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BALIKESİR ÜNİVERSİTESİ
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MATEMATİK ANABİLİM DALI

KARMA LEBESGUE UZAYLARINDA KARMA DÜZGÜNLÜK MODÜLÜ İLE YAKLAŞIM

UĞUR YİĞİTASLAN

YÜKSEK LİSANS TEZİ

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ETİK BEYAN

Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Tez Yazım Kurallarına uygun olarak tarafımca hazırlanan "**Karma Lebesgue Uzaylarında Karma Düzgünlük Modülü ile Yaklaşım**" başlıklı tezde;

- Tüm bilgi ve belgeleri akademik kurallar çerçevesinde elde ettiğimi,
- Kullanılan veriler ve sonuçlarda herhangi bir değişiklik yapmadığımı,
- Tüm bilgi ve sonuçları bilimsel araştırma ve etik ilkelere uygun şekilde sunduğumu,
- Yararlandığım eserlere atıfta bulunarak kaynak gösterdiğimi,

beyan eder, aksinin ortaya çıkması durumunda her türlü yasal sonucu kabul ederim.

Uğur YİĞİTASLAN

ÖZET

KARMA LEBESGUE UZAYLARINDA KARMA DÜZGÜNLÜK MODÜLÜ İLE YAKLAŞIM
YÜKSEK LISANS TEZI
UĞUR YİĞİTASLAN
BALIKESİR ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATIK ANABILIM DALI
(TEZ DANIŞMANI: PROF. DR. RAMAZAN AKGÜN)
BALIKESIR, MART - 2021

Bu tezde Karma Lebesgue Uzayında Karma Düzgünlük Modülü kullanılarak trigonometrik yaklaşımın temel eşitsizlikleri incelenmiştir.
Tez beş ana bölümden oluşmaktadır.
Birinci bölüm, giriş bölümüdür.
İkinci bölümde, tezde kullanılan ve gerekli olan tanımlar belirtilmiş, teoremler ise ispatsız şekilde verilmiştir.
Üçüncü bölümde Karma Lebesgue Uzayı açıklanmıştır.
Dördüncü bölümde Açısal yaklaşım, Fark operatörü, K fonksiyoneli, Vallee-Poussin ortalamaları, Karma Düzgünklük modülü ve türevlerinden bahsedilmiştir.
Beşinci ve son bölüm sonuç bölümüdür. Bu tezde elde edilen sonuçlara yer verilmiştir.

ANAHTAR KELİMELER: Karma lebesgue uzayı, Karma düzgünlük modülü, Açısal yaklaşım, Fark operatörleri, Valle-pousssin ortalamaları.

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ABSTRACT

MIXED LEBESGUE SPACES, MIXED MODULUS OF SMOOTHNESS AND APPROXİMATİON
MSC THESIS
UĞUR YİĞİTASLAN
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In this work, some trigonometric approximation results are investigated in Mixed Lebesgue Spaces with Mixed Moduli of Smoothness. The work consists of five main chapters.
The first chapter includes introduction.
In the second chapter, the necessary definitions utilized in this work were indicated. Besides, theorems were given without proof.
In the third chapter, Mixed Lebesgue Spaces were explained.
In the fourth chapter, angular trigonometric approximation, difference operator, K functional, special classes of functions, Vallee-Poussin sums, Mixed Moduli of Smoothness and its derivates were mentioned.
The fifth chapter, the last one includes conclusions of this work.

KEYWORDS: Mixed lebesgue spaces, Mixed moduli of smoothness, Angular approximation, difference operator, Vallee-poussin sums.

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İÇİNDEKİLER

Sayfa

SEMBOL LİSTESİ

:=	**:** Tanım olarak eşittir
□	: Doğal sayılar kümesi
□	: Reel sayılar kümesi
$\mathbf{L^p}\left(\mathbf{T}\right)$	: Lebesgue uzayı $\left(1 \le p \le \infty\right)$
$\mathbf{s_{m,\infty}}\left(\mathbf{f}\right)$	: $f \in L^1\left(\mathrm{T}^2\right)$ fonksiyonunun Fourier serisinin *x* e göre kısmi toplamı
$\mathbf{s_{\infty,n}}\left(\mathbf{f}\right)$	: $f \in L^1\left(\mathrm{T}^2\right)$ fonksiyonunun Fourier serisinin *y* e göre kısmi toplamı
$\mathbf{s_{m,n}}\left(\mathbf{f}\right)$	: $f \in L^1\left(\mathrm{T}^2\right)$ fonksiyonunun Fourier serisinin hem *x* e göre hem *y* ye göre kısmi toplamı
$\mathbf{D_m(t)}$	: Dirichlet Çekirdeği
$\mathbf{Y_{m_1,m_2}}\left(\mathbf{f}\right)_{\mathbf{L^p}\left(\mathbf{T^2}\right)}$	: İki boyutlu açısal yaklaşım
$f^{(\rho_1,\rho_2)}$	: $f\left(x,y\right)$nin *x* göre ρ_1 mertebeli ve *y* ye göre ρ_2 mertebeli Wely türevi
$\mathbf{\Delta_{h_1}^{\alpha_1}(f)}$	: $f\left(x,y\right)$ nin *x* ye göre pozitif α_1 dereceli h_1 adımlı fark operatörü
$\mathbf{\Delta_{h_2}^{\alpha_2}(f)}$	: $f\left(x,y\right)$ nin *y* ye göre pozitif α_2 dereceli h_2 adımlı fark operatörü
$\mathbf{C(\alpha_1,h_1)}$	: α_1 in h_1 li kombinasyonu
$\boldsymbol{\omega}_{\alpha_1,\alpha_2}\left(\mathbf{f},\boldsymbol{\delta}_1,\boldsymbol{\delta}_2\right)_{\mathbf{p_1p_2}}$	: $f\left(x,y\right)$ nin *x* e göre pozitif α_1 dereceli, *y* ye göre pozitif α_2 dereceli karma düzgünlük modülünü
$\mathbf{T_{m_1,\infty}}\left(\mathbf{x,y}\right)$	: Derecesi m_1 i geçmeyen *x* e göre trigonometrik polinom
$\mathbf{T_{\infty,m_2}}\left(\mathbf{x,y}\right)$	: Derecesi m_2 i geçmeyen *y* e göre trigonometrik polinom
$\mathbf{V_{m_1,\infty}}\left(\mathbf{f}\right)$	: *x* e göre $f\left(x,y\right)$ fonksiyonunun Fourier serisinin Vallee-Poussin ortalaması
$\mathbf{V_{\infty,m_2}}\left(\mathbf{f}\right)$	: *y* e göre $f\left(x,y\right)$ fonksiyonunun Fourier serisinin Vallee-Poussin ortalaması
$\mathbf{V_{m_1,m_2}}\left(\mathbf{f}\right)$	: Hem *x* hem *y* ye göre $f\left(x,y\right)$ fonksiyonunun Fourier serisinin Vallee-Poussin ortalaması
$\mathbf{\Lambda_p}$	: Lacunary Fourier serisine sahip $f\left(x,y\right)$fonksiyon sınıfı
$\boldsymbol{M_p}$	:Fourier katsayıları belli bir koşulu sağlayan fonskiyonların Sınıfı
a ≈ b	:Öyle $c_1 > 0, c_2 > 0$ vardır ki $c_1 a \le b \le c_2 a$ sağlanır
h.h.h.	: Hemen hemen her yerde

ÖNSÖZ

Bu tezin yazım sürecinde tecrübe ve bilgisinden her zaman yararlandığım, her durumda ve her zaman yardımını esirgemeyen, emeğini her zaman üzerinde hissettiğim, değerli hocam sayın Prof. Dr. Ramazan AKGÜN' e, yardımlarından ve desteklerinden ötürü sayın Araş. Gör. Dr. Ahmet Hamdi AVŞAR' a, yüksek lisans sürecini beraber atlattığımız arkadaşım Merve Nur BAĞCI' ya, yüksek lisans çalışmalarım boyunca yaşadığım tüm sıkıntıları unutmamı sağlayan, hayatı güzel kılan değerli eşim Nazlı' ya ve annem ile babama teşekkürü bir borç bilirim.

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1. GİRİŞ

Yaklaşım teorisinde çoğu zaman belirli bir sınıfa ait olan fonksiyonlara daha iyi özelliklere sahip, daha dar bir fonksiyon sınıfı üzerinden yaklaşım problemleri çalışılır. Daha iyi özellikli fonksiyonlar sınıfı olarak, çalışılan fonksiyon uzayının bir alt uzayı alınır. Bu çalışmada iki değişkenli Lebesgue uzayının genellemesi olan Karma Lebesgue uzayına ait fonksiyonlara trigonometrik açısal yaklaşım problemleri incelenmiş ve var olan sonuçlar derlenmiştir.

2. ÖN BİLGİLER

2.1 Tanım (Ölçülebilir Küme)

X boştan farklı bir küme olsun. X ’ in alt kümelerinin boş olmayan bir S koleksiyonu için eğer

i) $\varnothing, S \in X$

ii) $\forall \mathrm{E} \in S$ için $\mathrm{E}^c \in S$

iii) $\forall n \in \mathbb{N}$ için $\mathrm{E}_n \in S$ ise $\bigcup_{n=1}^{\infty} \mathrm{E}_n \in S$

koşulları sağlanıyor ise S koleksiyonuna X üzerinde bir $\sigma-$cebiri denir. Bu durumda (X, S) sıralı ikilisine ölçülebilir uzay, S koleksiyonundaki her bir kümeye de ölçülebilir küme denir.

2.2 Tanım (Lebesgue Uzayı)

$\mathrm{T} := [0, 2\pi]$, $1 \le p < \infty$ ve $n \in \mathbb{N}$ için

$$\| f \|_p = \left(\int_{\mathrm{T}^n} | f(x) |^p \, dx \right)^{1/P} < \infty$$

koşulunu sağlayan $f : \mathrm{T}^n \to \mathbb{C}$ ölçülebilir fonksiyonlarının kümesine Lebesgue uzayı denir ve $L^p(\mathrm{T}^n)$ ile gösterilir.

2.3 Tanım

$\mathrm{T} := [0, 2\pi]$ ve $\exists M > 0$ için

$|f(x)| \le M$ h.h.h.

olmak üzere $f : \mathrm{T} \to \mathbb{C}$ fonksiyonlarının hemen hemen her yerde eşit olma bağıntısına göre denklik sınıflarının kümesi $L^\infty(\mathrm{T})$ ile ifade edilir.

2.4 Tanım (Normlu Uzay)

N bir lineer uzay olsun. $\|.\|: N \to \mathbb{R}$ fonksiyonunun x vektörüne karşılık getirdiği negatif olmayan değer $\|x\|$ ile gösterilsin. Bu fonksiyon aşağıdaki üç özelliği sağlıyor ise $\|.\|$ fonksiyonuna N de bir norm, $\left(N,\|.\|\right)$ ikilisine de normlu uzay denir.

1. $\|x\| = 0_\theta \Leftrightarrow x = 0_\theta$,

2. $\forall\ \lambda \in \mathbb{R}$ $\|\lambda x\| = |\lambda|\|x\|$,

3. $\forall\ x, y$ elamanı için $\|x+y\| \le \|x\| + \|y\|$.

2.5 Tanım (Banach Uzayı)

Yukarıdaki tanım 2.4 için verilen normlu uzay tam ise $\left(N,\|.\|\right)$ normlu uzayına Banach uzayı denir.

2.6 Tanım (Hölder Eşitsizliği)

$1 < p < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1,\ f \in L^p(T),\ g \in L^q(T)$ için

$$\int_T f(x)\, g(x) dx \le \left(\int_T |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_T |g(x)|^q\, dx\right)^{\frac{1}{q}} = \|f\|_p \|g\|_q$$

dir.

2.7 Tanım (Minkowski Eşitsizliği)

$1 \le p < \infty,\ f, g \in L^p(T)$ için $f + g \in L^p\left(\mathrm{T}\right)$ olur ve

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

eşitsizliği sağlanır.

2.8 Tanım (Genelleştirilmiş Minkowski Eşitsizliği)

$1 \le p < \infty$, $K(x,y)$ hem *x* değişkenine hem de *y* değişkenine göre sürekli fonksiyon, $f \in L^p(T)$ ise

$$\int_T \left(\int_T \left| f(y) K(x,y) dy \right|^p dx \right)^{\frac{1}{p}} \le \int_T \left(\int_T \left| f(y) K(x,y) \right|^p dx \right)^{\frac{1}{p}} dy \le$$

$$\le \int_T \left| f(y) \right| \left(\int_T \left| K(x,y) \right|^p dx \right)^{\frac{1}{p}} dy .$$

2.9 Tanım

Negatif olmayan $F(f,\delta_1,\delta_2)$ ve $G(f,\delta_1,\delta_2)$ ifadeleri için $F(f,\delta_1,\delta_2) \lesssim G(f,\delta_1,\delta_2)$ demek $F(f,\delta_1,\delta_2) \le CG(f,\delta_1,\delta_2)$ özelliğini sağlayan f , δ_1 ve δ_2 den bağımsız bir C pozitif sayısının var olduğu anlamını taşıyacaktır. Öte yandan $F(f,\delta_1,\delta_2) \lesssim G(f,\delta_1,\delta_2)$ ve $G(f,\delta_1,\delta_2) \lesssim F(f,\delta_1,\delta_2)$ iken bu durumda $G(f,\delta_1,\delta_2) \approx F(f,\delta_1,\delta_2)$ notasyonunu kullanacağız.

2.10 Önerme (Jensen Eşitliği) [1]

$a_k \ge 0$ ve $0 < \alpha \le \beta < \infty$ olsun. Bu durumda

$$\left(\sum_{k=1}^{\infty} \alpha_k^{\beta} \right)^{1/\beta} \le \left(\sum_{k=1}^{\infty} \alpha_k^{\alpha} \right)^{1/\alpha} .$$

2.11 Önerme (Hardy Eşitliği) [2]

$a_k \ge 0$, $\beta_k \ge 0$ olsun.

a) Farz edelim ki $\sum_{k=1}^{\infty} a_k = \alpha_n \gamma_n$ olsun. Eğer $1 \le p < \infty$ ise

$$\sum_{k=1}^{\infty} a_k \left(\sum_{n=k}^{\infty} b_n \right)^p \le \sum_{k=1}^{\infty} a_k \left(b_k \gamma_k \right)^p .$$

Eğer $0 < p \le 1$ ise

$$\sum_{k=1}^{\infty} a_k \left(\sum_{n=k}^{\infty} b_n \right)^p \ge \sum_{k=1}^{\infty} a_k \left(b_k \gamma_k \right)^p .$$

b)Farz edelim ki $\sum_{k=1}^{\infty} a_k = \alpha_n \beta_n$ olsun. Eğer $1 \le p < \infty$ ise

$$\sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{k} b_n \right)^p \le \sum_{k=1}^{\infty} a_k \left(b_k \beta_k \right)^p .$$

Eğer $0 < p \le 1$ ise

$$\sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{k} b_n \right)^p \ge \sum_{k=1}^{\infty} a_k \left(b_k \beta_k \right)^p .$$

2.12 Teorem (Marcinkiewicz çarpan teoremi) [1]

$1 < p < \infty$, $f \in L_p^0\left(\mathrm{T}^2\right)$ nin Fourier serisi

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(a_{n_1,n_2} \cos n_1 x \cos n_2 y + b_{n_1,n_2} \sin n_1 x \cos n_2 y +$$

$$+ c_{n_1,n_2} \cos n_1 x \sin n_2 y + d_{n_1,n_2} \sin n_1 x \sin n_2 y \Big)$$

$$=: \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} A_{n_1,n_2}\left(x, y\right) \qquad (2.1)$$

dir.

$\left(\vartheta_{n_1,n_2}\right)_{n_1,n_2=1}^{\infty}$ sayı dizisi, sonlu bir M ve her $n_i \in \mathbb{N}, i = 1, 2$ için

$$\left|\vartheta_{n_1,n_2}\right| \le M ,$$

$$\sum_{m_1=2^{n_1-1}+1}^{2^{n_1}} \left| \vartheta_{m_1,n_2} - \vartheta_{m_1+1,n_2} \right| \le M ,$$

$$\sum_{m_2=2^{n_2-1}+1}^{2^{n_2}} \left|\vartheta_{n_1,m_2} - \vartheta_{n_1,m_2+1}\right| \le M \,,$$

$$\sum_{m_1=2^{n_1-1}+1}^{2^{n_1}} \sum_{m_2=2^{n_2-1}+1}^{2^{n_2}} \left|\vartheta_{m_1,m_2} - \vartheta_{m_1+1,n_2} - \vartheta_{m_1,n_2+1} + \vartheta_{n_1+1,n_2+1}\right| \le M$$ eşitsizliklerini sağlasın. Bu durumda $\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \vartheta_{n_1,n_2} A_{n_1,n_2}(x,y)$ trigonometrik serisi bir $\phi \in L_p^0(\mathrm{T}^2)$ fonksiyonunun Fourier serisidir ve $\|\phi\|_{L_p(\mathrm{T}^2)} \square \|f\|_{L_p(\mathrm{T}^2)}$.

2.13 Teorem (Litthewood Paley Teoremi) [1]

$1 < p < \infty$, $f \in L_p^0(\mathrm{T}^2)$ fonksiyonunun Fourier serisi (2.1) biçiminde olsun.

$$\Delta_{0,0} := A_{1,1}(x,y),$$

$m_1 \in \mathbb{N}$ için $\Delta_{m_1,0} := \sum_{\nu_1=2^{m_1-1}+1}^{2^{m_1}} A_{\nu_1,1}(x,y),$

$m_2 \in \mathbb{N}$ için $\Delta_{0,m_2} := \sum_{\nu_1=2^{m_2-1}+1}^{2^{m_2}} A_{\nu_1,1}(x,y),$

$m_1 \in \mathbb{N}$ ve $m_2 \in \mathbb{N}$ için $\Delta_{m_1,m_2} := \sum_{\nu_1=2^{m_1-1}+1}^{2^{m_1}} \sum_{\nu_1=2^{m_2-1}+1}^{2^{m_2}} A_{\nu_1,\nu_2}(x,y)$ olsun. Bu durumda

$$\|f\|_{L_p(\mathrm{T}^2)} \approx \left(\int_0^{2\pi}\int_0^{2\pi} \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} \Delta_{\nu_1,\nu_2}^2 \right)^{p/2} dxdy \right)^{1/p}.$$

2.14 Teorem (Hardy Littlwood-Paley teoremi) [3]

$f \in L_1^0(\mathrm{T}^2)$ fonksiyonunun Fourier serisi (2.1) deki gibi olsun.

a) $2 \le p < \infty$ ve

$$I := \left(\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left(\left| a_{n_1 n_2} \right| + \left| b_{n_1 n_2} \right| + \left| c_{n_1 n_2} \right| + \left| d_{n_1 n_2} \right| \right)^{p} \left(n_1 n_2 \right)^{p-2} \right)^{\frac{1}{p}} < \infty \text{ ise}$$

$$f \in L_p^0\left(\mathbb{T}^2\right) \text{ ve } \left\| f \right\|_{L_p\left(\mathbb{T}^2\right)} \asymp I .$$

b) $1 < p \le 2$ ve $f \in L_p^0\left(\mathbb{T}^2\right)$ için $I \le \left\| f \right\|_{L_p\left(\mathbb{T}^2\right)}$.

3. KARMA LEBESGUE UZAYI

Lebesgue uzayları, $p \geq 1$ olduğu durumda Banach uzaylarının örneğidir. Buna ek olarak da Karma Lebesgue uzayları, Lebesgue Uzaylarının bir genellemesidir.

Karma Lebesgue uzayları farklı değişkenler üzerinde farklı kontrol miktarları tanımlamamıza izin verir. Bu nedenle bağımlı değişkenler düşünüldüğünde doğal olarak bir dizi farklı nicelikte parametreler ortaya çıkar. İlk olarak Benedek ve Panzone **[4]** tarafından tanımlanmışlar ve Rubio de Francia, Ruiz ve Torrea **[5]** tarafından da araştırılmıştır.

