$$\left|R_n^1\left(z, f_{R_1}^+\right)\right| \leq \frac{c_9(p_1) E_n\left(f_{R_1}^+, G_{R_1}\right)_{p_1(\cdot)}}{R_1^{n+1}(R_1 - 1)} \sqrt{n \ln n} \tag{3.4}$$

with $c_9(p_1) > 0$. Now, let $z \in \Gamma_{r_2}$, $1 < r_2 < R_2$. Denoting

$$I_1^* := \frac{1}{2\pi} \int_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| dt$$

$$I_2^* := \frac{1}{2\pi} \int_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}}\right| dt$$

by virtue of (2.12), we see that

$$\left|R_n^2\left(z, f_{R_2}^-\right)\right| \leq I_1^* + I_2^*. \tag{3.5}$$

Using (2.2) and Hölder's inequality, we have

$$\begin{aligned}
I_1^* &= \frac{1}{2\pi} \int_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| |dt| \\
&= \frac{1}{2\pi} \int_{L_{R_2}} \left|f_{R_2}^-(\zeta) - p_n(1/\zeta)\right| \left|\sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\zeta)]^{k+1}}\right| |\varphi_2'(\zeta)| |d\zeta| \\
&\leq c_{10} \int_{L_{R_2}} \left|f_{R_2}^-(\zeta) - p_n(1/\zeta)\right| \left|\sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\zeta)]^{k+1}}\right| |d\zeta| \\
&\leq c_{11}(p_2) \left\|f_{R_2}^-(\cdot) - p_n(1/\cdot)\right\|_{L^{p_2(\cdot)}(L_{R_2})} \left\|\sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\cdot)]^{k+1}}\right\|_{L^{q_2(\cdot)}(L_{R_2})} \\
&\leq c_{12}(p_2) \left\|f_{R_2}^-(\cdot) - p_n(1/\cdot)\right\|_{L^{p_2(\cdot)}(L_{R_2})} \left\|\frac{|\varphi_2(z)|^{n+1}}{|\varphi_2(\cdot)|^{n+1}(|\varphi_2(\cdot)| - |\varphi_2(z)|)}\right\|_{L^{q_2(\cdot)}(L_{R_2})} \\
&\leq c_{13}(p_2) E_n\left(f_{R_2}^-, B_{R_2}^-\right)_{p_2(\cdot)} \frac{r_2^{n+1}}{R_2^{n+1}(R_2 - r_2)}.
\end{aligned} \tag{3.6}$$

Now, we estimate the integral I_2^*. By (2.13) we have

$$\begin{aligned} I_2^* &= \frac{1}{2\pi}\int\limits_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left|\sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}}\right| |dt| \\ &= \frac{1}{2\pi}\int\limits_{|t|=R_2} \left\{ \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \left| \sum_{k=n+1}^{\infty} \frac{1}{2\pi i} \int\limits_{|\tau|=r_2} \frac{\tau^k}{t^{k+1}} F_2(\tau, w)\, d\tau \right| \right\} |dt| \\ &\le \frac{1}{2\pi}\int\limits_{|t|=R_2} \left\{ \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \frac{1}{2\pi} \int\limits_{|\tau|=r_2} \left| \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} \right| |F_2(\tau, w)|\, |d\tau| \right\} |dt| \\ &= \frac{1}{2\pi}\int\limits_{|t|=R_2} \left\{ \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \frac{1}{2\pi} \int\limits_{|\tau|=r_2} \left| \frac{\tau^{n+1}}{t^{n+1}(t-\tau)} \right| |F_2(\tau, w)|\, |d\tau| \right\} |dt| . \end{aligned}$$

By Fubini's theorem

$$I_2^* \le \frac{1}{2\pi}\int\limits_{|\tau|=r_2} \left\{ |\tau|^{n+1} |F_2(\tau, w)| \frac{1}{2\pi} \int\limits_{|t|=R_2} \left|f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))\right| \frac{|t|^{-n-1}}{|t-\tau|} |dt| \right\} |d\tau|$$

and hence changing the variables and using Hölder's inequality, we obtain

$$\begin{aligned} I_2^* &\le \frac{r_2^{n+1}}{2\pi R_2^{n+1}} \int\limits_{|\tau|=r_2} \left\{ |F_2(\tau, w)| \frac{1}{2\pi} \int\limits_{L_{R_2}} \left|f_{R_2}^-(\zeta) - p_n(1/\zeta)\right| \frac{|\varphi_2'(\zeta)|}{|\varphi_2(\zeta) - \varphi_2(z)|} |d\zeta| \right\} |d\tau| \\ &\le \frac{c_{14}(p_2)\, r_2^{n+1}}{R_2^{n+1}} \int\limits_{|\tau|=r_2} \left\{ |F_2(\tau, w)| \left\|f_{R_2}^-(\cdot) - p_n(1/\cdot)\right\|_{L^{p_2(\cdot)}(L_{R_2})} \left\| \frac{|\varphi_2'(\cdot)|}{|\varphi_2(\cdot) - \varphi_2(z)|} \right\|_{L^{q_2(\cdot)}(L_{R_2})} \right\} |d\tau| . \end{aligned}$$