3.1 Tanım [6]

$i = 1, 2$, $1 \leq p_i < \infty$ olmak üzere L_{p_1, p_2} fonksiyon sınıfı x ve y 'ye göre 2π periyodik, ölçülebilir ve

$$\| f \|_{p_1 p_2} = \left(\int_0^{2\pi} \left(\int_0^{2\pi} | f(x, y)|^{p_1} dx \right)^{\frac{p_2}{p_1}} dy \right)^{\frac{1}{p_2}} < \infty$$

koşulunu sağlayan $f(x, y)$, $f : \mathrm{T}^2 \to \mathbb{R}$ fonksiyonlarından oluşur.

3.2 Tanım [6]

$L^0_{p_1, p_2}$ fonksiyon sınıfı

hemen hemen her x için $\int_0^{2\pi} f(x, y) dy = 0$,

hemen hemen her y için $\int_0^{2\pi} f(x, y) dx = 0$, koşullarını sağlayan $f \in L_{p_1,p_2}$ fonksiyonlarından oluşur.

3.3 Tanım [7]

$n \in \mathbb{N}$ olmak üzere $L^{\infty}\left(T^{n}\right)$ demek

$$\|f\|_{L^{\infty}(T)} = \inf\left\{M \geq 0 : \mu\left(\left\{x : |f(x)| > M\right\}\right) = 0\right\} < \infty$$

koşulunu sağlayan esaslı sınırlı $f : \mathbb{R}^{n} \to \mathbb{R}$ fonksiyonlar kümesi demektir.

4. KARMA DÜZGÜNLÜK MODÜLÜ

4.1 Açısal Yaklaşım

4.1.1 Tanım [6]

$S_{m_1,\infty}(f)$, $S_{\infty,m_2}(f)$, $S_{m_1,m_2}(f)$ ifadeleri $f(x,y)$ nin sırasıyla x e göre m_1 dereceli, y ye göre m_2 dereceli ve hem x göre m_1 hem de y ye göre m_2 dereceli Fourier serilerinin kısmi toplamları olsun. Başka bir deyişle

$$m\in \mathbb{N}\ , D_m(t)=\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{t}{2}}$$

olmak üzere

$$S_{m_1,\infty}(f)=\frac{1}{\pi}\int_0^{2\pi} f(x+t_1,y)D_{m_1}(t_1)dt_1\,,$$

$$S_{\infty,m_2}(f)=\frac{1}{\pi}\int_0^{2\pi} f(x,y+t_2)D_{m_2}(t_2)dt_2\,,$$

$$S_{m_1,m_2}(f)=\frac{1}{\pi^2}\int_0^{2\pi} f(x+t_1,y+t_2)D_{m_1}(t_1)D_{m_2}(t_2)dt_1dt_2\,,\left(m_i\in\mathbb{N}\,,i=1,2\right)$$

dir.

4.1.2 Tanım [8]

$f\in L^1(\mathbb{T})$ fonksiyonun kompleks Fourier serisi

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}, c_0=0$$

ise $\rho>0$ mertebeli kesirli integral

$$\varphi_\alpha(x)=\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e^{inx}}{(in)^\rho}$$

olmak üzere

$$I^\alpha f(x):=(f*\varphi_\rho)(x)=\frac{1}{2\pi}\int_0^{2\pi} f(t)\varphi_\rho(x-t)\,dt$$

biçiminde tanımlanır.

4.1.1 Tanım [8]

$\rho>0$ mertebeli f kesirli türev $n:=\lfloor\rho\rfloor+1$ olmak üzere

$$f^{(\rho)}(x):=\frac{d^n}{dx^n}I^{n-\rho}f(x)$$

biçiminde tanımlanır.

4.1.2 Tanım [8]

$f^{(\rho_1,\rho_2)}$ ile x e göre $\rho_1\geq 0$ ve y ye göre $\rho_2\geq 0$ dereceli $f\in L_1^0(\mathrm{T}^2)$ nin Wely anlamında kesirli türevini ifade edelim.

$W_p^{(\alpha_1,0)}$ Wely sınıfı yani $f^{(\alpha_1,0)}\in L_p^0(\mathrm{T}^2)$ koşulunu sağlayan $f\in L_p^0(\mathrm{T}^2)$ fonksiyonlarının koleksiyonudur.

Benzer şekilde $W_p^{(0,\alpha_2)}$ Wely sınıfı $f^{(0,\alpha_2)}\in L_p^0(\mathrm{T}^2)$ koşulunu sağlayan $f\in L_p^0(\mathrm{T}^2)$ fonksiyonlarından oluşur.

Ayrıca $W_p^{(\alpha_1,\alpha_2)}$ sınıfı $f^{(\alpha_1,\alpha_2)}\in L_p^0(\mathrm{T}^2)$ koşulunu sağlayan $f\in L_p^0(\mathrm{T}^2)$ fonksiyonlarından oluşur.

4.1.5 Tanım [9]

$Y_{m_1,m_2}(f)_{p_1p_2}$, $f\in L_{p_1p_2}$ fonksiyonuna en iyi iki boyutlu açısal yaklaşımı veren iki değişkenli trigonometrik polinom olsun. Yani;

$$Y_{m_1,m_2}(f)_{p_1p_2} = \inf_{T_{m_1,\infty},T_{\infty,m_2}} \left\| f - T_{m_1,\infty} - T_{\infty,m_2} \right\|_{p_1p_2}.$$

Burada $T_{m,\infty}(x,y) \in L_{p_1p_2}$ x e göre derecesi m_1 i geçmeyen trigonometrik polinom, $T_{\infty,m_2}(x,y) \in L_{p_1p_2}$ y ye göre derecesi m_2 yi geçmeyen trigonometrik polinomdur.

4.1.6 Önerme [9]

$1 < p < \infty$, $f \in L_p^0(\mathrm{T}^2), n_i \in \mathbb{N}, i = 1,2$ alalım. Bu durumda

$$\left\| f - s_{m_1,\infty}(f) - s_{\infty,m_2}(f) + s_{m_1,m_2}(f) \right\|_{L_p(\mathrm{T}^2)} \ll Y_{n_1,n_2}(f)_{L_p(\mathrm{T}^2)}.$$

4.1.3 Önerme [10]

$1 < p < q < \infty$, $f \in L_p^0(\mathrm{T}^2)$, $\theta = \frac{1}{p} - \frac{1}{q}$, $N \in \mathbb{N}, i = 1,2$ alalım. Bu durumda

$$Y_{2^{N_1-1},2^{N_2-1}}(f)_{L_q(\mathrm{T}^2)} \le \left\{ \sum_{\nu_1=N_1}^{\infty} \sum_{\nu_2=N_2}^{\infty} 2^{(\nu_1+\nu_2)\theta q} Y^q_{2^{\nu_1-1},2^{\nu_2-1}}(f)_{L_p(\mathrm{T}^2)} \right\}^{1/q}.$$

4.1.8 Önerme [1]

$1 < p < \infty$, $f \in L_p^0(\mathrm{T})$ alalım. Bu durumda

$\left\| s_n(f) \right\|_{L_p(\mathrm{T})} \ll \left\| f \right\|_{L_p(\mathrm{T})}$, $n \in \mathbb{N}$ olur.

4.2 Fark Operatörleri

4.2.1 Tanım [6]

$f \in L_{p_1p_2}$ olmak üzere x e göre pozitif α_1 dereceli h_1 adımlı ve y ye göre α_2 dereceli h_2 adımlı fark operatörleri

$$\Delta_{h_1}^{\alpha_1}(f) = \sum_{\nu_1=0}^{\infty} (-1)^{\nu_1} C(\alpha_1, h_1) f(x + (\alpha_1 - \nu_1)h_1, y)$$

$$\Delta_{h_2}^{\alpha_2}(f) = \sum_{\nu_2=0}^{\infty} (-1)^{\nu_2} C(\alpha_2, h_2) f(x, y + (\alpha_2 - \nu_2)h_2)$$

olarak tanımlanır. Burada

$C(\alpha,0)=1$, $C(\alpha,1)=\alpha$

$C(\alpha,v)=\dfrac{\alpha(\alpha-1)\ldots(\alpha-v+1)}{v!}$,($v\geq 2$) binom katsayılarıdır.

4.2.2 Önerme [11,12]

$1<p<\infty$, $\alpha>0$ alalım ve T_n derecesi en fazla n , $n\in\mathbb{N}$ olan trigonometrik polinom olsun. Bu durumda

a) $\forall\ \ 0<|h|\leq\dfrac{\pi}{n}$ için $\left\|\Delta_h^{\alpha}T_n\right\|_{L_p(\mathrm{T})}\leq n^{-\alpha}\left\|T_n^{(\alpha)}\right\|_{L_p(\mathrm{T})}$,

b) $\left\|T_n^{(\alpha)}\right\|_{L_p(\mathrm{T})}\leq n^{\alpha}\left\|\Delta_{\pi/n}^{\alpha}T_n\right\|_{L_p(\mathrm{T})}$.

4.2.3 Önerme [12]

$\alpha>0$ olsun. Bu durumda

$\sum\limits_{v=0}^{\infty}(-1)^{v}C(\alpha,v)=0$.

4.2.4 Önerme [11,12]

$f\in L_{p_1p_2}^{0}$, $g\in L_{p_1p_2}^{0}$,$1\leq p_i\leq\infty$, $\alpha,\beta>0$ olsun. Bu durumda

a) $\Delta_h^{\alpha}(f+g)=\Delta_h^{\alpha}f+\Delta_h^{\alpha}g$,

b) $\Delta_h^{\alpha}\left(\Delta_h^{\beta}f\right)=\Delta_h^{\alpha+\beta}f$,

c) $\left\|\Delta_h^{\alpha}f\right\|_{p_1p_2}\lesssim\|f\|_{p_1p_2}$.

4.2.5 Önerme [13]

$a_1>0$, $T_{n_1\infty}(x_1,x_2)\in L_{p_1p_2}$, $1\leq p_i\leq\infty, i=1,2$ x_1 değişkenine göre derecesi $n_1(n_1\in N)$ yi aşmayan trigonometrik bir polinom olsun. Bu durumda aşağıdaki eşitsizlikler geçerlidir.

a)Eğer $0<|h_1|\le \dfrac{\pi}{n_1}$ ise $\left\|\Delta_{h_1}^{\alpha_1} T_{n_1\infty}\right\|_{p_1p_2} \ll n_1^{-\alpha_1}\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2}$,

b) $\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2} \ll n_1^{\alpha_1}\left\|\Delta_{\pi/n_1}^{\alpha_1} T_{n_1\infty}\right\|_{p_1p_2}$,

c) $\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2} \ll n_1^{\alpha_1}\left\|T_{n_1\infty}\right\|_{p_1p_2}$.

İspat a)

Öncelikle $0<a_1\le 1, 0<|h_1|\le \dfrac{\pi}{n_1}$ olsun.

[12] numaralı kaynağın 397. sayfasındaki sonuca göre hemen hemen her x_2 için

$$\Delta_{h_1}^{\alpha_1} T_{n_1\infty}\left(x_1-\frac{\alpha_1}{2}h_1, x_2\right)=\sum_{k=-\infty}^{+\infty} d_k\left(n_1\right) T_{n_1\infty}^{(\alpha_1,0)}\left(x_1+\frac{k\pi}{n_1}, x_2\right)$$

ifadesi $\sum\limits_{k=-\infty}^{+\infty}\left|d_k\left(n_1\right)\right| \ll n_1^{-a_1}$ iken sağlanır.

Minkowski eşitsizliğini uygularsak,

$$\left\|\Delta_{h_1}^{\alpha_1} T_{n_1\infty}\right\|_{p_1p_2} \ll \left\|\Delta_{h_1}^{\alpha_1} T_{n_1\infty}\left(x_1-\frac{\alpha_1}{2}h_1, x_2\right)\right\|_{p_1p_2} \ll \sum_{k=-\infty}^{+\infty}\left|d_k\left(n_1\right)\right|\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2} \ll n_1^{-\alpha_1}\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2}.$$

Şimdi $a_1>1$ alalım. $0<\beta_1\le 1, m_1\in N$ olmak üzere $a_1=m_1+\beta_1$ bulunabilir. 4.2.4. önermesinin c şıkkını uygulayıp son eşitsizliği m_1 kez uygularsak

$$\left\|\Delta_{h_1}^{\alpha_1} T_{n_1\infty}\right\|_{p_1p_2} = \left\|\Delta_{h_1}^{1}\left(\Delta_{h_1}^{m_1-1+\beta_2} T_{n_1\infty}\right)\right\|_{p_1p_2} \ll n_1^{-\alpha_1}\left\|\Delta_{h_1}^{m_1-1+\beta_2} T_{n_1\infty}^{(1,0)}\right\|_{p_1,p_2}$$

$\ll \ldots \ll n_1^{-m_1}\left\|\Delta_{h_1}^{\beta_1} T_{n_1\infty}^{(m_1,0)}\right\|_{p_1p_2} \ll n_1^{-\alpha_1}\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2}$ elde ederiz ve dolayısıyla (a) ispatlanmış olur.

İspat b)

[12] numaralı kaynağın 392 sayfasındaki sonuca göre

$$T_{n_1\infty}^{(\alpha_1,0)}(x_1,x_2)=\sum_{k=-\infty}^{+\infty} b_k(n_1)\Delta_{\frac{\pi}{n_1}}^{\alpha_1} T_{n_1\infty}\left(x_1+\frac{k\pi}{n_1}-\frac{\alpha_1\pi}{2n_1},x_2\right)$$

ifadesi hemen hemen her x_2 için $\sum\limits_{k=-\infty}^{+\infty}\left|b_k(n_1)\right| \ll n_1^{\alpha_1}$ iken sağlanır.

Minkowski eşitsizliği kullanılırsa

$$\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2} \ll \sum_{k=-\infty}^{+\infty}\left|b_k(n_1)\right|\left\|\Delta_{\pi/n_1}^{\alpha_1} T_{n_1\infty}\left(x_1+\frac{k\pi}{n_1}-\frac{\alpha_1\pi}{2n_1},x_2\right)\right\|_{p_1p_2} \ll n_1^{\alpha_1}\left\|\Delta_{\pi/n_1}^{\alpha_1} T_{n_1\infty}\right\|_{p_1p_2}$$

elde edilir ve bu da ispatı tamamlar.

İspat c)

4.2.4 önermesinin c şıkkına göre ifade $\left\|T_{n_1\infty}^{(\alpha_1,0)}\right\|_{p_1p_2} \ll n_1^{\alpha_1}\left\|T_{n_1\infty}\right\|_{p_1p_2}$ olur ve bu da c şıkkının ispatını tamamlar.

4.2.6 Önerme [13]

$\alpha_2>0$, $T_{\infty,n_2}(x_1,x_2)\in L_{p_1p_2}, 1\le p_i\le\infty, i=1,2$ x_2 değişkenine göre derecesi $n_2\left(n_2\in N\right)$ yi aşmayan trigonometrik polinom olsun. Bu durumda aşağıdaki eşitsizlikler geçerlidir.

a)Eğer $0<\left|h_2\right|\le\dfrac{\pi}{n_2}$ ise $\left\|\Delta_{h_2}^{\alpha_2} T_{\infty,n_2}\right\|_{p_1p_2} \ll n_2^{-\alpha_2}\left\|T_{\infty,n_2}^{\alpha_2}\right\|_{p_1p_2}$,

b) $\left\|T_{\infty,n_2}^{(0,\alpha_2)}\right\|_{p_1p_2} \ll n_2^{\alpha_2}\left\|\Delta_{\pi/n_2}^{\alpha_2} T_{\infty,n_2}\right\|_{p_1p_2}$,

c) $\left\|T_{\infty,n_2}^{(0,\alpha_2)}\right\|_{p_1p_2} \ll n_2^{\alpha_2}\left\|T_{\infty,n_2}\right\|_{p_1p_2}$.

4.3 K Fonksiyoneli

4.3.1 Önerme [13]

$f\in L_{p_1p_2}^0$ fonksiyonu ve $t_i>0, i=1,2$ için karma K fonksiyoneli

$$K_{\alpha_1,\alpha_2}(f,t_1,t_2)_{p_1p_2} := \inf_{g_1\in W_{p_1p_2}^{(\alpha_1,0)},\, g_2\in W_{p_1p_2}^{(0,\alpha_2)},\, g\in W_{p_1p_2}^{(\alpha_1,\alpha_2)}}\Big[\left\|f-g_1-g_2-g\right\|_{p_1p_2}$$

$$+t_1^{\alpha_1}\left\|g_1^{(\alpha_1,0)}\right\|_{p_1p_2}+t_2^{\alpha_2}\left\|g_2^{(0,\alpha_2)}\right\|_{p_1p_2}+t_1^{\alpha_1}t_2^{\alpha_2}\left\|g_1^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}\Bigg]$$

olarak tanımlanır.

4.4 Bazı Özel Fonksiyon Sınıfları

4.4.1 Tanım [8]

$1<p<\infty$ olmak üzere M_p sınıfı öyle $f\in L_p^0\left(\mathrm{T}^2\right)$ fonksiyonlarından oluşur ki f nin Fourier serisi $\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty}\alpha_{\nu_1\nu_2}\cos\nu_1 x\cos\nu_2 y$ dir ve her ν_1 ve ν_2 tamsayısı için

$$\alpha_{\nu_1\nu_2}-\alpha_{\nu_1+1,\nu_2}-\alpha_{\nu_1,\nu_2+1}+\alpha_{\nu_1+1+\nu_2+1}\geq 0$$

eşitsizliği sağlanır.

4.4.2 Not [8]

Bu eşitsizlik bize $m_1\leq m_2$ için $\alpha_{n,m_1}\geq\alpha_{n,m_2}$ ve $n_1\leq n_2$ için $\alpha_{n_1,m}\geq\alpha_{n_2,m}$ eşitsizliğini verir.

4.4.3 Tanım [8]

$1<p<\infty$ olmak üzere, Λ_p fonksiyon sınıfı Fourier serisi, $\lambda_{\mu_1,\mu_2}\in\mathbb{R}$

$$\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty}\lambda_{\mu_1,\mu_2}\cos 2^{\mu_1}x\cos 2^{\mu_2}y$$

biçiminde olan $f\in L_p^0\left(\mathrm{T}^2\right)$ fonksiyonlarından oluşur.

4.5 Vallee-Poussin Ortalamaları

4.5.1 Tanım [13]

$V_{m_1,\infty}(f), V_{\infty,m_2}(f), V_{m_1,m_2}(f)$ ile sırasıyla x e göre, y ye göre, hem x hem y ye göre $f(x,y)$ fonksiyonunun Fourier serisinin Vallee-Poussin ortalamalarını gösterelim. Yani

$$V_0^0(t) = D_0(t), V_n^{2n}(t) = \frac{D_n(t) + \ldots + D_{2n-1}(t)}{n}, n = 1,2,\ldots,\ D_m(t) = \frac{\sin\left(m + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}, m = 0,1,2\ldots;$$

olmak üzere

$$V_{m_1,\infty}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x+t_1, y) V_{m_1}^{2m_1}(t_1)\,dt_1,$$

$$V_{\infty,m_2}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x, y+t_2) V_{m_2}^{2m_2}(t_2)\,dt_2,$$

$$V_{m_1,m_2}(f) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x+t_1, y+t_2) V_{m_1}^{2m_1}(t_1) V_{m_2}^{2m_2}(t_2)\,dt_1 dt_2 \quad (m_i = 0,1,2,\ldots, i = 1,2).$$

4.5.1 Önerme [14]

$f \in L_{p_1p_2}^0$, $1 \le p_i \le \infty$, $k_i \in N$, $n_i \in \{0,1,2,\ldots\}$, $i = 1,2$ olsun. Bu durumda

$$\left\| f - V_{n_1,\infty}(f) - V_{\infty,n_2}(f) + V_{n_1,n_2}(f) \right\|_{p_1p_2} \ll Y_{n_1,n_2}(f)_{p_1p_2} \ll \omega_{k_1,k_2}\left(f, \frac{\pi}{n_1+1}, \frac{\pi}{n_2+1}\right)_{p_1p_2}$$ olur.

4.5.2 Önerme [13]

$f \in L_{p_1p_2}^0$, $1 \le p_i \le \infty$, $n_i \in \{0,1,2,\ldots\}$, $i = 1,2$ olsun. Bu durumda

a) $\left\| V_{n_1,\infty}(f) \right\|_{p_1p_2} \ll \| f \|_{p_1p_2}$,

b) $\left\| V_{\infty,n_2}(f) \right\|_{p_1p_2} \ll \| f \|_{p_1p_2}$.

İspat a)

$m = 0,1,2,\ldots$ için $\frac{1}{\pi}\int_0^{2\pi} \left| V_m^{2m}(t) \right| dt \le 3$ olduğundan genelleştirilmiş Minkowski eşitsizliğini uygulayarak

$$\left\|V_{n_1,\infty}(f)\right\|_{p_1p_2}=\left\|\frac{1}{\pi}\int_0^{2\pi} f\left(x_1+t_1,x_2\right)V_{n_1}^{2n_1}\left(t_1\right)dt_1\right\|_{p_1p_2}\leq\frac{1}{\pi}\int_0^{2\pi}\left|V_{n_1}^{2n_1}\left(t_1\right)\right|\left\|f\left(x_1+t_1,x_2\right)\right\|_{p_1p_2}\leq 3\left\|f\right\|_{p_1p_2}$$

elde edilir.

İspat b)

$m=0,1,2,...$ için $\frac{1}{\pi}\int_0^{2\pi}\left|V_m^{2m}(t)\right|dt\leq 3$ olduğundan genelleştirilmiş Minkowski eşitsizliğini uygulayarak

$$\left\|V_{\infty,n_2}(f)\right\|_{p_1p_2}=\left\|\frac{1}{\pi}\int_0^{2\pi} f\left(x_1,x_2+t_2\right)V_{n_2}^{2n_2}\left(t_2\right)dt_2\right\|_{p_1p_2}\leq\frac{1}{\pi}\int_0^{2\pi}\left|V_{n_2}^{2n_2}\left(t_1\right)\right|\left\|f\left(x_1,x_2+t_2\right)\right\|_{p_1p_2}\leq 3\left\|f\right\|_{p_1p_2}$$

elde edilir.

4.5.3 Önerme [13]

$f\in L_{p_1p_2}, 1\leq p_i\leq\infty, m_i\in\{0,1,2,...\}, \alpha_i>0, i=1,2$ olsun. Bu durumda

a) $\left\|V_{2^{m_1+1},\infty}(f)-V_{2^{m_1},\infty}(f)\right\|_{p_1p_2}\asymp 2^{-m_1\alpha_1}\left\|V_{2^{m_1+1},\infty}^{(\alpha_1,0)}(f)-V_{2^{m_1},\infty}^{(\alpha_1,0)}(f)\right\|_{p_1p_2}$;

b) $\left\|V_{\infty,2^{m_2+1}}(f)-V_{\infty,2^{m_2}}(f)\right\|_{p_1p_2}\asymp 2^{-m_2\alpha_2}\left\|V_{\infty,2^{m_2+1}}^{(0,\alpha_2)}(f)-V_{\infty,2^{m_2}}^{(0,\alpha_2)}(f)\right\|_{p_1p_2}$;

c) $\left\|V_{2^{m_1},2^{m_2}}(f)-V_{2^{m_1+1},2^{m_2}}(f)-V_{2^{m_1},2^{m_2+1}}(f)+V_{2^{m_1+1},2^{m_2+1}}(f)\right\|_{p_1p_2}\asymp$

$\asymp 2^{-m_1\alpha_1-m_2\alpha_2}\left\|\left(V_{2^{m_1},2^{m_2}}(f)-V_{2^{m_1+1},2^{m_2}}(f)-V_{2^{m_1},2^{m_2+1}}(f)+V_{2^{m_1+1},2^{m_2+1}}(f)\right)^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$.

4.5.4 Önerme [13]

$f\in L_{p_1p_2}^0, 1\leq p_i\leq\infty, \alpha_i>0, n_i=\mathbb{N}, i=1,2$ alalım. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{2n_1-1},\frac{\pi}{2n_2-1}\right)\approx n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|V_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2}+n_1^{-\alpha_1}\left\|V_{n_1,\infty}^{(\alpha_1,0)}\left(f-V_{\infty,n_2}(f)\right)\right\|_{p_1p_2}$$

$$+n_2^{-\alpha_2}\left\|V_{\infty,n_2}^{(0,\alpha_2)}\left(f-V_{n_1,\infty}(f)\right)\right\|_{p_1p_2}+\left\|f-V_{n_1,\infty}(f)-V_{\infty.n_2}(f)+V_{n_1,n_2}(f)\right\|_{p_1p_2}.$$

İspat

4.2.4 önermesinin a ve b şıkları kullanılarak ve norm fonksiyonunun özellikleriyle, her h_i ve $n_i \in N$, $i = 1, 2$ için

$$\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(f)\right)\right\|_{p_1 p_2} \le \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(f - V_{n_1,\infty}(f) - V_{\infty,n_2}(f) + V_{n_1,n_2}(f)\right)\right)\right\|_{p_1 p_2} +$$

$$+\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{n_1,\infty}\left(f - V_{\infty,n_2}\right)\right)\right)\right\|_{p_1 p_2} + \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{\infty,n_2}\left(f - V_{n_1,\infty}\right)\right)\right)\right\|_{p_1 p_2} +$$

$$+\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{n_1,n_2}(f)\right)\right)\right\|_{p_1 p_2} := I_1 + I_2 + I_3 + I_4 .$$

Öncelikle I_1 üstten değerlendirelim. $\varphi(x, y) = f - V_{n_1,\infty}(f) - V_{\infty,n_2}(f) + V_{n_1,n_2}(f)$ olsun. 4.2.4 önermesinin c şıkkını kullanırsak

$$I_1 = \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(\varphi)\right)\right\|_{p_1 p_2} \ll \left\|\Delta_{h_2}^{\alpha_2}(\varphi)\right\|_{p_1 p_2} \ll \|\varphi\|_{p_1 p_2} .$$

Bu nedenle $I_1 \ll \left\|f - V_{n_1,\infty}(f) - V_{\infty,n_2}(f) + V_{n_1,n_2}(f)\right\|_{p_1 p_2}$.

Şimdi I_2 yi üstten değerlendirelim. $\psi = f - W_{\infty,n_2}(f)$ olsun. 4.2.4 önermeye göre

$$I_2 = \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{n_1,\infty}(\psi)\right)\right)\right\|_{p_1 p_2} \ll \left\|\Delta_{h_1}^{\alpha_1}\left(V_{n_1,\infty}(\psi)\right)\right\|_{p_1 p_2} .$$

4.2.5 önerme yardımıyla $0 < |h_1| \le \dfrac{\pi}{2n_1 - 1}$ için aşağıdaki eşitsizlikler geçerlidir:

$$\left\|\Delta_{h_1}^{\alpha_1}\left(V_{n_1,\infty}(\psi)\right)\right\|_{p_1 p_2} \ll n_1^{-\alpha_1}\left\|V_{n_1,\infty}^{(\alpha_1,0)}(\psi)\right\|_{p_1 p_2} .$$

Bu nedenle $0 < |h_1| \le \dfrac{\pi}{2n_1 - 1}$ için $I_2 \ll n_1^{-\alpha_1}\left\|V_{n_1,\infty}^{(\alpha_1,0)}\left(f - V_{\infty,n_2}(f)\right)\right\|_{p_1 p_2}$.

Genellersek $0 < |h_1| \le \dfrac{\pi}{2n_1 - 1}$, $0 < |h_2| \le \dfrac{\pi}{2n_2 - 1}$ için

$$I_3 \square \; n_2^{-\alpha_2}\left\|V_{\infty,n_2}^{(0,\alpha_2)}\left(f-V_{n_1,\infty}(f)\right)\right\|_{p_1p_2};\; I_4 \square \; n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|V_{n_1,n_2}^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$$ geçerlidir.

Bu sebeple

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{2n_1-1},\frac{\pi}{2n_2-1}\right)_{p_1p_2} \square \; \left\|f-V_{n_1,\infty}(f)-V_{\infty,n_2}(f)+V_{n_1,n_2}(f)\right\|_{p_1p_2}+$$

$$+n_1^{-\alpha_1}\left\|V_{n_1,\infty}^{(\alpha_1,0)}\left(f-V_{\infty,n_2}(f)\right)\right\|_{p_1p_2}+n_2^{-\alpha_2}\left\|V_{\infty,n_2}^{(0,\alpha_2)}(f-)V_{n_1,\infty}(f)\right\|_{p_1p_2}+$$

$$+n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|V_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2}. \tag{4.1}$$

Böylece 4.5.5 teoremin üst eşitsizliği ispatlanmış olur.

Şimdi alt eşitsizliği ispatlayalım. 4.5.2 önerme nedeniyle ve doğal sayı mertebeli düzgünlük modülünün özellikleri kullanılırsa

$$A_1 := \left\|f-V_{n_1,\infty}(f)-V_{\infty,n_2}(f)+V_{n_1,n_2}(f)\right\|_{p_1p_2} \square$$

$$\square \; \omega_{[\alpha_1]+1},\omega_{[\alpha_2]+1}\left(f,\frac{\pi}{n_1+1},\frac{\pi}{n_2+1}\right)_{p_1p_2} \square \; \omega_{[\alpha_1]+1},\omega_{[\alpha_2]+1}\left(f,\frac{\pi}{2n_1-1},\frac{\pi}{2n_2-1}\right)_{p_1p_2}.$$

4.2.4 önermesinin b şıkkını uygularsak

$$A_1 \le \sup_{|h_i|\le\frac{\pi}{2n_i-1},i=1,2}\left\|\Delta_{h_1}^{[\alpha_1]+1-\alpha_1}\left(\Delta_{h_2}^{[\alpha_2]+1-\alpha_2}\left(\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\right)\right)\right)\right\|_{p_1p_2}.$$

4.2.4 önermesinin c şıkkını uygularsak

$$A_1 \le \sup_{|h_i|\le\frac{\pi}{2n_i-1},i=1,2}\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(f)\right)\right\|_{p_1p_2}=\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{2n_1-1},\frac{\pi}{2n_2-1}\right)_{p_1p_2}.$$

Şimdi A_2 yi değerlendirelim.

$$A_2=\left\|V_{n_1,\infty}^{(\alpha_1,0)}\left(f-V_{\infty,n_2}(f)\right)\right\|_{p_1p_2}.$$

$\gamma(x,y) = f(x,y) - V_{\infty,n_2} f(x,y)$ olsun. Bu durumda 4.2.5 önermesinin b şıkkını uygularsak $\left\| V_{n_1,\infty}^{(\alpha_1,0)}(\gamma) \right\|_{p_1 p_2} \ll n_1^{\alpha_1} \left\| \Delta_{\frac{\pi}{2n_1-1}}^{\alpha_1} \left(V_{n_1,\infty}(\gamma) \right) \right\|_{p_1 p_2}$ ve dolayısıyla

$$A_2 \ll n_1^{\alpha_1} \left\| V_{n_1,\infty} \left(\Delta_{\frac{\pi}{2n_1-1}}^{\alpha_1} (\gamma) \right) \right\|_{p_1 p_2} .$$

4.5.3 önermesinin a şıkkını uygularsak

$$A_2 \ll n_1^{\alpha_1} \left\| \Delta_{\frac{\pi}{2n_1-1}}^{\alpha_1} \left(f - V_{\infty,n_2}(f) \right) \right\|_{p_1 p_2} .$$

$\Delta_{\frac{\pi}{2n_1-1}}^{\alpha_1}(f) := F$ olsun. Bu durumda $A_2 \ll n_1^{\alpha_1} \left\| F - V_{\infty,n_2}(F) \right\|_{p_1 p_2}$.

$V_{0,\infty}(F) = V_{0,n_2}(F) = 0$ olduğundan

$$A_2 \ll n_1^{\alpha_1} \left\| F - V_{0,\infty}(F) - V_{\infty,n_2}(F) + V_{0,n_2}(F) \right\|_{p_1 p_2} .$$

4.5.2 önermesini hesaba katarsak ve doğal sayı mertebeli düzgünlük modülünün özelliklerini kullanırsak

$$A_2 \ll n_1^{\alpha_1} \omega_{[\alpha_1]+1,[\alpha_2]+1} \left(F, \pi, \frac{\pi}{n_2+1} \right)_{p_1 p_2} \ll n_1^{\alpha_1} \omega_{[\alpha_1]+1,[\alpha_2]+1} \left(F, \pi, \frac{\pi}{2n_2-1} \right)_{p_1 p_2} .$$

4.2.4 önermesinin b şıkkına göre

$$A_2 \ll n_1^{\alpha_1} \sup_{|h_1| \le \pi, |h_2| \le \frac{\pi}{n_2+1}} \left\| \Delta_{h_1}^{[\alpha_1]+1} \left(\Delta_{h_2}^{[\alpha_2]+1-\alpha_2} \left(\Delta_{h_2}^{\alpha_2}(F) \right) \right) \right\|_{p_1 p_2} .$$

4.2.4 önermesinin c şıkkı nedeniyle

$$A_2 \ll n_1^{\alpha_1} \sup_{|h_2| \le \frac{\pi}{2n_2-1}} \left\| \Delta_{h_2}^{\alpha_2}(F) \right\|_{p_1 p_2} = n_1^{\alpha_1} \sup_{|h_2| \le \frac{\pi}{2n_2-1}} \left\| \Delta_{h_2}^{\alpha_2} \left(\Delta_{\frac{\pi}{2n_2-1}}^{\alpha_1}(f) \right) \right\|_{p_1 p_2} \ll$$

$$\ll n_1^{\alpha_1} \omega_{\alpha_1,\alpha_2} \left(f, \frac{\pi}{2n_1-1}, \frac{\pi}{2n_2-1} \right)_{p_1 p_2} .$$

Aynı şekilde devam edip genellersek

$$\left\| V_{\infty,n_2}^{(0,\alpha_2)}\left(f - V_{n_1,\infty}(f)\right)\right\|_{p_1p_2} \ll n_2^{\alpha_2}\omega_{\alpha_1,\alpha_2}\left(f, \frac{\pi}{2n_1-1}, \frac{\pi}{2n_2-1}\right)_{p_1p_2},$$

$$\left\| V_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2} \ll n_1^{\alpha_1}n_2^{\alpha_2}\omega_{\alpha_1,\alpha_2}\left(f, \frac{\pi}{2n_1-1}, \frac{\pi}{2n_2-1}\right)_{p_1p_2} \quad \text{elde edilir.}$$

Böylece

$$\begin{aligned}
&\left\| f - V_{n_1,\infty}(f) - V_{\infty,n_2}(f) + V_{n_1,n_2}(f)\right\|_{p_1p_2} + n_1^{-\alpha_1}\left\| V_{n_1,\infty}^{(\alpha_1,0)}\left(f - V_{\infty,n_2}(f)\right)\right\|_{p_1p_2} \\
&+ n_2^{-\alpha_2}\left\| V_{\infty,n_2}^{(0,\alpha_2)}\left(f - V_{n_1,\infty}(f)\right)\right\|_{p_1p_2} + n_1^{-\alpha_1}n_2^{-\alpha_2}\left\| V_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2} \ll \\
&\ll \omega_{\alpha_1,\alpha_2}\left(f, \frac{\pi}{2n_1-1}, \frac{\pi}{2n_2-1}\right)_{p_1p_2}.
\end{aligned} \tag{4.2}$$

(4.1) ve (4.2) birlikte düşünülürse 2.9 tanıma göre ispat gerçekleşir.

4.6 Karma Düzgünlük Modülü

4.6.1 Tanım [6]

$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}$ ile x e göre pozitif α_1 dereceli, y ye göre pozitif α_2 dereceli karma kesirli düzgünlük modülünü gösterelim. Yani

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2} = \sup_{|h_i|<\delta_i,\, i=1,2}\left\| \Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(f)\right)\right\|_{p_1p_2}$$

dir.

4.6.2 Teorem [8]

Karma Düzgünlük Modülünün özellikleri aşağıda gösterilmiştir.

$f, g \in L_p(\mathrm{T}^2)$, $1 < p < \infty$, $a_i > 0$, $i = 1,2$ alalım.

1) $\omega_{\alpha_1,\alpha_2}(f,\delta_1,0)_{L_p(\mathrm{T}^2)} = \omega_{\alpha_1,\alpha_2}(f,0,\delta_2)_{L_p(\mathrm{T}^2)} = \omega_{\alpha_1,\alpha_2}(f,0,0)_{L_p(\mathrm{T}^2)} = 0$,

2) $\omega_{\alpha_1,\alpha_2}(f+g,\delta_1,\delta_2)_{L_p(\mathrm{T}^2)} \le \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathrm{T}^2)} + \omega_{\alpha_1,\alpha_2}(g,\delta_1,\delta_2)_{L_p(\mathrm{T}^2)}$,

3) $0 < \delta_i \le t_i, i = 1,2$ için $\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathrm{T}^2)} \le \omega_{\alpha_1,\alpha_2}(f,t_1,t_2)_{L_p(\mathrm{T}^2)}$,

4) $0 < \delta_i \le t_i, i = 1,2$ için $\dfrac{\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(\mathrm{T}^2)}}{\delta_1^{\alpha_1}\delta_2^{\alpha_2}} \le \dfrac{\omega_{\alpha_1,\alpha_2}(f,t_1,t_2)_{L_p(\mathrm{T}^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}}$,

5) $\lambda_i > 1, i = 1,2$ için $\omega_{\alpha_1,\alpha_2}(f, \lambda_1\delta_1, \lambda_2\delta_2)_{L_p(T^2)} \le \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(T^2)}$,

6) $0 < \alpha_i < \beta_i, i = 1,2$ için $\omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(T^2)} \le \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(T^2)}$,

7) $0 < \alpha_i < \beta_i$, $0 < \delta_i \le \frac{1}{2}$, $i = 1,2$ için,

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{L_p(T^2)} \le \delta_1^{\alpha_1}\delta_2^{\alpha_2}\int_{\delta_1}^{1}\int_{\delta_2}^{1}\frac{\omega_{\beta_1,\beta_2}(f,\delta_1,\delta_2)_{L_p(T^2)}}{t_1^{\alpha_1}t_2^{\alpha_2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2},$$

8) $\beta_i, r_i > 0, i = 1,2$ için

$$\omega_{\beta_1+r_1,\beta_2+r_2}(f,\delta_1,\delta_2)_{L_p(T^2)} \le \delta_1^{r_1}\delta_2^{r_2}\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p(T^2)},$$

9) $\beta_i, r_i > 0, i = 1,2$ için

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p(T^2)} \le \int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1}t_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}(f,t_1,t_2)_{L_p(T^2)}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

4.6.3 Teorem [13]

Karma Düzgünlük Modülü ile K-Fonksiyonelinin denkliği aşağıdaki gibi gösterilir.