From this, using Theorem 2.5, in the case of $i = 2$ we have

$$I_2^* \le \frac{c_{15}(p_2)\, r_2^{n+1}}{R_2^{n+1}(R_2 - r_2)} E_n\left(f_{R_2}^-, B_{R_2}^-\right)_{p_2(\cdot)} \sqrt{\frac{r_2^2}{r_2^4 - 1} \ln \frac{r_2^2}{r_2^2 - 1}},$$

which, in combination with (3.5) and (3.6), implies that

$$\left|R_n^2\left(z, f_{R_2}^-\right)\right| \le \frac{c_{16}(p_2)\, r_2^{n+1} E_n\left(f_{R_2}^-, B_{R_2}^-\right)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - r_2)} \sqrt{\frac{r_1^2}{r_1^4 - 1} \ln \frac{r_1^2}{r_1^2 - 1}}.$$

Setting in this estimate $r_2 = 1 + \frac{1}{n}$, we obtain for $z \in K$ the inequality

$$\left|R_n^2\left(z, f_{R_2}^-\right)\right| \le \frac{c_{17}(p_2)\, E_n\left(f_{R_2}^-, B_{R_2}^-\right)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - 1)} \sqrt{n \ln n} \tag{3.7}$$

with $c_{17}(p_2) > 0$. Finally, combining the relations (3.4), (3.7), and (2.5) we have

$$|R_n(z,f)| \leq c(p_1,p_2)\sqrt{n\ln n}\left[\frac{E_n\left(f^+_{R_1},G_{R_1}\right)_{p_1(\cdot)}}{R_1^{n+1}(R_1-1)}+\frac{E_n\left(f^-_{R_2},B^-_{R_2}\right)_{p_2(\cdot)}}{R_2^{n+1}(R_2-1)}\right].$$

□

Now we can proof Theorem 1.8.

Proof [Proof of Theorem 1.8] From Theorem 1.6, 2.3 and 2.4 we have

$$|R_n(z,f)| \leq c(p_1,p_2)\sqrt{n\ln n}\left[\frac{E_n\left(f^+_{R_1},G_{R_1}\right)_{p_1(\cdot)}}{R_1^{n+1}(R_1-1)}+\frac{E_n\left(f^-_{R_2},B^-_{R_2}\right)_{p_2(\cdot)}}{R_2^{n+1}(R_2-1)}\right]$$

$$\leq c_1(p_1,p_2)\sqrt{n\ln n}\left[\frac{\Omega\left(f^+_{R_1}\circ\varphi_1^{-1},1/n\right)_{p_{1,1}(\cdot),\mathbb{T}}}{R_1^{n+1}(R_1-1)}+\frac{\Omega\left(f^-_{R_2}\circ\varphi_2^{-1},1/n\right)_{p_{2,2}(\cdot),\mathbb{T}}}{R_2^{n+1}(R_2-1)}\right]. \tag{3.8}$$

Using the subadditivity property of modulus and (2.3), and also Lemma (2.2), we obtain that

$$\begin{aligned}\Omega\left(f^+_{R_1}\circ\varphi_1^{-1},1/n\right)_{p_{1,1}(\cdot),\mathbb{T}} &= \Omega\left(f_1/2+S_{\mathbb{T}}(f_1),1/n\right)_{p_{1,1}(\cdot),\mathbb{T}}\\ &\leq c_{18}(p_{1,1})\left[\Omega(f_1,1/n)_{p_{1,1}(\cdot),\mathbb{T}}+\Omega\left(S_{\mathbb{T}}(f_1),1/n\right)_{p_{1,1}(\cdot),\mathbb{T}}\right]\\ &\leq c_{19}(p_{1,1})\,\Omega(f_1,1/n)_{p_{1,1}(\cdot),\mathbb{T}}\end{aligned}$$

and

$$\begin{aligned}\Omega\left(f^-_{R_2}\circ\varphi_2^{-1},1/n\right)_{p_{2,2}(\cdot),\mathbb{T}} &= \Omega\left(f_2/2-S_{\mathbb{T}}(f_2),1/n\right)_{p_{2,2}(\cdot),\mathbb{T}}\\ &\leq c_{20}(p_{2,2})\left[\Omega(f_2,1/n)_{p_{2,2}(\cdot),\mathbb{T}}+\Omega\left(S_{\mathbb{T}}(f_2),1/n\right)_{p_{2,2}(\cdot),\mathbb{T}}\right]\\ &\leq c_{21}(p_{2,2})\,\Omega(f_2,1/n)_{p_{2,2}(\cdot),\mathbb{T}}\,.\end{aligned}$$

Now, these inequalities and (3.8) imply that

$$|R_n(z,f)| \leq c(p_{1,1},p_{2,2})\sqrt{n\ln n}\left[\frac{\Omega(f_1,1/n)_{p_{1,1}(\cdot),\mathbb{T}}}{R_1^{n+1}(R_1-1)}+\frac{\Omega(f_2,1/n)_{p_{2,2}(\cdot),\mathbb{T}}}{R_2^{n+1}(R_2-1)}\right].$$

□

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