$f \in L^0_{p_1p_2}, 1 \le p_i \le \infty$, $\alpha_i > 0, 0 < \delta_i \le \pi$, $i = 1,2$ alalım. Bu durumda

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2} \approx K_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}.$$

İspat

Her $\delta_i \in (0,\pi]$ için öyle n_i, $i = 1,2$ doğal sayısı vardır ki $\frac{\pi}{2n_i+1} < \delta_i \le \frac{\pi}{2n_i-1}, i = 1,2.$

Eğer $f \in L^0_{p_1p_2}$ ise

$$\left(V_{n_1+1,\infty}(f) - V_{n_1+1,n_2+1}(f)\right) \in W^{(\alpha_1,0)}_{p_1p_2}, \left(V_{\infty,n_2+1}(f) - V_{n_1+1,n_2+1}(f)\right) \in W^{(0,\alpha_2)}_{p_1p_2}, V_{n_1+1,n_2+1}(f) \in W^{(\alpha_1,\alpha_2)}_{p_1p_2}.$$

Bu nedenle

$$K_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2} \le \left\| f - \left(V_{n_1+1,\infty}(f) - V_{n_1+1,n_2+1}(f)\right) - \left(V_{\infty,n_2+1}(f) - V_{n_1+1,n_2+1}(f)\right) - V_{n_1+1,n_2+1}(f)\right\|_{p_1p_2} +$$

$$+\delta_1^{\alpha_1}\left\|V^{(\alpha_1,0)}_{n_1+1,\infty}(f) - V^{(\alpha_1,0)}_{n_1+1,n_2+1}(f)\right\|_{p_1p_2} + \delta_2^{\alpha_2}\left\|V^{(0,\alpha_2)}_{\infty,n_2+1}(f) - V^{(0,\alpha_2)}_{n_1+1,n_2+1}(f)\right\|_{p_1p_2} + \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\|V^{(\alpha_1,\alpha_2)}_{n_1+1,n_2+1}(f)\right\|_{p_1p_2}$$

$$= \left\| f - V_{n_1+1,\infty}(f) - V_{\infty,n_2+1}(f) + V_{n_1+1,n_2+1}(f)\right\|_{p_1p_2} + \delta_1^{\alpha_1}\left\|V^{(\alpha_1,0)}_{n_1+1,\infty}\left(f - V_{\infty,n_2+1}(f)\right)\right\|_{p_1p_2}$$

$$+\delta_2^{\alpha_2}\left\|V_{\infty,n_2+1}^{(0,\alpha_2)}\left(f-V_{n_1+1,\infty}(f)\right)\right\|_{p_1p_2}+\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\|V_{n_1+1,n_2+1}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2}$$

$$\ll \left\|f-V_{n_1+1,\infty}(f)-V_{\infty,n_2+1}(f)+V_{n_1+1,n_2+1}(f)\right\|_{p_1p_2}+(n_1+1)^{-\alpha_2}\left\|V_{n_1+1,\infty}^{(\alpha_1,0)}\left(f-V_{\infty,n_2+1}(f)\right)\right\|_{p_1p_2}$$

$$+(n_2+1)^{-\alpha_2}\left\|V_{\infty,n_2+1}^{(0,\alpha_2)}\left(f-V_{n_1+1,\infty}(f)\right)\right\|_{p_1p_2}+(n_1+1)^{-\alpha_2}(n_2+1)^{-\alpha_2}\left\|V_{n_1+1,n_2+1}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2}.$$

4.5.5 teoremi uygularsak ve karma düzgünlük modülünün tanımı göz önüne alınırsa

$$K_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}\ll \omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{2n_1-1},\frac{\pi}{2n_2-1}\right)_{p_1p_2}\ll \tag{4.3}$$

$$\ll \omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}$$

Şimdi son eşitsizliğin tersini ispatlayalım. 4.2.4 önermesinin a şıkkı ve normun özelliklerine göre her $g_1\in W_{p_1p_2}^{(\alpha_1,0)}, g_2\in W_{p_1p_2}^{(0,\alpha_2)}, g\in W_{p_1p_2}^{(\alpha_1,\alpha_2)}$ için

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}\ll \omega_{\alpha_1,\alpha_2}(f-g_1-g_2-g,\delta_1,\delta_2)_{p_1p_2}+\omega_{\alpha_1,\alpha_2}(g_1,\delta_1,\delta_2)_{p_1p_2}+$$

$$+\omega_{\alpha_1,\alpha_2}(g_2,\delta_1,\delta_2)_{p_1p_2}+\omega_{\alpha_1,\alpha_2}(g,\delta_1,\delta_2)_{p_1p_2}:=J_1+J_2+J_3+J_4.$$

4.2.4 önermesinin c şıkkı bize $J_1\ll \left\|f-g_1-g_2-g\right\|_{p_1p_2}$ eşitsizliğini verir.

Şimdi J_2 yi hesaplamaya çalışalım. Her $\delta_i\in(0,\pi]$ için negatif olmayan n_i tamsayıları vardır öyle ki

$$\frac{\pi}{2^{n_i+2}-1}<\delta_i\le\frac{\pi}{2^{n_i+1}-1}, i=1,2.$$

$B_2=\omega_{\alpha_1,\alpha_2}\left(g_1,\frac{\pi}{2^{n_1+1}-1},\frac{\pi}{2^{n_2+1}-1}\right)_{p_1p_2}$ ifadesini düşünelim. 4.2.4 önermesinin a ve c şıklarına göre

$$B_2\ll \omega_{\alpha_1,\alpha_2}\left(g_1-V_{2^{n_1},\infty}(g_1),\frac{\pi}{2^{n_1+1}-1},\frac{\pi}{2^{n_2+1}-1}\right)_{p_1p_2}+\omega_{\alpha_1,\alpha_2}\left(V_{2^{n_1},\infty}(g_1),\frac{\pi}{2^{n_1+1}-1},\frac{\pi}{2^{n_2+1}-1}\right)_{p_1p_2}$$

$$\left\|g_1-V_{2^{n-1},\infty}(g_1)\right\|_{p_1p_2}+\sup_{|h_1|\le\frac{\pi}{2^{n_1+1}-1}}\left\|\Delta_{h_1}^{\alpha_1}\left(V_{2^{n_1},\infty}(g_1)\right)\right\|_{p_1p_2}:=J_{21}+J_{22}.$$

4.2.5 önermesinin a şıkkını uygulayıp sonra 4.5.3 önermesinin a şıkkını uygularsak her $0 < |h_i| \le \frac{\pi}{2^{n_1+1}-1}$ için

$$\left\| \Delta_{h_1}^{\alpha_1} V_{2^{n_1},\infty}(g_1) \right\|_{p_1 p_2} \ll 2^{-n_1\alpha_1} \left\| V_{2^{n_1},\infty}^{(\alpha_1,0)}(g_1) \right\|_{p_1 p_2} = 2^{-n_1\alpha_1} \left\| V_{2^{n_1},\infty}\left(g_1^{(\alpha_1,0)}\right) \right\|_{p_1 p_2} \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}$$

bulunur ve böylece $J_{22} \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}$.

4.5.4 önermesinin a şıkkı ve 4.5.3 önermesinin a şıkkını kullanırsak ve yukarıda elde edilen hesaplamalar göz önünde bulundurulursa aşağıdaki eşitsizliklerin geçerli olduğunu elde edebiliriz.

$$\left\| g_1 - V_{2^{n_1},\infty}(g_1) \right\|_{p_1 p_2} \ll \sum_{m_1=n_1}^{\infty} \left\| V_{2^{m_1},\infty}(g_1) - V_{2^{m_1+1},\infty}(g_1) \right\|_{p_1 p_2}$$

$$\ll \sum_{m_1=n_1}^{\infty} \frac{1}{2^{m_1\alpha_1}} \left\| \left(V_{2^{m_1},\infty}(g_1) - V_{2^{m_1+1},\infty}(g_1) \right)^{(\alpha_1,0)} \right\|_{p_1 p_2} \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}.$$

Böylece $J_{21} \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}$.

J_{21} ve J_{22} için elde edilen değerlendirmeleri birleştirirsek $B_2 \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}$.

Karma düzgünlük modülünün tanımı yardımıyla

$$\omega_{\alpha_1,\alpha_2}(g_1, \delta_1, \delta_2)_{p_1 p_2} \ll \omega_{\alpha_1,\alpha_2}\left(g_1, \frac{\pi}{2^{n_1+1}-1}, \frac{\pi}{2^{n_2+1}-1} \right)_{p_1 p_2} \quad \text{olduğundan}$$

$$J_2 \ll 2^{-n_1\alpha_1} \left\| g_1^{(\alpha_1,0)} \right\|_{p_1 p_2}.$$

Benzer şekilde $J_3 \ll 2^{-n_2\alpha_2} \left\| g_2^{(0,\alpha_2)} \right\|_{p_1 p_2}$.

Şimdi J_4 ü üstten değerlendirelim. Her $\delta_i \in (0,\pi]$ için negatif olmayan öyle n_i tamsayıları vardır ki $\frac{\pi}{2^{n_i+2}-1} < \delta_i \le \frac{\pi}{2^{n_i+1}-1}, i = 1,2.$

Şimdi $B_3 = \omega_{\alpha_1,\alpha_2}\left(g, \frac{\pi}{2^{n_1+1}-1}, \frac{\pi}{2^{n_2+1}-1} \right)_{p_1 p_2}$ olsun.

4.2 önermesinin a ve c şıklarına göre

$$B_3 \ll \omega_{\alpha_1,\alpha_2}\left(g - V_{2^{n_1},2^{n_2}}(g), \frac{\pi}{2^{n_1+1}-1}, \frac{\pi}{2^{n_2+1}-1}\right)_{p_1p_2} +$$

$$+\omega_{\alpha_1,\alpha_2}\left(V_{2^{n_1},2^{n_2}}(g), \frac{\pi}{2^{n_1+1}-1}, \frac{\pi}{2^{n_2+1}-1}\right) \ll$$

$$\ll \left\| g - V_{2^{n_1},2^{n_2}}(g) \right\|_{p_1p_2} + \sup_{|h_i| \le \frac{\pi}{2^{n_i+1}-1}, i=1,2} \left\| \Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{2^{n_1},2^{n_2}}(g)\right)\right)\right\|_{p_1p_2} := J_{31} + J_{32}.$$

4.2.5 önermesinin a şıkkı, 4.2.6 önermesinin a şıkkı ve sonra önerme 4.5.3 önermesinin a ve b şıkları kullanılırsa $0 < |h_i| \le \frac{\pi}{2^{n_i+1}-1}, i = 1,2$ için

$$\left\| \Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(V_{2^{n_1},2^{n_2}}(g)\right)\right)\right\|_{p_1p_2} \ll 2^{-n_1\alpha_1} \left\| \Delta_{h_2}^{\alpha_2}\left(V_{2^{n_1}2^{n_2}}^{(\alpha_1,0)}(g)\right)\right\|_{p_1p_2} \ll$$

$$\ll 2^{-n_1\alpha_1-n_2\alpha_2} \left\| V_{2^{n_1}2^{n_2}}\left(g^{(\alpha_1,\alpha_2)}\right)\right\|_{p_1p_2} \ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\| g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$$ olur ve buradan da

$$J_{32} \ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\| g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}.$$

4.5.4 önermesinin c şıkkı ve 4.5.3 önermesinin a ve b şıklarını göz önünde bulundurursak

$$\left\| g - V_{2^{n_1},2^{n_2}}(g_1)\right\|_{p_1p_2} \ll \sum_{m_2=n_2}^{\infty}\sum_{m_1=n_1}^{\infty}\left\| V_{2^{m_1},2^{m_2}}(g) - V_{2^{m_1+1},2^{m_2}}(g) - V_{2^{m_1},2^{m_2+1}}(g) + V_{2^{m_1+1},2^{m_2+1}}(g)\right\|_{p_1p_2} \ll$$

$$\ll \sum_{m_2=n_2}^{\infty}\sum_{m_1=n_1}^{\infty}\frac{1}{2^{m_1\alpha_1+m_2\alpha_2}}\left\| \left(V_{2^{m_1},2^{m_2}}(g) - V_{2^{m_1+1},2^{m_2}}(g) - V_{2^{m_1},2^{m_2+1}}(g) + V_{2^{m_1+1},2^{m_2+1}}(g)\right)^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2} \ll$$

$$\ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\| g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}.$$

Böylece $J_{31} \ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\| g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$.

J_{31} ve J_{32} için elde edilen değerlendirmeleri birleştirirsek $B_3 \ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\| g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$.

Karma düzgünlük modülünün tanımı gereği

$$\omega_{\alpha_1,\alpha_2}\left(g,\delta_1,\delta_2\right)_{p_1p_2} \ll \omega_{\alpha_1,\alpha_2}\left(g,\frac{\pi}{2^{n_1+1}-1},\frac{\pi}{2^{n_2+1}-1}\right)_{p_1p_2},$$

Ve sonra $J_4 \ll 2^{-n_1\alpha_1-n_2\alpha_2}\left\|g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}$.

J_1, J_2, J_3, J_4 için elde edilen değerlendirmeler birleştirilerek

$$\omega_{\alpha_1,\alpha_2}\left(g,\delta_1,\delta_2\right)_{p_1p_2} \ll \left\|f-g_1-g_2-g\right\|_{p_1p_2}+\delta_1^{\alpha_1}\left\|g_1^{(\alpha_1,0)}\right\|_{p_1p_2}+$$
$$+\delta_2^{\alpha_2}\left\|g_2^{(0,\alpha_2)}\right\|_{p_1p_2}+\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\|g^{(\alpha_1,\alpha_2)}\right\|_{p_1p_2}.$$

Son eşitsizlik her $g_1 \in W_{p_1p_2}^{(\alpha_1,0)}, g_2 \in W_{p_1p_2}^{(0,\alpha_2)}, g \in W_{p_1p_2}^{(\alpha_1,\alpha_2)}$ için sağlandığından dolayı infimum alarak

$$\omega_{\alpha_1,\alpha_2}\left(g_1,\delta_1,\delta_2\right)_{p_1p_2} \ll K_{\alpha_1,\alpha_2}\left(g_1,\delta_1,\delta_2\right)_{p_1p_2} \tag{4.4}$$

(4.3) ve (4.4) eşitsizliklerini 2.9 tanım yardımıyla düşünülürse ispat tamamlanır.

4.6.4 Teorem [8]

$1<p<\infty$ $f\in L_p^0\left(\mathrm{T}^2\right)$, $n_1,n_2=0,1,2,\ldots$, $\alpha_1,\alpha_2>0$ alalım. Bu durumda

$$Y_{n_1,n_2}\left(f\right)_{L_p\left(\mathrm{T}^2\right)} \ll \omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{n_1+1},\frac{\pi}{n_2+1}\right)_{L_p\left(\mathrm{T}^2\right)} \ll$$
$$\ll \frac{1}{(n_1+1)^{\alpha_1}(n_2+1)^{\alpha_2}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1-1}\nu_2^{\alpha_2-1}Y_{\nu_1-1,\nu_2-1}\left(f\right)_{L_p\left(\mathrm{T}^2\right)}.$$

4.6.5 Önerme [15]

$f\in L_{p_1p_2}^0$, $1\le p_i\le\infty$, $\alpha_i>0$, $m_i\in\mathrm{N}$, $i=1,2$ olsun. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{m_1},\frac{1}{m_2}\right)_{p_1p_2} \approx m_1^{-\alpha_1}m_2^{-\alpha_2}\left\|S_{m_1,m_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{p_1p_2}+m_1^{-\alpha_1}\left\|S_{m_1,\infty}^{(\alpha_1,0)}(f-S_{\infty,m_2}(f))\right\|_{p_1p_2}$$
$$+m_2^{-\alpha_2}\left\|S_{\infty,m_2}^{(0,\alpha_2)}(f-S_{\infty,m_2}(f))\right\|_{p_1p_2}+\left\|f-S_{m_1,\infty}(f)-S_{\infty,m_2}(f)+S_{m_1,m_2}(f)\right\|_{p_1p_2}.$$

4.6.6 Teorem (Karma Düzgünlük Modülünün Yapısal Karakteristiği) [8]

$1<p<\infty$, $f\in L_p^0\left(\mathrm{T}^2\right)$, $\alpha_i>0$, $n_i\in\mathbb{N}, i=1,2$ alalım. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{n_1},\frac{\pi}{n_2}\right)_{L_p(\mathrm{T}^2)} \approx n_1^{-\alpha_1} n_2^{-\alpha_2}\left\|s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{L_p(\mathrm{T}^2)} + n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}\left(f-s_{\infty,n_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)} +$$

$$+n_2^{-\alpha_2}\left\|s_{\infty,n_2}^{(0,\alpha_2)}\left(f-s_{n_1,\infty}(f)\right)\right\|_{L_p(\mathrm{T}^2)} + \left\|f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\right\|_{L_p(\mathrm{T}^2)}.$$

İspat

Normun özelliklerini kullanırsak her h_i ve $n_i \in \mathbb{N}$ için , $i=1,2$

$$\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)} \le \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\right)\right)\right\|_{L_p(\mathrm{T}^2)} +$$

$$+\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(s_{n_1,\infty}\left(f-s_{\infty,n_2}\right)\right)\right)\right\|_{L_p(\mathrm{T}^2)} + \left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(s_{\infty,n_2}\left(f-s_{n_1,\infty}\right)\right)\right)\right\|_{L_p(\mathrm{T}^2)} +$$

$$+\left\|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(s_{n_1,n_2}(f)\right)\right)\right\|_{L_p(\mathrm{T}^2)} =: I_1+I_2+I_3+I_4.$$

Öncelikle I_1 i üstten değerlendirelim. $\varphi(x,y) := f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)$ olarak tanımlansın. 4.2.4 önermesinin c şıkkına göre hemen hemen her *y* için

$$\left(\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(\varphi)\right)\right|^p\right)^{\frac{1}{p}} \le \left(\int_0^{2\pi}\left|\Delta_{h_2}^{\alpha_2}(\varphi)\right|^p\right)^{\frac{1}{p}} .$$

Buradan

$$\int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(\varphi)\right)\right|^p dxdy \lesssim \int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{h_2}^{\alpha_2}(\varphi)\right|^p dxdy .$$

Böylelikle

$$I_1 \lesssim \left\|\Delta_{h_2}^{\alpha_2}(\varphi)\right\|_{L_p(\mathrm{T}^2)} := I_5 .$$

4.2.4 önermesinin c şıkkını kullanırsak

hemen hemen her *x* için $\left(\int_0^{2\pi}\left|\Delta_{h_2}^{\alpha_2}(\varphi)^p\,dy\right|\right)^{\frac{1}{p}} \le \left(\int_0^{2\pi}|\varphi|^p\,dy\right)^{\frac{1}{p}}.$

Buradan

$$\int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{h_2}^{\alpha_2}(\varphi)^p\,dydx\right| \ll \int_0^{2\pi}\int_0^{2\pi}|\varphi|^p\,dydx \text{ ve } I_5 \ll \|\varphi\|_{L_p(\mathrm{T}^2)}.$$

Böylelikle $I_1 \ll \left\|f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f)\right\|_{L_p(\mathrm{T}^2)}$.

Benzer şekilde I_2 üstten değerlendirebiliriz, Bunun için $\psi := f - s_{\infty,n_2}(f)$ olarak tanımlayalım. 4.2.4 önermesinin c şıkkı tarafından

Hemen hemen her x için

$$\left(\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(s_{\infty,n_2}(\psi)\right)\right)\right|^p dy\right)^{\frac{1}{p}} \le \left(\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(s_{n_1,\infty}(\psi)\right)\right|^p dy\right)^{\frac{1}{p}}.$$

Bu nedenle

$$\int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}\left(s_{n_1,\infty}(\psi)\right)\right)\right|^p dydx \ll \int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(s_{n_1,\infty}(\psi)\right)\right|^p dydx$$

ve $I_2 \ll \left\|\Delta_{h_1}^{\alpha_1}\left(s_{n_1,\infty}(\psi)\right)\right\|_{L_p(\mathrm{T}^2)} := I_6$.

Şimdi 4.2.2 önermesinin a şıkkını kullanarak, hemen hemen her y ve her $h_1 \in \left(0, \frac{\pi}{n_1}\right)$ için

$$\left(\int_0^{2\pi}\left|\Delta_{h_1}^{\alpha_1}\left(s_{n_1,\infty}(\psi)\right)\right|^p dx\right)^{\frac{1}{p}} \le n_1^{-\alpha_1}\left(\int_0^{2\pi}\left|s_{n_1,\infty}^{(\alpha_1,0)}(\psi)\right|^p dx\right)^{\frac{1}{p}}.$$

Buradan

$$\int_0^{2\pi}\int_0^{2\pi} \le \left|\Delta_{h_1}^{\alpha_1}\left(s_{n_1,\infty}(\psi)\right)\right|^p \ll n_1^{-\alpha_1}\int_0^{2\pi}\int_0^{2\pi}\left|s_{n_1,\infty}^{(\alpha_1,0)}(\psi)\right|^p dxdy \text{ elde ederiz.}$$

Buradan da $I_6 \ll n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}(\psi)\right\|_{L_p(\mathrm{T}^2)}$.

Böylece

$0<|h_1|<\dfrac{\pi}{n_1}$ için $I_2 \lesssim n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}\left(f-s_{\infty,n_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)}$ olduğunu elde ettik.

Benzer şekilde $0<|h_1|<\dfrac{\pi}{n_1}$, $0<|h_2|<\dfrac{\pi}{n_2}$ için

$$I_3 \lesssim n_2^{-\alpha_2}\left\|s_{\infty,n_2}^{(0,\alpha_2)}\left(f-s_{n_1,\infty}(f)\right)\right\|_{L_p(\mathrm{T}^2)},$$

$$I_4 \lesssim n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{L_p(\mathrm{T}^2)}.$$

Sonuç olarak

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{n_1},\frac{\pi}{n_2}\right)_{L_p(\mathrm{T}^2)} \lesssim \left\|f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\right\|_{L_p(\mathrm{T}^2)}+$$

$$+n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}\left(f-s_{\infty,n_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)}+n_2^{-\alpha_2}\left\|s_{\infty,n_2}^{(0,\alpha_2)}\left(f-s_{n_1,\infty}(f)\right)\right\|_{L_p(\mathrm{T}^2)}+$$

$+n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{L_p(\mathrm{T}^2)}$ ve üst değerlendirme gösterilmiş oldu.

Alt değerlendirmeyi ispatlayalım. Bunun için 4.1.6 önerme, 4.6.4 teorem ve tamsayı dereceli karma modülün özelliklerini kullanıp.

$$A_1:=\left\|f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\right\|_{L_p(\mathrm{T}^2)} \asymp Y_{n_1,n_2}(f)_{L_p(\mathrm{T}^2)} \asymp$$

$$\asymp \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{\pi}{n_1+1},\frac{\pi}{n_2+1}\right)_{L_p(\mathrm{T}^2)} \asymp \omega_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{\pi}{n_1},\frac{\pi}{n_2}\right)_{L_p(\mathrm{T}^2)}.$$

4.2.4 önermesinin b şıkkına göre

$$A_1 \le \sup_{|h_i|\le\frac{\pi}{n_i},i=1,2}\left\|\Delta_{h_1}^{\lfloor\alpha_1\rfloor+1-\alpha_1}\left(\Delta_{h_2}^{\lfloor\alpha_2\rfloor+1-\alpha_2}\left(\Delta_{h_1}^{\alpha_1}\left(\Delta_{h_2}^{\alpha_2}(f)\right)\right)\right)\right\|_{L_p(\mathrm{T}^2)}.$$

4.2.4 önermesinin c şıkkından

$$A_1 \le \sup_{|h_i| \le \frac{\pi}{n_i}, i=1,2} \left\| \Delta_{h_1}^{\alpha_1} \left(\Delta_{h_2}^{\alpha_2} \left(f \right) \right) \right\|_{L_p(\mathrm{T}^2)} = \omega_{\alpha_1,\alpha_2} \left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} .$$

Şimdi

$A_2 := \left\| s_{n_1,\infty}^{(\alpha_1,0)} \left(f - s_{\infty,n_2}(f) \right) \right\|_{L_p(\mathrm{T}^2)}$ yi hesaplayalım.

$\gamma(x,y) := f(x,y) - s_{\infty,n_2}(f)$ tanımlayıp 4.2.2 önermesinin b şıkkı kullanılırsa hemen hemen her y için

$$\left(\int_0^{2\pi} \left| s_{n_1,\infty}(\gamma) \right|^p dx \right)^{\frac{1}{p}} \ll \left(\int_0^{2\pi} \left| \Delta_{\pi/n_1}^{\alpha_1} \left(s_{n_1,\infty}(\gamma) \right) \right|^p dx \right)^{\frac{1}{p}} .$$

Bundan dolayı

$$\int_0^{2\pi} \int_0^{2\pi} \left| s_{n_1,\infty}(\gamma) \right|^p dxdy \ll n_1^{\alpha_1 p} \int_0^{2\pi} \int_0^{2\pi} \left| \Delta_{\pi/n_1}^{\alpha_1} \left(s_{n_1,\infty}(\gamma) \right) \right|^p dxdy ,$$

ve

$$A_2 \ll n_1^{\alpha_1} \left\| s_{n_1,\infty} \left(\Delta_{\pi/n_1}^{\alpha_1}(\gamma) \right) \right\|_{L_p(\mathrm{T}^2)} .$$

4.1.10 önermeye göre

$$A_2 \ll n_1^{\alpha_1} \left\| \Delta_{\pi/n_1}^{\alpha_1} \left(f - s_{\infty,n_2}(f) \right) \right\|_{L_p(\mathrm{T}^2)} .$$

$\Delta_{\pi/n_1}^{\alpha_1}(f) := F$ olarak tanımlarsak $A_2 \ll n_1^{\alpha_1} \left\| F - s_{\infty,n_2}(f) \right\|_{L_p(\mathrm{T}^2)}$.

$s_{0,\infty}(F) = s_{0,n_2}(F) = 0$ olduğundan dolayı

$$A_2 \ll n_1^{\alpha_1} \left\| F - s_{0,\infty}(F) - s_{\infty,n_2}(F) + s_{0,n_2}(F) \right\|_{L_p(\mathrm{T}^2)} .$$

Bu nedenle 4.1.6 önerme, 4.6.4 teorem ve karma düzgünlük modülünün özelliklerinden

$$A_2 \ll n_1^{\alpha_1} \omega_{\lfloor \alpha_1 \rfloor + 1, \lfloor \alpha_2 \rfloor + 1} \left(F, \pi, \frac{\pi}{n_2+1} \right)_{L_p(\mathrm{T}^2)} \ll n_1^{\alpha_1} \omega_{\lfloor \alpha_1 \rfloor + 1, \lfloor \alpha_2 \rfloor + 1} \left(F, \pi, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} .$$

4.2.4 önermesinin b şıkkını uygularsak

$$A_2 \ll n_1^{\alpha_1} \sup_{|h_i|\le \pi, i=1,2} \left\| \Delta_{h_1}^{\lfloor \alpha_1 \rfloor + 1} \left(\Delta_{h_1}^{\lfloor \alpha_2 \rfloor + 1 - \alpha_2} \left(\Delta_{h_2}^{\alpha_2} (F) \right) \right) \right\|_{L_p(\mathrm{T}^2)} .$$

Şimdi 4.2.4 önermesinin c şıkkından

$$A_2 \ll n_1^{\alpha_1} \sup_{|h_2|\le \frac{\pi}{n_2}} \left\| \Delta_{h_2}^{\alpha_2} (F) \right\|_{L_p(\mathrm{T}^2)} = n_1^{\alpha_1} \sup_{|h_2|\le \frac{\pi}{n_2}} \left\| \Delta_{h_2}^{\alpha_2} \left(\Delta_{\frac{\pi}{n_1}}^{\alpha_1} (f) \right) \right\|_{L_p(\mathrm{T}^2)} \ll n_1^{\alpha_1} \omega_{\alpha_1, \alpha_2} \left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} .$$

Benzer şekilde

$$A_3 := \left\| s_{\infty, n_2}^{(0,\alpha_2)} \left(f - s_{n_1,\infty}(f) \right) \right\|_{L_p(\mathrm{T}^2)} \ll n_2^{\alpha_2} \omega_{\alpha_1,\alpha_2} \left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} \quad \text{ve}$$

$$A_4 := \left\| s_{n_1, n_2}^{(\alpha_1,\alpha_2)} (f) \right\|_{L_p(\mathrm{T}^2)} \ll n_1^{\alpha_1} n_2^{\alpha_2} \omega_{\alpha_1,\alpha_2} \left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)}$$

olur.

Sonuç olarak

$$\left\| f - s_{n_1,\infty}(f) - s_{\infty,n_2}(f) + s_{n_1,n_2}(f) \right\|_{L_p(\mathrm{T}^2)} + n_1^{-\alpha_1} \left\| s_{n_1,\infty}^{(\alpha_1,0)} \left(f - s_{\infty,n_2}(f) \right) \right\|_{L_p(\mathrm{T}^2)} +$$

$$+ n_2^{-\alpha_2} \left\| s_{\infty,n_2}^{(0,\alpha_2)} \left(f - s_{n_1,\infty}(f) \right) \right\|_{L_p(\mathrm{T}^2)} + n_1^{-\alpha_1} n_2^{-\alpha_2} \left\| s_{n_1,n_2}^{(\alpha_1,\alpha_2)} (f) \right\|_{L_p(\mathrm{T}^2)} \ll \omega_{\alpha_1,\alpha_2} \left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)}$$

olur yani alt değerlendirme ispatlanmıştır.

4.6.7 Önerme [8]

$1 < p < \infty$, $f \in M_p$, $r_i \ge 0, i = 1,2$ alalım. Bu durumda

$$\|f\|_{L_p(\mathrm{T}^2)} \approx \left(\sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} a_{\nu_1,\nu_2}^{p} (\nu_1 \nu_2)^{p-2} \right)^{1/p}$$

ve

$$\left\| f^{(r_1,r_2)} \right\|_{L_p(\mathrm{T}^2)} \approx \left(\sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} a_{\nu_1,\nu_2}^{p} \nu_1^{r_1 p+p-2} \nu_2^{r_2 p+p-2} \right)^{1/p} .$$

4.6.8 Önerme [8]

$1 < p < \infty$, $f \in \Lambda_p$ alalım. Bu durumda

$$\| f \|_{L_p(\mathrm{T}^2)} \approx \left(\sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} \lambda_{\mu_1,\mu_2}^{2} \right)^{1/2} .$$

4.6.9 Teorem [8]

$1 < p < \infty, f \in L_p^0(\mathrm{T}^2)$ ve f nin Fourier serisi (2,1) ile verilsin.
$\theta := \min(2, p), \tau := \max(2, p)$, $\alpha_1 > 0, \alpha_2 > 0$, $n_1 \in \mathbb{N}, n_2 \in \mathbb{N}$ olsun.

Bu durumda

$1 < s < \infty$ için

$$\mathrm{I}(s) := \frac{1}{n_1^{\alpha_1}} \frac{1}{n_2^{\alpha_2}} \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} \rho_{\nu_1,\nu_2}^{s} \nu_1^{(\alpha_1+1)s-2} \nu_2^{(\alpha_2+1)s-2} \right\}^{1/s} +$$

$$+ \frac{1}{n_1^{\alpha_1}} \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} \rho_{\nu_1,\nu_2}^{s} \nu_1^{(\alpha_1+1)s-2} \nu_2^{s-2} \right\}^{1/s} +$$

$$+ \frac{1}{n_2^{\alpha_2}} \left\{ \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=1}^{n_2} \rho_{\nu_1,\nu_2}^{s} \nu_1^{s-2} \nu_2^{(\alpha_2+1)s-2} \right\}^{1/s} + \left\{ \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} \rho_{\nu_1,\nu_2}^{s} (\nu_1 \nu_2)^{s-2} \right\}^{1/s}$$

durumunda

$$\mathrm{I}(\theta) \ll \omega_{\alpha_1,\alpha_2}\left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} \ll \mathrm{I}(\tau)$$

sağlanır.

İspat

4.6.6 teoremden

$$I := \omega_{\alpha_1,\alpha_2}\left(f, \frac{\pi}{n_1}, \frac{\pi}{n_2} \right)_{L_p(\mathrm{T}^2)} \approx n_1^{-\alpha_1} n_2^{-\alpha_2} \left\| s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f) \right\|_{L_p(\mathrm{T}^2)} +$$

$$+n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}\left(f-s_{\infty,n_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)}+n_2^{-\alpha_2}\left\|s_{\infty,n_2}^{(0,\alpha_2)}\left(f-s_{n_1,\infty}(f)\right)\right\|_{L_p(\mathrm{T}^2)}+$$

$$+\left\|\left(f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}(f)\right)\right\|_{L_p(\mathrm{T}^2)}.$$

Öncelikle $2\le p<\infty$ durumuna bakalım. 2.12 teorem ve 2.14 teoremin a şıkkını dikkate alırsak

$$I \ll n_1^{-\alpha_1}n_2^{-\alpha_2}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\rho_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p}+n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty}\rho_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\right)^{1/p}$$

$$+n_2^{-\alpha_2}\left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2}\rho_{\nu_1,\nu_2}^{p}\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p}+\left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}\rho_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right)^{1/p}.$$

Bu sebeple $p\ge 2$ için teoremin üst değerlendirmesini göstermiş oluruz.

Eğer $1<p<2$ ise Hölder Eşitsizliğinden

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \ll \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)}.$$

Böylelikle üst değerlendirmenin ispatı da yapılmış olur. Şimdi alt değerlendirmeyi ispatlayalım. Eğer $1<p\le 2$ ise 2.12 teorem ve 2.14 teoremin b şıkkını uygularsak

$$I \ll n_1^{-\alpha_1}n_2^{-\alpha_2}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\rho_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p}$$

$$+n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty}\rho_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\right)^{1/p}+n_2^{-\alpha_2}\left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2}\rho_{\nu_1,\nu_2}^{p}\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p}$$

$$+\left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}\rho_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right)^{1/p}.$$

Şimdi $2<p<\infty$ için Hölder eşitsizliğini uygularsak yeniden

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p\left(\mathbb{T}^2\right)} \asymp \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p\left(\mathbb{T}^2\right)}$$ değerlendirmesi çıkar.

4.6.10 Teorem [8]

$1<s<\infty$, $1<p<\infty$, $f\in M_p$ $\alpha_1>0,\alpha_2>0,n_1\in\mathbb{N},n_2\in\mathbb{N}$ olsun.

Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{\pi}{n_1},\frac{\pi}{n_2}\right)_{L_p\left(\mathbb{T}^2\right)} \approx \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{p}\,\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\right\}^{1/p}$$

$$+\frac{1}{n_1^{\alpha_1}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\,\nu_1^{(\alpha_1+1)p-2}\nu_2^{p-2}\right\}^{1/p}+\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{p}\,\nu_1^{p-2}\nu_2^{(\alpha_2+1)p-2}\right\}^{1/p}$$

$$+\left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right\}^{1/p}.$$

İspat

$$I_1 := n_1^{-\alpha_1}n_2^{-\alpha_2}\left\|s_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f)\right\|_{L_p\left(\mathbb{T}^2\right)},$$

$$I_2 := n_1^{-\alpha_1}\left\|s_{n_1,\infty}^{(\alpha_1,0)}\left(f-s_{\infty,n_2}(f)\right)\right\|_{L_p\left(\mathbb{T}^2\right)},$$

$$I_3 := n_2^{-\alpha_2}\left\|s_{\infty,n_2}^{(0,\alpha_2)}\left(f-s_{n_1,\infty}\right)\right\|_{L_p\left(\mathbb{T}^2\right)},$$

$$I_4 := \left\|f-s_{n_1,\infty}(f)-s_{\infty,n_2}(f)+s_{n_1,n_2}\right\|_{L_p\left(\mathbb{T}^2\right)}$$

ve

$$A_1 := n_1^{-\alpha_1}n_2^{-\alpha_2}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{p}\,\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p},$$

$$A_2 := n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\,\nu_1^{(\alpha_1+1)p-2}\,\nu_2^{\,p-2}\right)^{1/p},$$

$$A_3 := n_2^{-\alpha_2}\left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{p}\,\nu_1^{\,p-2}\,\nu_2^{(\alpha_2+1)p-2}\right)^{1/p},$$

$$A_4 := \left(\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right)^{1/p}$$ olarak tanımlayalım.

Böylece I_1, I_2, I_3, I_4 ve A_1, A_2, A_3, A_4 arasındaki değerlendirmeler kanıtlayalım. 4.6.7 teoremden

$$I_1 \approx A_1 \tag{4.5}$$

I_2 ve A_2 yi değerlendirmek için

$$a_{\nu_1,\nu_2}^{*} := \begin{cases} 1 \le \nu_1 \le n_1, \nu_2 > n_2 \text{ için } a_{\nu_1,\nu_2} \\ 1 \le \nu_1 \le n_1, 1 \le \nu_2 \le n_2 \text{ için } a_{\nu_1,\nu_2} \end{cases}$$ olmak üzere

$$\eta_1(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty} a_{\nu_1,\nu_2}^{*}\cos\nu_1 x\cos\nu_2 y \tag{4.6}$$

ve

$$\eta_2(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{*}\cos\nu_1 x\cos\nu_2 y \tag{4.7}$$

fonksiyonlarına bakalım. Bu durumda

$$s_{n_1,\infty}^{(\alpha_1,0)}\left(f - s_{\infty,n_2}(f)\right) = \left(\eta_1 - \eta_2\right)^{(\alpha_1,0)}.$$

Önce I_2 yi değerlendirelim.

$$I_2 \ll n_1^{-\alpha_1}\left\{\left\|\eta_1^{(\alpha_1,0)}\right\|_{L_p(\mathrm{T}^2)} + \left\|\eta_1^{(\alpha_2,0)}\right\|_{L_p(\mathrm{T}^2)} + \right\}$$ olduğunu görmek kolaydır.

4.6.7 önermeden

$$I_2 \le n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}\left(a_{\nu_1,\nu_2}^{*}\right)^p \nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-2}\right)^{1/p} + n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\left(a_{\nu_1,\nu_2}^{*}\right)^p \nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-2}\right)^{1/p}$$

$$\ll n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-2}\right)^{1/p} + n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1} a_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-1}\right)^{1/p}$$

$$\ll A_2 + n_1^{-\alpha_1}n_2^{-\alpha_2}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=\left\lfloor \frac{n_2}{2}\right\rfloor+1}^{n_2} a_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}\nu_2^{(\alpha_2+1)p-2}\right)^{1/p} \ll A_2 + A_1. \tag{4.9}$$

A_2 yi üstten değerlendirirsek,

$$A_2 \ll n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}\left(a_{\nu_1,\nu_2}^{*}\right)^p \nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-2}\right)^{1/p}.$$

4.6.7 önermeden ve (4.6) ve (4.7) ifadelerinden

$$A_2 \ll n_1^{-\alpha_1}\left\|\eta_1^{(\alpha_1,0)}\right\|_{L_p\left(\mathrm{T}^2\right)} \ll I_2 + n_1^{-\alpha_1}\left\|\eta_2^{(\alpha_1,0)}\right\|_{L_p\left(\mathrm{T}^2\right)}$$

ve

$$A_2 \ll I_2 + n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}\left(a_{\nu_1,\nu_2}^{*}\right)^p \nu_1^{(\alpha_1+1)p-2}\nu_2^{\;p-2}\right)^{1/p}$$

$$\ll I_2 + n_1^{-\alpha_1}\left(\sum_{\nu_1=1}^{n_1} a_{\nu_1,\nu_2}^{p}\nu_1^{(\alpha_1+1)p-2}n_2^{\;p-1}\right)^{\frac{1}{p}} \ll I_2 + A_1.$$

Şimdi (4.5) yi kullanarak

$$A_2 \ll I_2 + I_1 \tag{4.10}$$

Benzer şekilde

$$I_3 \ll A_1 + A_3$$

ve

$A_3 \ll I_3 + I_1$.

Böylece $A_1 + A_2 + A_3 \approx I_1 + I_2 + I_3$

Olduğu gösterilmiş olur.

Şimdi

$$b_{\nu_1,\nu_2} := \begin{cases} \nu_1 > n_1, \nu_2 > n_2 \text{ için } a_{\nu_1,\nu_2} \\ \nu_1 > n_1, 1 \le \nu_2 \le n_2 \text{ için } a_{\nu_1,\nu_2} \\ 1 \le \nu_1 \le n_1, \nu_2 > n_2 \text{ için } a_{\nu_1,\nu_2} \\ 1 \le \nu_1 \le n_1, 1 \le \nu_2 \le n_2 \text{ için } a_{\nu_1,\nu_2} \end{cases} \qquad \text{olmak üzere}$$

$$B := f - s_{n_1,\infty} - s_{\infty,n_2} + s_{n_1,n_2} \text{ ve} \tag{4.8}$$

$$\varphi_1(x,y) := \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} b_{\nu_1,\nu_2} \cos \nu_1 x \cos \nu_2 y,$$

$$\varphi_2(x,y) := \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} b_{\nu_1,\nu_2} \cos \nu_1 x \cos \nu_2 y,$$

$$\varphi_3(x,y) := \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=1}^{n_2} b_{\nu_1,\nu_2} \cos \nu_1 x \cos \nu_2 y,$$

$$\varphi_4(x,y) := \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} b_{\nu_1,\nu_2} \cos \nu_1 x \cos \nu_2 y \text{ olsun.}$$

Buradan $B = \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4$.

Öncelikle

$$I_4 = \|B\|_{L_p(\mathrm{T}^2)} \ll \|\varphi_1\|_{L_p(\mathrm{T}^2)} + \|\varphi_2\|_{L_p(\mathrm{T}^2)} + \|\varphi_3\|_{L_p(\mathrm{T}^2)} + \|\varphi_4\|_{L_p(\mathrm{T}^2)}.$$

4.6.7 önerme kullanılırsa

$$\|\varphi_1\|_{L_p(\mathrm{T}^2)} \approx \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} b_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2} \approx \sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}$$

$$+\sum_{\nu_1=n_1+1}^{\infty} a_{\nu_1,\nu_2}^{p}\nu_1^{\,p-2}n_2^{p-1}+\sum_{\nu_2=n_1+1}^{\infty} a_{\nu_1,\nu_2}^{p}\nu_2^{\,p-2}n_1^{p-1}+a_{n_1,n_2}^{p}\left(n_1,n_2\right)^{p-1} =: J_1+J_2+J_3+J_4\,.$$

$\|\varphi_2\|_{L_p(\mathrm{T}^2)}$ yi değerlendirmek için,

$$b_{\nu_1,\nu_2}^{*} := \begin{cases} 1\le \nu_1 \le n_1, \nu_2 > n_2 \text{ için } \mathrm{b}_{\nu_1,\nu_2} \\ 1_2 \le \nu_1 \le n_1, 1\le \nu_2 \le n_2 \text{ için } \mathrm{b}_{\nu_1,\nu_2} \end{cases}$$

olmak üzere

$$\varphi_{21}(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty} b_{\nu_1,\nu_2}^{*}\cos\nu_1 x\cos\nu_2 y\,,$$

$\varphi_{22}(x,y) := \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} b_{\nu_1,\nu_2}^{*}\cos\nu_1 x\cos\nu_2 y$ fonksiyonlarını ele alalım.

Akabinde $\|\varphi_2\|_{L_p(\mathrm{T}^2)} \ll \|\varphi_{21}\|_{L_p(\mathrm{T}^2)} + \|\varphi_{22}\|_{L_p(\mathrm{T}^2)}$. 4.6.7 önermeyi kullanırsak

$$\|\varphi_{21}\|_{L_p(\mathrm{T}^2)} \approx \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{\infty}\left(b_{\nu_1,\nu_2}^{*}\right)^{p}\left(\nu_1\nu_2\right)^{p-2} \ll \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} b_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2} + L_p(\mathrm{T})$$

$$+\sum_{\nu_1=1}^{n_1} b_{\nu_1,\nu_2}^{p}\nu_1^{\,p-2}n_2^{p-1} \ll \sum_{\nu_2=n_2+1}^{\infty} a_{n_1,n_2}^{p}\nu_2^{\,p-2}n_1^{p-1}+a_{n_1,n_2}^{p}\left(n_1n_2\right)^{p-1} =: J_3+J_4\,,$$

$$\|\varphi_{22}\|_{L_p(\mathrm{T}^2)} \approx \sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\left(b_{\nu_1,\nu_2}^{*}\right)^{p}\left(\nu_1\nu_2\right)^{p-2} \ll a_{n_1,n_2}^{p}\left(n_1n_2\right)^{p-1} \ll J_4\,.$$

Bu sebeple $\|\varphi_2\|_{L_p(\mathrm{T}^2)}^{p} \ll J_3+J_4$. Aynı şekilde $\|\varphi_3\|_{L_p(\mathrm{T}^2)}^{p} \ll J_2+J_4$ ve $\|\varphi_4\|_{L_p(\mathrm{T}^2)}^{p} \ll J_4$. Elde edilen değerlendirmeleri birleştirerek

$$I_4 \ll \left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right\}^{1/p} + \left\{\sum_{\nu_1=n_1+1}^{\infty} a_{\nu_1,\nu_2}^{p}\nu_1^{\,p-2}\right\}^{1/p} n_2^{1-1/p}$$

$$+\left\{\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p} \nu_2^{\,p-2}\right\}^{1/p} n_1^{1-(1/p)} + a_{n_2,n_2}\left(n_1 n_2\right)^{1-(1/p)}.$$

$n_1=0$ ve/veya $n_2=0$ ise son eşitsizlik yine geçerli olur. Bu son eşitsizlikler yardımıyla

$$I_4 \ll A_1 + A_2 + A_3 + A_4. \tag{4.9}$$

A_4 ü üstten değerlendirelim.

$A_4 \ll \left\{\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} b_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right\}^{1/p}$ olduğunu biliyoruz.

$A_4 \ll \left\|\varphi_1\right\|_{L_p\left(\mathrm{T}^2\right)}$ ve (4.8) den de

$$A_4 \ll \left\|B\right\|_{L_p\left(\mathrm{T}^2\right)} + \left\|\varphi_2\right\|_{L_p\left(\mathrm{T}^2\right)} + \left\|\varphi_3\right\|_{L_p\left(\mathrm{T}^2\right)} + \left\|\varphi_4\right\|_{L_p\left(\mathrm{T}^2\right)}.$$

Sonra $A_4 \le J_1 + J_2 + J_3 + J_4 \le I_4 + \left\{\sum_{\nu_1=1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right\} n_2^{1-(1/p)}$

$$+\left\{\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p}\left(\nu_1\nu_2\right)^{p-2}\right\}^{\frac{1}{p}} n_1^{1-\frac{1}{p}} + a_{n_1,n_2}\left(n_1 n_2\right)^{1-\frac{1}{p}} \ll I_4 + A_2 + A_3 + A_1.$$

(4.9) yi kullanırsak

$$A_4 \le I_1 + I_2 + I_3 + I_4.$$

Böylece (7.10) ve (7.12) den

$$I_1 + I_2 + I_3 + I_4 \approx A_1 + A_2 + A_3 + A_4.$$

4.6.6 teoremden dolayı

$\omega_{\alpha_1,\alpha_2}\left(f, \dfrac{1}{n_1}, \dfrac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)}^{p} \approx I_1 + I_2 + I_3 + I_4$ olur yani teorem ispatlanmıştır.

4.6.11 Teorem [8]

$1 < p < \infty,\ f \in \Lambda_p, \alpha_1 > 0, \alpha_2 > 0, n_i = 0,1,2,\ldots, i = 1,2$ olsun.

Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_p \approx \frac{1}{2^{n_1\alpha_1+n_2\alpha_2}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2}\lambda_{\mu_1,\mu_2}^2\,2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\right\}^{1/2}+\frac{1}{2^{n_1\alpha_1}}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=n_2+1}^{\infty}\lambda_{\mu_1,\mu_2}^2\,2^{2\mu_1\alpha_1}\right\}^{1/2}$$

$$+\frac{1}{2^{n_2\alpha_2}}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=0}^{n_2}\lambda_{\mu_1,\mu_2}^2\,2^{2\mu_2\alpha_2}\right\}^{1/2}+\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty}\lambda_{\mu_1,\mu_2}^2\right\}^{1/2}.$$

İspat

4.6.6 teoremden dolayı

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_{L_p(\mathbb{T}^2)}^p \approx 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\left\|s_{2^{n_1},2^{n_2}}^{(\alpha_1,\alpha_2)}\right\|_{L_p(\mathbb{T}^2)}$$

$$+2^{-n_1\alpha_1}\left\|s_{2^{n_1},\infty}^{(\alpha_1,0)}\left(f-s_{\infty,2^{n_2}}(f)\right)\right\|_{L_p(\mathbb{T}^2)}+2^{-n_2\alpha_2}\left\|s_{\infty,2^{n_2}}^{(0,\alpha_2)}\left(f-s_{2^{n_1},\infty}(f)\right)\right\|_{L_p(\mathbb{T}^2)}$$

$$+\left\|f-s_{2^{n_1},\infty}(f)-s_{\infty,2^{n_2}}(f)+s_{2^{n_1},2^{n_2}}(f)\right\|_{L_p(\mathbb{T}^2)}.$$

2.12 teoremden ve 4.6.8 önermeden

$$I\approx 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=0}^{n_2}\lambda_{\mu_1,\mu_2}^2\,2^{2(\alpha_1\mu_1+\alpha_2\mu_2)}\right\}^{\frac{1}{2}}+2^{-n_1\alpha_1}\left\{\sum_{\mu_1=0}^{n_1}\sum_{\mu_2=n_1+1}^{\infty}\lambda_{\mu_1,\mu_2}^2\,2^{2\alpha_1\mu_1}\right\}^{\frac{1}{2}}$$

$$+2^{-n_2\alpha_2}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=0}^{n_2}\lambda_{\mu_1,\mu_2}^2\,2^{2\alpha_2\mu_2}\right\}^{\frac{1}{2}}+\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_1+1}^{\infty}\lambda_{\mu_1,\mu_2}^2\right\}^{\frac{1}{2}}$$ elde edilir. Bu da ilgili denkliği verir.

4.6.12 Teorem [8]

$1<p<\infty,\ f\in M_p,\alpha_i>0,n_i\in\mathbb{N},i=1,2$ olsun. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathbb{T}^2)} \approx \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y_{\nu_1-1,\nu_2-1}^{p}(f)_{L_p(\mathbb{T}^2)}\right\}^{\frac{1}{p}}.$$

İspat

$I^p:=\frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y_{\nu_1-1,\nu_2-1}^{p}(f)_{L_p(\mathbb{T}^2)}$ olarak tanımlayalım.

4.6.4 teoremden

$$I^p = \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^p_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1+1},\frac{1}{\nu_2+1}\right)_{L_p(\mathrm{T}^2)}.$$

4.6.10 teoremden

$$I^p \ll \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_1+1)^{(\lfloor\alpha_1\rfloor+1)p}(\nu_2+1)^{(\lfloor\alpha_2\rfloor+1)p}}\left[\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=1}^{\nu_2} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{(\lfloor\alpha_2\rfloor+2)p-2}\right]$$

$$+\frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_1+1)^{(\lfloor\alpha_1\rfloor+1)p}}\left[\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=\nu_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}\right]$$

$$+\frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{(\nu_2+1)^{(\lfloor\alpha_2\rfloor+1)p}}\left[\sum_{\mu_1=\nu_1+1}^{\infty}\sum_{\mu_2=1}^{\nu_2} a^p_{\mu_1,\mu_2}\mu_2^{(\lfloor\alpha_2\rfloor+2)p-2}\mu_1^{p-2}\right]$$

$$+\frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\left[\sum_{\mu_1=\nu_1+1}^{\infty}\sum_{\mu_2=\nu_2+1}^{\infty} a^p_{\mu_1,\mu_2}(\mu_1\mu_2)^{p-2}\right] =: A_1+A_2+A_3+A_4 .$$

Toplamların sırasını değiştirirsek

$$A_1 \ll \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1} a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{(\alpha_2+1)p-2} =: B_1 .$$

A_2 yi değerlendirelim.

$$A_2 \ll \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\nu_2^{\alpha_2 p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=1}^{n_2} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}$$

$$+\frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1} \nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\nu_2^{\alpha_2 p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2} =: A_{11}+A_{12} .$$

Toplamların sırasını değiştirirsek $\lfloor\alpha_1\rfloor+1>\alpha_1$ ve $\alpha_2>0$ olduğu için

$$A_{11} \ll \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=1}^{n_2+1} a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{(\alpha_2+1)p-2} = B_1 ,$$

$$A_{12} \ll \frac{1}{n_1^{\alpha_1 p}} \sum_{\nu_1=1}^{n_1+1} \nu_1^{(\alpha_1-\lfloor\alpha_1\rfloor-1)p-1}\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\lfloor\alpha_1\rfloor+2)p-2}\mu_2^{p-2}$$

olur. Toplamların sırasını değiştirirsek

$$A_{12} \ll \frac{1}{n_1^{\alpha_1 p}} \sum_{\mu_1=1}^{n_1+1}\sum_{\mu_2=n_2+1}^{\infty} a^p_{\mu_1,\mu_2}\mu_1^{(\alpha_1+1)p-2}\mu_2^{(\alpha_2+1)p-2} =: B_2$$

Böylece $A_2 \ll B_1 + B_2$ olur. Benzer şekilde

$B_3 := \dfrac{1}{n_2^{\alpha_2 p}} \displaystyle\sum_{\mu_1=n_1+1}^{\infty} \sum_{\mu_2=1}^{n_2+1} a_{\mu_1,\mu_2}^{p} \mu_1^{p-2} \mu_2^{(\alpha_1+1)p-2}$ ve $B_4 := \displaystyle\sum_{\mu_1=n_1+1}^{\infty} \sum_{\mu_2=n_2+1}^{\infty} a_{\mu_1,\mu_2}^{p} \left(\mu_1 \mu_2\right)^{p-2}$ olmak üzere

$A_3 \ll B_1 + B_3$ ve $A_4 \ll B_1 + B_2 + B_3 + B_4$ da olur.

A_1, A_2, A_3, A_4 için geçerli olan değerlendirmeleri birleştirilerek,

$$I^{p} \ll \frac{1}{n_1^{\alpha_1 p}} \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\mu_1,\mu_2}^{p} \nu_1^{(\alpha_1+1)p-2} \nu_2^{(\alpha_2+1)p-2} + \frac{1}{n_1^{\alpha_1 p}} \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p} \nu_1^{(\alpha_1+1)p-2} \nu_2^{p-2}$$

$$+ \frac{1}{n_2^{\alpha_2 p}} \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^{p} \nu_1^{p-2} \nu_2^{(\alpha_2+1)p-2} + \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} a_{\mu_1,\mu_2}^{p} \left(\nu_1 \nu_1\right)^{p-2} .$$

4.6.10 teoremden $I \ll \omega_{\alpha_1,\alpha_2}\left(f, \dfrac{1}{n_1}, \dfrac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)}$ olup alt değerlendirmenin ispatı verilmiş olur.

Üst değerlendirmeyi elde edelim. 4.6.10 teoremden

$$a_{n_1,n_2}^{p} \left(n_1 n_2\right)^{p-1} \ll \frac{1}{n_1^{(\alpha_1]+1)p} n_2^{(\alpha_2]+1)p}} \sum_{\nu_1=\lfloor n_1/2+1 \rfloor}^{n_1} \sum_{\nu_2=\lfloor n_2/2+1 \rfloor}^{n_2} a_{\nu_1,\nu_2}^{p} \nu_1^{(\lfloor \alpha_1 \rfloor+2)p-2} \nu_2^{(\lfloor \alpha_2 \rfloor+2)p-2}$$

$$\ll \omega_{\lfloor \alpha_1 \rfloor+1, \lfloor \alpha_2 \rfloor+1}^{p}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} .$$

Bununla beraber

$$n_1^{p-1} \sum_{\nu_2=n_2+1}^{\infty} a_{n_1,\nu_2}^{p} \nu_2^{p-2} \ll \frac{1}{n_1^{(\lfloor \alpha_1 \rfloor+1)p}} \sum_{\nu_1=\left\lfloor \frac{n_1}{2} \right\rfloor+1}^{n_1} \sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^{p} \nu_1^{(\lfloor \alpha_1 \rfloor+2)p-2} \nu_2^{p-2}$$

$$\ll \omega_{\lfloor \alpha_1 \rfloor+1, \lfloor \alpha_2 \rfloor+1}^{p}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} ,$$

$$n_2^{p-1} \sum_{\nu_1=n_1+1}^{\infty} a_{n_1,\nu_2}^{p} \nu_1^{p-2} \ll \omega_{\lfloor \alpha_1 \rfloor+1, \lfloor \alpha_2 \rfloor+1}^{p}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} ,$$

ve

$$\sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=n_2+1}^{\infty} a_{n_1,\nu_2}^{p} \nu_1^{p-2} \nu_2^{p-2} \ll \omega_{\lfloor \alpha_1 \rfloor+1, \lfloor \alpha_2 \rfloor+1}^{p}\left(f, \frac{1}{n_1}, \frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} .$$

4.6.10 teoremi tekrar kullanırsak

$$\omega^{p}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \ll \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\left(\nu_1^{p-1}\nu_2^{p-1}a^{p}_{\nu_1,\nu_2}\right)+$$

$$+\frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1}\nu_1^{\alpha_1 p-1}\left(\nu_1^{p-1}\sum_{\nu_2=n_2+1}^{\infty}a^{p}_{n_1,\nu_2}\nu_2^{\,p-2}\right)+\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_2=1}^{n_2}\nu_2^{\alpha_2 p-1}\left(\nu_2^{p-1}\sum_{\nu_1=n_1+1}^{\infty}a^{p}_{\nu_1,\nu_2}\nu_1^{\,p-2}\right)+$$

$$+\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}a^{p}_{n_1,\nu_2}\left(\nu_1\nu_2\right)^{p-2}.$$

Böylelikle,

$$\omega^{p}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \ll \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^{p}_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathrm{T}^2)}+$$

$$+\frac{1}{n_1^{\alpha_1 p}}\sum_{\nu_1=1}^{n_1}\nu_1^{\alpha_1 p-1}\omega^{p}_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathrm{T}^2)}+$$

$$+\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_2=1}^{n_2}\nu_2^{\alpha_2 p-1}\omega^{p}_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathrm{T}^2)}+$$

$$+\omega^{p}_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathrm{T}^2)}.$$

Karma düzgünlük modülünün özelliklerinden

$$\omega^{p}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \ll \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}\omega^{p}_{\lfloor\alpha_1\rfloor+1,\lfloor\alpha_2\rfloor+1}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p(\mathrm{T}^2)}.$$

4.6.4. teoremden

$$\omega^{p}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \ll$$

$$\ll \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\frac{\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}}{\nu_1^{(\lfloor\alpha_1\rfloor+1)p}\nu_2^{(\lfloor\alpha_2\rfloor+1)p}}\left(\sum_{\mu_1=1}^{\nu_1}\sum_{\mu_2=1}^{\nu_2}\mu_1^{\lfloor\alpha_1\rfloor}\mu_2^{\lfloor\alpha_2\rfloor}Y_{\mu_1-1,\mu_2-1}(f)_{L_p(\mathrm{T}^2)}\right)^{p}.$$

2.11 önerme yardımıyla

$$\omega^{p}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \asymp \frac{1}{n_1^{\alpha_1 p}}\frac{1}{n_2^{\alpha_2 p}}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1 p-1}\mu_2^{\alpha_2 p-1}Y_{\mu_1-1,\mu_2-1}(f)_{L_p(\mathrm{T}^2)}.$$

Böylece Teorem ispatlanmış olur.

4.6.13 Teorem [8]

$1<p<\infty,\ f\in\Lambda_p,\alpha_i>0,n_i\in\mathbb{N},i=1,2$ olsun. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)} \approx \frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{2\alpha_1-1}\nu_2^{2\alpha_2-1}Y^2_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathrm{T}^2)}\right\}^{1/2}.$$

İspat

$$I:=2^{-(m_1\alpha_1+m_2\alpha_2)}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2}2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}Y^2_{2^{\mu_1}-1,2^{\mu_2}-1}(f)_{L_p(\mathrm{T}^2)}\right\}^{1/2}$$

olmak üzere

$$\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_{L_p(\mathrm{T}^2)}\approx I$$

Denkliğini göstermek yeterlidir.

4.1.6 önerme ve 4.6.8 önermeyi uygularsak

$$I\approx 2^{-(m_1\alpha_1+m_2\alpha_2)}\left\{\sum_{\mu_1=1}^{m_1}\sum_{\mu_2=1}^{m_2}2^{2(\mu_1\alpha_1+\mu_2\alpha_2)}\left\| f-s_{2^{\mu_1}-1,\infty}(f)-s_{\infty,2^{\mu_2}-1}(f)+s_{2^{\mu_1}-1,2^{\mu_2}-1}(f)\right\|_{L_p(\mathrm{T}^2)}\right\}^{1/2}.$$

$$+2^{-m_2\alpha_2}\left\{\sum_{\mu_1=m_1+1}^{m_1}\sum_{\mu_2=1}^{m_2}\lambda^2_{\mu_1,\mu_2}2^{2\mu_2\alpha_2}\right\}^{1/2}+\left\{\sum_{\mu_1=m_1+1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty}\lambda^2_{\mu_1,\mu_2}\right\}^{1/2}.$$

4.6.11 teoremi uygularsak sonuç olarak $I\approx\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_{L_p(\mathrm{T}^2)}$.

4.6.14 Teorem [8]

$1<p<\infty,\ f\in L_p^0\left(\mathrm{T}^2\right),\sigma:=\max(2,p),\theta:=\min(2,p),\alpha_i>0,n_i\in\mathbb{N},i=1,2$ olsun. Bu durumda

$$\frac{1}{n_1^{\alpha_1}}\frac{1}{n_2^{\alpha_2}}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\sigma-1}\nu_2^{\alpha_2\sigma-1}Y^{\sigma}_{\nu_1-1,\nu_2-1}(f)_{L_p(\mathrm{T}^2)}\right\}\ll\ \omega_{\beta_1,\beta_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p(\mathrm{T}^2)}$$

$$\lesssim \frac{1}{n_1^{\alpha_1}} \frac{1}{n_2^{\alpha_2}} \left\{ \sum_{\nu_1=1}^{n_1+1} \sum_{\nu_2=1}^{n_2+1} \nu_1^{\alpha_1\theta-1} \nu_2^{\alpha_2\theta-1} Y_{\nu_1-1,\nu_2-1}^{\theta} \left(f \right)_{L_p\left(\mathrm{T}^2\right)} \right\}^{\frac{1}{\theta}} .$$

4.6.15 Teorem [8]

$1 < p < \infty,\ f \in L_p^0\left(\mathrm{T}^2\right), \tau := \max(2, p), \theta := \min\left(2, p\right), 0 < \alpha_i < \beta_i, \delta_i \in \left(0, \frac{1}{2}\right), i = 1,2$ olsun.

Bu durumda

$$\delta_1^{\alpha_1} \delta_2^{\alpha_2} \left\{ \int_{\delta_1}^{1} \int_{\delta_2}^{1} \left[t_1^{-\alpha_1} t_2^{-\alpha_2} \omega_{\beta_1,\beta_2} \left(f, t_1, t_2 \right)_{L_p\left(\mathrm{T}^2\right)} \right]^{\tau} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{1/\tau} \lesssim \omega_{\alpha_1,\alpha_2} \left(f, \delta_1, \delta_2 \right)_{L_p\left(\mathrm{T}^2\right)} \lesssim$$

$$\lesssim \delta_1^{\alpha_1} \delta_2^{\alpha_2} \left\{ \int_{\delta_1}^{1} \int_{\delta_2}^{1} \left[t_1^{-\alpha_1} t_2^{-\alpha_2} \omega_{\beta_1,\beta_2} \left(f, t_1, t_2 \right)_{L_p\left(\mathrm{T}^2\right)} \right]^{\theta} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{1/\theta} .$$

İspat

$$I := \delta_1^{\alpha_1} \delta_2^{\alpha_2} \left\{ \int_{\delta_1}^{1} \int_{\delta_2}^{1} \left[t_1^{-\alpha_1} t_2^{-\alpha_2} \omega_{\beta_1,\beta_2} \left(f, t_1, t_2 \right)_{L_p\left(\mathrm{T}^2\right)} \right]^{\tau} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{1/\tau}$$

Olarak tanımlayalım.

$\delta_i \in \left(0, \frac{1}{2}\right)$ verildiğinde $\frac{1}{n_i+1} \leq \delta_i \leq \frac{1}{n_i}, i = 1,2$ koşulunu sağlayan n_i tamsayılarını bulabiliriz. Bu durumda

$$I^{\tau} \lesssim n_1^{-\alpha_1\tau} n_2^{-\alpha_2\tau} \sum_{\mu_1=1}^{n_1} \sum_{\mu_2=1}^{n_2} \mu_1^{\alpha_1\tau-1} \mu_2^{\alpha_2\tau-1} \omega_{\beta_1,\beta_2}^{\tau} \left(f, \frac{1}{\mu_1}, \frac{1}{\mu_2} \right)_{L_p\left(\mathrm{T}^2\right)} .$$

4.6.14. teoremden

$$I^{\tau} \lesssim n_1^{-\alpha_1\tau} n_2^{-\alpha_2\tau} \sum_{\mu_1=1}^{n_1+1} \sum_{\mu_2=1}^{n_2+1} \mu_1^{\alpha_1\tau-\beta_1\tau-1} \mu_2^{\alpha_2\tau-\beta_2\tau-1} \left\{ \sum_{\nu_1=1}^{\mu_1+1} \sum_{\nu_2=1}^{\mu_2+1} \nu_1^{\beta_1\theta-1} \nu_2^{\beta_2\theta-1} Y_{\nu_1-1,\nu_2-1}^{\theta} \left(f \right)_{L_p\left(\mathrm{T}^2\right)} \right\}.$$

$\tau / \theta \geq 1$ olduğunu ele alıp 2.11 önermeyi uygularsak

$$I^{\tau} \lesssim n_1^{-\alpha_1\tau} n_2^{-\alpha_2\tau} \sum_{\mu_1=1}^{n_1+1} \sum_{\mu_2=1}^{n_2+1} \mu_1^{\alpha_1\tau-\beta_1\tau-1} \mu_2^{\alpha_2\tau-\beta_2\tau-1} Y_{\mu_1-1,\mu_2-1}^{\tau} \left(f \right)_{L_p\left(\mathrm{T}^2\right)} .$$

4.6.14 teorem tarafından

$$I^{\tau} \ll \omega^{\tau}_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p\left(\mathbb{T}^2\right)},$$

dolayısıyla

$$I \ll \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1+1},\frac{1}{n_2+1}\right)_{L_p\left(\mathbb{T}^2\right)} \ll \omega_{\alpha_1,\alpha_2}\left(f,t_1,t_2\right)_p \ll$$

$$\ll \omega_{\alpha_1,\alpha_2}\left(f,t_1,t_2\right)_p .$$

Böylelikle eşitsizliğin ilk kısmı eşitlendi.

Şimdi öteki kısmı inceleyelim. $\delta_i \in \left(0,\frac{1}{2}\right)$ verildiğinde $\frac{1}{n_i+1} \le \delta_i \ll \frac{1}{n_i}, i=1,2$ şartını sağlayan n_i tamsayısı alalım. 4.6.14 teorem yardımıyla

$$\omega_{\alpha_1,\alpha_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathbb{T}^2\right)} \ll \omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1+1},\frac{1}{n_2+1}\right)_{L_p\left(\mathbb{T}^2\right)}$$

$$\ll n_1^{-\alpha_1\tau} n_2^{-\alpha_2\tau}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}Y^{\theta}_{\nu_1-1,\nu_2-1}\left(f\right)_{L_p\left(\mathbb{T}^2\right)}\right\}^{1/\theta} .$$

4.6.4 teoremden

$$\omega_{\alpha_1,\alpha_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathbb{T}^2\right)} \le n_1^{-\alpha_1\tau} n_2^{-\alpha_2\tau}\left\{\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1\theta-1}\nu_2^{\alpha_2\theta-1}\omega^{\theta}_{\beta_1,\beta_2}\left(f,\frac{1}{\nu_1},\frac{1}{\nu_2}\right)_{L_p\left(\mathbb{T}^2\right)}\right\}^{1/\theta}$$

$$\ll \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathbb{T}^2\right)}\right]^{\theta}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/\theta}$$ olur ve bu da teoremin ispatını bitirir.

4.6.16 Teorem [8]

$f \in M_p, 1<p<\infty, 0<\alpha_i<\beta_i, \delta_i \in \left(0,\frac{1}{2}\right), i=1,2$ olsun. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathbb{T}^2\right)} \approx \delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathbb{T}^2\right)}\right]^{p}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/p} .$$

İspat

$$A:=\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathrm{T}^2\right)}\right]^p\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/p}$$ olarak tanımlayıp

$\frac{1}{n_i+1}\le\delta_i \square \frac{1}{n_i}, i=1,2$ şartını sağlayan n_i tamsayılarını alalım. Böylece

$$A^p\approx n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=}^{n_2}\mu_1^{\alpha_1 p-1}\mu_2^{\alpha_2 p-1}\omega_{\beta_1,\beta_2}^p\left(f,\frac{1}{\mu_1},\frac{1}{\mu_2}\right).$$

4.6.12 teoremden

$$A^p\approx n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}\mu_1^{\alpha_1 p-\beta_1 p-1}\mu_2^{\alpha_2 p-\beta_2 p-1}\sum_{\nu_1=1}^{\mu_1+1}\sum_{\nu_2=1}^{\mu_2+1}\nu_1^{\beta_1 p-1}\nu_2^{\beta_2 p-1}Y_{\nu_1-1,\nu_2-1}^p\left(f\right)_{L_p\left(\mathrm{T}^2\right)}$$

$$\approx n_1^{-\alpha_1 p}n_2^{-\alpha_2 p}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}\nu_1^{\alpha_1 p-1}\nu_2^{\alpha_2 p-1}Y_{\nu_1-1,\nu_2-1}^p\left(f\right)_{L_p\left(\mathrm{T}^2\right)}.$$

Son olarak 4.6.12 teoremden den $A^p\approx\omega_{\alpha_1,\alpha_2}\left(f,\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)}$ olur. Bu da teoremin ispatını bitirir.

4.6.17 Teorem [8]

$f\in\Lambda_p, 1<p<\infty, 0<\alpha_i<\beta_i, \delta_i\in\left(0,\frac{1}{2}\right), i=1,2$ olsun. Bu durumda

$$\omega_{\alpha_1,\alpha_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathrm{T}^2\right)}\approx\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{-\alpha_2}\omega_{\beta_1,\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathrm{T}^2\right)}\right]^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/2}.$$

İspat

$$B:=\delta_1^{\alpha_1}\delta_2^{\alpha_2}\left\{\int_{\delta_1}^{1}\int_{\delta_2}^{1}\left[t_1^{-\alpha_1}t_2^{\alpha_2}\omega_{\beta_1,\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathrm{T}^2\right)}\right]^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/2}$$ tanımlayıp

$\frac{1}{2^{n_i+1}}\le\delta_i\le\frac{1}{2^{n_i}}, i=1,2$ olacak $n_1,n_2\in\square$ alalım. Bu durumda

$$B^2\approx 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}2^{2\mu_1\alpha_1+\mu_2\alpha_2}\omega_{\beta_1,\beta_2}^2\left(f\,\frac{1}{2^{n_1}},\frac{1}{2^{n_2}}\right)_{L_p\left(\mathrm{T}^2\right)}.$$

4.6.13 teoremi kullanırsak

$$B^2 \approx 2^{-n_1\alpha_1}2^{-n_2\alpha_2}\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2}2^{2\mu_1\alpha_1+\mu_2\alpha_2-2\mu_1\beta_1-2\mu_2\beta_2}\sum_{\nu_1=1}^{\mu_1+1}\sum_{\nu_2=1}^{\mu_2+1}2^{2\nu_1\beta_1+2-\beta_2}Y^2_{2^{\nu_1}-1,2^{\nu_2}-1}\left(f\right)_{L_p\left(\mathrm{T}^2\right)}$$

$$\approx 2^{-2\left(n_1\alpha_1+n_2\alpha_2\right)}\sum_{\nu_1=1}^{n_1+1}\sum_{\nu_2=1}^{n_2+1}2^{2\left(\nu_1\alpha_1+\nu_2\alpha_2\right)}Y^2_{2^{\nu_1}-1,2^{\nu_2}-1}\left(f\right)_{L_p\left(\mathrm{T}^2\right)}.$$

4.6.13 teoremden den $B \approx \omega_{\alpha_1,\alpha_2}\left(f\,\dfrac{1}{n_1},\dfrac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} \approx \omega_{\alpha_1,\alpha_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathrm{T}^2\right)}$

4.7 Karma Düzgünlük Modülü ve Türevler

4.7.1 Teorem [16]

$1<p<\infty$ olsun. $f, f^{(k)} \in L_p\left(\mathrm{T}\right)$ ve $r,k \in \mathbb{N}$ olmak üzere

$\omega_{r+k}\left(f,\delta\right) \lesssim \delta^k\omega_r\left(f^{(k)},\delta\right)_{L_p(\mathrm{T})}$ ve bunun zayıf tersi

$\omega_r\left(f^{(k)},\delta\right)_{L_p(\mathrm{T})} \lesssim \displaystyle\int_0^{\delta}\frac{\omega_{r+k}\left(f,u\right)_{L_p(\mathrm{T})}}{u^{k+1}}du$ eşitsizlikleri sağlanır.

4.7.2 Teorem [8]

$j=1,2, \beta_j, r_j>0$ olmak üzere

$$\delta_1^{-r_1}\delta_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}\left(f,\delta_1,\delta_2\right) \lesssim \omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p\left(\mathrm{T}^2\right)}$$

$$\lesssim \int_0^{\delta_1}\int_0^{\delta_2}t_1^{-r_1}t_2^{-r_2}\omega_{\beta_1+r_1,\beta_2+r_2}\left(f,\delta_1,\delta_2\right)_{L_p\left(\mathrm{T}^2\right)}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\ .$$

4.7.3 Teorem [8]

$1<p<\infty$, $f \in L_p^0\left(\mathrm{T}^2\right)$, $\theta := \min\left(2,p\right), \tau := \max\left(2,p\right)$, $\beta_1,\beta_2,\tau_1,\tau_2>0$, $\delta_1,\delta_2 \in \left(0,\dfrac{1}{2}\right)$ olsun.

Eğer

$\left(\displaystyle\int_0^1\int_0^1 t_1^{-r_1\theta-1}t_2^{-r_2\theta-1}\omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathrm{T}^2\right)}dt_1dt_2\right)^{\frac{1}{\theta}} < \infty$ ise f

nin Wely anlamında karma türevi $f^{(r_1,r_2)} \in L_p\left(\mathrm{T}^2\right)$ özelliğini sağlar ve

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p(\mathrm{T}^2)} \ll \left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\theta-1}t_2^{-r_2\theta-1}\omega_{r_1+\beta_1,r_2+\beta_2}^{\theta}\left(f,t_1,t_2\right)_{L_p(\mathrm{T}^2)}dt_1dt_2\right\}^{\frac{1}{\theta}} \text{ olur.}$$

Eğer $f\in L_p\left(\mathrm{T}^2\right)$nin Wely anlamında $f^{(r_1,r_2)}\in L_p\left(\mathrm{T}^2\right)$ türevi varsa o zaman

$$\left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\tau-1}t_2^{-r_2\tau-1}\omega_{r_1+\beta_1,r_2+\beta_2}^{\tau}\left(f,t_1,t_2\right)_{L_p(\mathrm{T}^2)}dt_1dt_2\right\}^{\frac{1}{\tau}} \ll \omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p(\mathrm{T}^2)}.$$

İspat

$2^{-n_i}<\delta_i<2^{-n_i+1}, i=1,2$ olacak şekilde n_1,n_2 tamsayılarını seçelim. 2.12 teorem, 2.13 teorem ve 4.6.6 teoremden,

$A_{\nu_1\nu_2}$ (2.1) de ki gibi olmak üzere

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p \ll \left\|\sum_{\nu_1=2^{n_1+1}}^{\infty}\sum_{\nu_2=2^{n_2+1}}^{\infty}\nu_1^{r_1}\nu_2^{r_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p$$

$$+2^{-n_1\beta_1}\left\|\sum_{\nu_1=1}^{2^{n_1}}\sum_{\nu_2=2^{n_2+1}}^{\infty}\nu_1^{r_1+\beta_1}\nu_2^{r_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p$$

$$+2^{-n_2\beta_2}\left\|\sum_{\nu_1=2^{n_1+1}}^{\infty}\sum_{\nu_2=1}^{2^{n_2}}\nu_1^{r_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p$$

$$+2^{-n_1\beta_1}2^{-n_2\beta_2}\left\|\sum_{\nu_1=2^{n_1+1}}^{2^{n_1}}\sum_{\nu_2=2^{n_2+1}}^{2^{n_2}}\nu_1^{r_1+\beta_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p$$

$$=: I_1+I_2+I_3+I_4 .$$

I_2 yi değerlendirelim:

$$2^{2\nu_j r_j}\approx\left(\sum_{\xi=n_j+1}^{\nu_j}2^{\xi_j r_j\theta}\right)^{\frac{2}{\theta}}, j=1,2$$

ve Minkowski eşitsizliği kullanılırsa

$$I_2 \ll 2^{-n_1\beta_1}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=0}^{n_1}\sum_{\nu_2=0}^{\infty}2^{2\nu_1(r_1+\beta_1)+2\nu_2 r_2}\Delta_{\nu_1\nu_2}^2\right]^{\frac{p}{2}}dx_1dx_2\right\}^{\frac{1}{p}}.$$

Buradan

$$I_2 \ll 2^{-n_1\beta_1}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\xi=n_2+1}^{\infty}2^{\xi_2 r_2\theta}\left(\sum_{\nu_2=\xi_2}^{\infty}\sum_{\nu_1=0}^{n_1}2^{2\nu_1(r_1+\beta_1)}\Delta^2_{\nu_1\nu_2}\right)^{\frac{\theta}{2}}\right]^{\frac{p}{\theta}}dx_1dx_2\right\}^{\frac{1}{p}}.$$

Toplamlar ve integraller için tekrar Minkowski eşitsizliğini uygularsak

$$I_2 \ll \left\{2^{-n_1\beta_1\theta}\sum_{\xi=n_2+1}^{\infty}2^{\xi_2 r_2\theta}\left(\int_0^{2\pi}\int_0^{2\pi}\left\langle\sum_{\nu_1=0}^{n_1}\sum_{\nu_2=\xi_2}^{\infty}2^{2\nu_1(r_1+\beta_1)}\Delta^2_{\nu_1\nu_2}\right\rangle^{p/2}dx_1dx_2\right)^{\theta/p}\right\}^{1/\theta}.$$

Şimdi $2^{-n_j\beta_j\theta}\approx\sum_{\xi=n_j+1}^{\infty}2^{\xi_j r_j\theta},\ j=1,2$ (4.10)

Olduğunu kullanıp 2.12 teorem, 2.13 teorem ve 4.6.6 teorem den

$$I_2 \ll \left\{\sum_{\xi_1=n_1}^{\infty}2^{-\xi_1\beta_1\theta}\sum_{\xi_2=n_2}^{\infty}2^{\zeta_2 r_2\theta}\left\|\sum_{\nu_1=1}^{2^{\xi_1}}\sum_{\nu_2=2^{\xi_2+1}}^{\infty}\nu_1^{r_1+\beta_1}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\theta}\right\}^{\frac{1}{\theta}}$$

$$\ll \left\{\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1 r_1\theta}2^{\xi_2 r_2\theta}\omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}\left(f,\frac{1}{2^{\xi_1}},\frac{1}{2^{\xi_2}}\right)_p\right\}^{\frac{1}{\theta}}.$$

Benzer sebeple I_3 için

$$I_3 \ll \left\{\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\zeta_1 r_1\theta}2^{-\xi_2\beta_2\theta}\left\|\sum_{\nu_1=2^{\xi_1+1}=1}^{\infty}\sum_{\nu_2=1}^{2^{\xi_2}}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\theta}\right\}^{\frac{1}{\theta}}$$

$$\ll \left\{\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty}2^{\xi_1 r_1\theta}2^{\xi_2 r_2\theta}\omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}\left(f,\frac{1}{2^{\xi_1}},\frac{1}{2^{\xi_2}}\right)_p\right\}^{\frac{1}{\theta}}.$$

Şimdi I_1 i değerlendirelim:

$$I_1 \ll \left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}2^{2\nu_1 r_1+2\nu_2 r_2}\Delta^2_{\nu_1\nu_2}\right]^{\frac{p}{2}}dx_1dx_2\right\}^{\frac{1}{p}}$$

$$\ll \left\{ \sum_{\xi_2=n_2}^{\infty} 2^{\xi_2 r_2 \theta} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=n_1+1}^{\infty} \sum_{\nu_2=\zeta_2+1}^{\infty} 2^{2\nu_1 r_1} \Delta_{\nu_1\nu_2}^2 \right\rangle^{p/2} dx_1 dx_2 \right)^{\theta/p\nu} \right\}^{1/\theta}$$

$$\ll \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1 \theta} 2^{\xi_2 r_2 \theta} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=\xi_1+1}^{\infty} \sum_{\nu_2=\xi_2+1}^{\infty} \Delta_{\nu_1\nu_2}^2 \right\rangle^{p/2} dx_1 dx_2 \right)^{\theta/p} \right\}^{1/p}$$

$$\ll \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1 \theta} 2^{\xi_2 r_2 \theta} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=2^{\xi_2}+1}^{\infty} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p^{\theta} \right\}^{1/p}$$

$$\ll \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1 \theta} 2^{\xi_2 r_2 \theta} \omega_{r_1+\beta_1, r_2+\beta_2}^{\theta} \left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}} \right)_p \right\}^{1/\theta}.$$

I_4 ü değerlendirmek istersek

$$I_4 \ll 2^{-n_1\beta_1} 2^{-n_2\beta_2} \left\{ \int_0^{2\pi}\int_0^{2\pi} \left[\sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} 2^{2\nu_1(r_1+\beta_1)+2(r_2+\beta_2)} \Delta_{\nu_1\nu_2}^2 \right]^{p/2} dx_1 dx_2 \right\}^{1/p} \ll$$

$$\ll \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{-\xi_1\beta_1\theta} 2^{-\xi_2\beta_2\theta} \left\| \sum_{\nu_1=0}^{2\xi_1} \sum_{\nu_2=0}^{2\xi_2} \nu_1^{r_1+\beta_1} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}(x_1, x_2) \right\|_p^{\theta} \right\}^{1/\theta} \ll$$

$$\ll \left\{ \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1 \theta} 2^{\xi_2 r_2 \theta} \omega_{r_1+\beta_1, r_2+\beta_2}^{\theta} \left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}} \right)_p \right\}^{1/\theta}.$$

$I_j,\ j = 1, 2, 3, 4$ için sonuçları birleştirirsek

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)}, \delta_1, \delta_2 \right)_p \ll \left\{ \sum_{\xi_1=n_1+1}^{\infty} \sum_{\xi_2=n_2+1}^{\infty} 2^{\xi_1 r_1 \theta} 2^{\xi_2 r_2 \theta} \omega_{r_1+\beta_1, r_2+\beta_2}^{\theta} \left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}} \right)_p \right\}^{\frac{1}{\theta}}.$$

Şimdi ters eşitsizliği ispatlayalım.

$$K := \int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\tau-1} t_2^{-r_2\tau-1} \omega_{r_1+\beta_1, r_2+\beta_2}^{\tau} (f, t_1, t_2)_{L_p(\mathrm{T}^2)} dt_1 dt_2$$

$$\ll \sum_{\xi_1=n_1}^{\infty} \sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1 \tau} 2^{\xi_2 r_2 \tau} \omega_{r_1+\beta_1, r_2+\beta_2}^{\tau} \left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}} \right)_p.$$

2.12 teorem, 2.13 teorem ve 4.6.6 teoremi kullanırsak

$$K := \int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\tau-1} t_2^{-r_2\tau-1} \omega^{\tau}_{r_1+\beta_1, r_2+\beta_2}\left(f, t_1, t_2\right)_{L_p\left(\mathrm{T}^2\right)} dt_1 dt_2$$

$$\ll \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{\xi_2 r_2 \tau} \omega^{\tau}_{r_1+\beta_1, r_2+\beta_2}\left(f, \frac{1}{2^{\xi_1}}, \frac{1}{2^{\xi_2}}\right)_p$$

$$K \ll \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{\xi_2 r_2 \tau} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=2^{\xi_2}+1}^{\infty} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$+\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{-\xi_2 r_2 \tau} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=1}^{2^{\xi_2}} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$+\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{-\xi_1 r_1\tau} 2^{\xi_2 r_2 \tau} \left\| \sum_{\nu_1=1}^{2^{\xi_1}} \sum_{\nu_2=2^{\xi_2}+1}^{\infty} \nu_1^{r_1+\beta_1} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$+\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{-\xi_1 r_1\tau} 2^{-\xi_2 r_2 \tau} \left\| \sum_{\nu_1=1}^{2^{\xi_1}} \sum_{\nu_2=1}^{2^{\xi_2}} \nu_1^{r_1+\beta_1} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$=: K_1 + K_2 + K_3 + K_4 .$$

K_2 yi değerlendirelim:

$$K_2 \ll \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{-\xi_2 r_2 \tau} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=1}^{2^{n_2}} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$+\sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau} 2^{-\xi_2 r_2 \tau} \left\| \sum_{\nu_1=2^{\xi_1}+1}^{\infty} \sum_{\nu_2=2^{\xi_2}+1}^{2^{\xi_2}} \nu_2^{r_2+\beta_2} A_{\nu_1\nu_2}\left(x_1, x_2\right)\right\|_p^{\tau}$$

$$=: K_{21} + K_{22} .$$

(4.10) ve 2.12 teorem ve 2.13 teoremi kullanırsak

$$K_{21} \ll 2^{-n_2\beta_2\tau} \sum_{\xi_1=n_1}^{\infty} 2^{\xi_1 r_1 \tau} \left(\int_0^{2\pi}\int_0^{2\pi} \left\langle \sum_{\nu_1=\xi_1+1}^{\infty} \sum_{\nu_2=0}^{n_2} 2^{2\nu_2\left(r_2+\beta_2\right)} \Delta^2_{\nu_1\nu_2} \right\rangle^{p/2} dx_1 dx_2 \right)^{\tau/p} .$$

Minkowski Eşitsizliğini, (8.3) ifadesini, ve 2.12 teorem ve 2.13 teoremi ve 4.6.6 teoremi kullanarak

$$K_{21} \ll 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\xi_1=n_1}^{\infty}2^{\xi_1 r_1\tau}\left\langle\sum_{\nu_1=\xi_1+1}^{\infty}\sum_{\nu_2=0}^{n_2}2^{2\nu_2(r_2+\beta_2)}\Delta^2_{\nu_1\nu_2}\right\rangle^{\tau/2}\right]^{p/\tau}dx_1dx_2\right\}^{\tau/p} \ll$$

$$\ll 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\left\langle\sum_{\xi_1=n_1}^{\nu_1}2^{\xi_1 r_1\tau}\right\rangle^{2/\tau}\sum_{\nu_2=0}^{n_2}2^{2\nu_2(r_2+\beta_2)}\Delta^2_{\nu_1\nu_2}\right]^{p/2}dx_1dx_2\right\}^{\tau/p} \ll$$

$$\ll 2^{-n_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=0}^{n_2}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\Delta^2_{\nu_1\nu_2}\right]^{p/2}dx_1dx_2\right\}^{\tau/p} \ll$$

$$\ll 2^{-n_2\beta_2\tau}\left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=1}^{2^{n_2}}\nu_1^{r_1}\nu_2^{r_2+\beta_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau} \ll \omega^{\tau}_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p .$$

K_{22} yi üstten değerlendirirsek

$$K_{22} \ll \sum_{\xi_2=n_2}^{\infty}2^{-\xi_2\beta_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\xi_2}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\Delta^2_{\nu_1\nu_2}\right]^{\frac{p}{2}}dx_1dx_2\right\}^{\frac{\tau}{p}} \ll$$

$$\ll \left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\delta_2=n_2}^{\infty}2^{-\xi_2\beta_2\tau}\left\langle\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty}2^{2\nu_1 r_1}2^{2\nu_2(r_2+\beta_2)}\Delta^2_{\nu_1\nu_2}\right\rangle^{\frac{\tau}{2}}\right]^{\frac{p}{\tau}}dx_1dx_2\right\}^{\frac{\tau}{p}} .$$

Dahası

$$K_{22} \ll \left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_2=n_2+1}^{\infty}2^{2\nu_2(r_2+\beta_2)}\left\langle\sum_{\xi_2=n_2}^{\infty}2^{-\xi_2\beta_2\tau}\right\rangle^{\frac{2}{\tau}}\sum_{\nu_1=n_1+1}^{\infty}2^{2\nu_1 r_1}\Delta^2_{\nu_1\nu_2}\right]^{\frac{p}{2}}dx_1dx_2\right\}^{\frac{\tau}{p}}$$

$$\ll \left\|\sum_{\nu_1=2^{n_1+1}}^{\infty}\sum_{\nu_2=2^{n_2+1}}^{\infty}\nu_1^{r_1}\nu_2^{r_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}$$

$$\ll \omega^{\tau}_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p .$$

Benzer şekilde

$$K_3 \ll 2^{-n_1\beta_1\tau}\left\|\sum_{\nu_1=1}^{2^{n_1}}\sum_{\nu_2=2^{n_2}+1}^{\infty}\nu_1^{r_1+\beta_1}\nu_2^{r_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}+\left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=2^{n_2}+1}^{\infty}\nu_1^{r_1}r_2^{r_2}A_{\nu_1\nu_2}(x_1,x_2)\right\|_p^{\tau}$$

$$\ll \omega^{\tau}_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p .$$

Son olarak K_1 ve K_4 için:

$$K_1 \ll \sum_{\xi_1=n_1}^{\infty}\sum_{\xi_2=n_2}^{\infty} 2^{\xi_1 r_1\tau}2^{\xi_2 r_2\tau}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[\sum_{\nu_1=\xi_1+1}^{\infty}\sum_{\nu_2=\xi_2+1}^{\infty}\Delta^2_{\nu_1\nu_2}\right]^{\frac{p}{2}}dx_1dx_2\right\}^{\frac{\tau}{p}} \ll$$

$$\ll \left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=2^{n_2}+1}^{\infty}\nu_1^{r_1}r_2^{r_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p^{\tau} \ll \omega^{\tau}_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p$$

ve

$$K_4 \ll \left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=2^{n_2}+1}^{\infty}\nu_1^{r_1}r_2^{r_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p^{\tau} + 2^{-n_2\beta_2\tau}\left\|\sum_{\nu_1=2^{n_1}+1}^{\infty}\sum_{\nu_2=1}^{\infty}\nu_1^{r_1}r_2^{r_2+\beta_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p^{\tau} +$$

$$+2^{-n_1\beta_1\tau}\left\|\sum_{\nu_1=1}^{2^{n_1}}\sum_{\nu_2=2^{n_2}+1}^{\infty}\nu_1^{r_1+\beta_1}r_2^{r_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p^{\tau} +$$

$$+2^{-n_1\beta_1\tau}2^{-n_2\beta_2\tau}\left\|\sum_{\nu_1=1}^{2^{n_1}}\sum_{\nu_2=1}^{2^{n_2}}\nu_1^{r_1+\beta_1}r_2^{r_2+\beta_2}A_{\nu_1\nu_2}\left(x_1,x_2\right)\right\|_p^{\tau} \ll \omega^{\tau}_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_p .$$

4.7.4 Teorem [8]

$1<p<\infty$, $f\in M_p$, $\beta_1,\beta_2,r_1,r_2>0$, $\delta_1,\delta_2\in\left(0,\frac{1}{2}\right)$ olsun. Bu durumda

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\delta_1,\delta_2\right)_{L_p\left(\mathrm{T}^2\right)} \approx \left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1p-1}t_2^{-r_2p-1}\omega^{p}_{r_1+\beta_1,r_2+\beta_2}\left(f,t_1,t_2\right)_{L_p\left(\mathrm{T}^2\right)}dt_1dt_2\right\}^{\frac{1}{p}} .$$

İspat

4.6.7 önerme, 4.6.8 önerme ve 4.6.10. teorem, 4.6.11 teorem yardımıyla aşağıdaki iki değerlendirmeyi elde ederiz.

$1<p<\infty$, $f\in M_p$, $i=1,2,\beta_i,r_i>0,n_1,n_2\in\mathbb{N}$ olsun.

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)},\frac{1}{n_1},\frac{1}{n_2}\right)_{L_p\left(\mathrm{T}^2\right)} \approx \frac{1}{n_1^{\beta_1}n_2^{\beta_2}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a^{p}_{\nu_1,\nu_2}\nu_1^{(\beta_1+r_1+1)p-2}\nu_2^{(\beta_2+r_2+1)p-2}\right\}^{\frac{1}{p}}$$

$$+\frac{1}{n_1^{\beta_1}}\left\{\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^p \nu_1^{(\beta_1+r_1+1)p-2} \nu_2^{(\beta_2+r_2+1)p-2}\right\}^{\frac{1}{p}}$$

$$+\frac{1}{n_2^{\beta_2}}\left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2}^p \nu_1^{(r_1+1)p-2} \nu_2^{(\beta_2+r_2+1)p-2}\right\}^{\frac{1}{p}}$$

$$+\left\{\sum_{\nu_1=n_1+1}^{\infty}\sum_{\nu_2=n_2+1}^{\infty} a_{\nu_1,\nu_2}^p \nu_1^{(r_1+1)p-2} \nu_2^{(r_2+1)p-2}\right\}^{\frac{1}{p}}.$$

4.7.5 Teorem [8]

$1 < p < \infty$, $f \in \Lambda_p$, $\beta_1, \beta_2, r_1, r_2 > 0$, $\delta_1, \delta_2 \in \left(0, \frac{1}{2}\right)$ olsun. Bu durumda

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)}, \delta_1, \delta_2\right)_{L_p(\mathrm{T}^2)} \approx \left\{\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-2r_1-1} t_2^{-2r_2-1} \omega_{r_1+\beta_1, r_2+\beta_2}^2\left(f, t_1, t_2\right)_{L_p(\mathrm{T}^2)} dt_1 dt_2\right\}^{\frac{1}{2}}.$$

İspat

4.6.7. önerme, 4.6.8. önerme ve 4.6.10. teorem, 4.6.11 teorem yardımıyla aşağıdaki iki değerlendirme elde edilebilir.

$1 < p < \infty$, $f \in \Lambda_p$, $i = 1,2, \beta_i, r_i > 0, n_1, n_2 \in \mathbb{N}$ olsun.

$$\omega_{\beta_1,\beta_2}\left(f^{(r_1,r_2)}, \frac{1}{2^{n_1}}, \frac{1}{2^{n_2}}\right)_p \approx \frac{1}{2^{n_1\beta_1+n_2\beta_2}}\left\{\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=1}^{n_2} \lambda_{\mu_1,\mu_2}^2 \, 2^{2(\mu_1(\beta_1+r_1)+\mu_2(\beta_2+r_2))}\right\}^{\frac{1}{2}}$$

$$+\frac{1}{2^{n_1\beta_1}}\left\{\sum_{\mu_1=1}^{n_1}\sum_{\mu_2=n_2+1}^{\infty} \lambda_{\mu_1,\mu_2}^p \, 2^{2(\mu_1(\beta_1+r_1)+\mu_2 r_2)}\right\}^{\frac{1}{2}}$$

$$+\frac{1}{2^{n_2\beta_2}}\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=1}^{n_2} \lambda_{\mu_1,\mu_2}^p \, 2^{2(\mu_1 r_1+\mu_2(\beta_2+r_2))}\right\}^{\frac{1}{2}}$$

$$+\left\{\sum_{\mu_1=n_1+1}^{\infty}\sum_{\mu_2=n_2+1}^{\infty} \lambda_{\nu_1,\nu_2}^p \, 2^{2(\mu_1 r_1+\mu_2 r_2)}\right\}^{\frac{1}{2}}.$$

5. SONUÇ VE ÖNERİLER

Sürekli 2π periyotlu fonksiyonlar ve L^p $(1 \leq p < \infty)$ uzayına ait fonksiyonlardaki karma düzgünlük modülü ile yaklaşımın aynen karma Lebesgue uzaylarında olduğu görülmüştür. Karma Lebesgue uzayları, Lebesgue uzaylarının bir genellemesi olduğundan bu beklenen bir sonuçtur. Bunun için gerekli kaynaklar taranmış, daha önce yapılan çalışmalar incelenmiştir. Hangi yöntemlerin kullanıldığı ayrıntılı şekilde açıklanmıştır. Bir sonraki çalışmalar için rehber niteliğinde olan bu tez sayesinde Türkçe bir kaynak ortaya koyulmuştur, bu ve konudaki Türkçe kaynak eksikliği giderilmeye çalışılmıştır.

6. KAYNAKLAR

[1] S. M. Nikol'skii, "*Approximation of Functions of Several Variables and Embedding Theorems*" Springer-Verlag, 1975.

[2] L. Leindler,"Generalization of inequalities of Hardy and Littlewood", Acta Sci. Math. 31, pp. 279-285, 1970.

[3] M. I. Dyachenko, "Some problems in the theory of multiple trigonometric series," Russian Math. Surveys 47(5) 1992, 103-171; translated from Uspekhi Mat. Nauk 47(5), pp. 97-162, 1992.

[4] Bedenek, A.; Panzone, R. The space L^p, with mixed norm. Duke Math. J. 28 191 pp. 301-324, 1961.

[5] Rubio de Francia, J.L.; Ruiz, F.J.; Torrea, J.L. Calderon-Zygmund theory for operator-valued kernels. Advances in Mathematics 62, 1, pp. 7-48, 1986.

[6] M. K. Potapov and B. V. Simonov, "Properties of Mixed Moduli of Smoothness of Functions with Lacunary Fourier Coefficients", Moskow University Mathematics Bulletin, 69, 1, pp. 5-15, 2014.

[7] Erika L. Ward, "New Estimates in Harmonic Analysis for Mixed Lebesgue Spaces", 2010.

[8] M. K. Potapov, B. V. Simonov and S. Yu. Tikhanov "*Mixed Moduli of Smoothness in* L_p, $1 < p < \infty$", vol. 8, pp. 1-57, 2013.

[9] M. K. Popatov, "On angular" approximation, Proc. Conf. Constructive Function Theory"(Budapest, 1969), Akad. Kiado, Budapest, pp. 371-379, 1971.

[10] M. K. Popatov, Imbedding of classes of functions with a dominating mixed modulus of smoothness, Trudy Mat. Inst. Steklov., 131, pp. 199-210, 1974.

[11] P. L. Butzer, H. Dyckhoff, E.Görlich, R. L. Stens, Best trigonometric approximation, fractional order derivatives and Lipschitz classes, Canad. J. Math. 29(4), pp. 781-793, 1977.

[12] R. Taberski, Differences, moduli and derivatives of frantional orders, Comment. Math. Prace Mat. 19(2), pp. 389-400, 1976-1977.

[13] M.K. Potapov and B. V. Simonov, Properties of Mixed Moduli of Smoothness of Positive Order in a Mixed Metric, Moskow University Mathematics Bullettin, vol 69, no:6, pp. 258-266, 2014.

[14] M. K. Popatov, “Approximation by Angle and Embedding Theorems,” Math. Balk. 2, 183, 1972.

[15] V. B. Simonov, “Relations for Moduli of Smoothness and Partial Fourier Series and Embedding Theorems of Nikol’skii Classes”, Doklady Russ. Akad. Nauk 437 (6), 751, 2011.

[16] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer, 1993.