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Approximation Theorems in Weighted Lebesgue Spaces with Variable Exponent

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Abstract. In this work, approximation properties of de la Vallée-Poussin means are investigated in weighted Lebesgue spaces with variable exponent where weight function belongs to Muckenhoupt class. For this purpose direct, inverse and simultaneous theorems of approximation theory are proved and constructive characterizations of functions are obtained in weighted Lebesgue spaces with variable exponent.

1. Introduction

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [1, \infty)$ be a Lebesgue measurable 2π periodic function. We suppose that the considered exponent functions $p(\cdot)$ satisfy the condition

$$1 < p_- := ess \inf_{x \in \mathbb{T}} p(x) \leq ess \sup_{x \in \mathbb{T}} p(x) := p^+ < \infty.$$

If there exist a positive constant c such that

$$\left|p(x) - p(y)\right| \ln\left(1/\left|x-y\right|\right) \leq c, \quad x, y \in \mathbb{T}, \quad 0 < \left|x-y\right| \leq 1/2, \tag{1}$$

then we say that $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. The Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ with variable exponent is defined as the set of all Lebesgue measurable 2π periodic functions f such that $\rho_{p(\cdot)}(f) := \int_0^{2\pi} \left|f(x)\right|^{p(x)} dx < \infty$. $L^{p(\cdot)}(\mathbb{T})$ becomes a Banach space equipped with the norm $\left\|f\right\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$ as long as $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$.

For a given weight ω, we define the weighted variable Lebesgue space $L^{p(\cdot)}_{\omega}(\mathbb{T})$ as the set of all measurable 2π periodic functions f such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. The norm of $L^{p(\cdot)}_{\omega}(\mathbb{T})$ can be defined as $\left\|f\right\|_{p(\cdot),\omega} := \left\|f\omega\right\|_{p(\cdot)}$.

The subspace $W^{p(\cdot),r}_{\omega}(\mathbb{T})$, $r = 1, 2, ...,$ consist of the functions $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$ such that $f^{(r-1)}$ is absolutely continuous on $\mathbb{T}$ and $f^{(r)} \in L^{p(\cdot)}_{\omega}(\mathbb{T})$. This subspace of $L^{p(\cdot)}_{\omega}(\mathbb{T})$ is called weighted Sobolev space with variable exponent. Also, $W^{p(\cdot),r}_{\omega}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ becomes a Banach space with the norm

$$\left\|f\right\|_{W^{p(\cdot),r}_{\omega}(\mathbb{T})} := \left\|f\right\|_{p(\cdot),\omega} + \left\|f^{(r)}\right\|_{p(\cdot),\omega}.$$

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Let $f \in L^1(\mathbb{T})$ and Fourier series of f is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where a_k and b_k, $k = 0, 1, 2, \ldots$, are Fourier coefficients of f such that

$$a_k := a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos ktdt \text{ and } b_k := b_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sin ktdt.$$

Let

$$S_n(f) = S_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 1, 2, \ldots,$$

be the nth partial sums of Fourier series of f. Also, de la Vallée-Poussin means of f is defined as

$$V_m^n(f) = V_m^n(f, x) := \frac{1}{m+1} \sum_{k=n}^{n+m} S_k(f, x).$$

Definition 1.1. *For a given exponent $p(\cdot)$ we say that $\omega \in A_{p(\cdot)}(\mathbb{T})$ if*

$$\sup_{B_j} |B_j|^{-1} \|\omega \chi_{B_j}\|_{p(\cdot)} \|\omega^{-1} \chi_{B_j}\|_{q(\cdot)} < \infty, \quad 1/p(\cdot) + 1/q(\cdot) = 1,$$

where supremum is taken over all open intervals $B_j \subset \mathbb{T}$ with the characteristic functions χ_{B_j}.

Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. The K-functional $K_r(f, \delta)_{p(\cdot),\omega}$, $r = 1, 2, \ldots$, for $\delta > 0$ is defined as

$$K_r(f, \delta)_{p(\cdot),\omega} := \inf_{g \in W_\omega^{p(\cdot),r}(\mathbb{T})} \left\{ \|f - g\|_{p(\cdot),\omega} + \delta^r \|g^{(r)}\|_{p(\cdot),\omega} \right\}.$$

Definition 1.2. *Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let*

$$\Delta_t^r f(x) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} f(x + st), \quad r = 1, 2, \ldots, .$$

We define the rth modulus of smoothness as

$$\Omega_r(f, \delta)_{p(\cdot),\omega} := \sup_{0 < h \le \delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f dt \right\|_{p(\cdot),\omega}, \quad \delta > 0.$$

The correctness of this definition follows from the boundedness of the maximal operator

$$M(f) : f \to Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(t)| dt$$

for any $x \in \mathbb{T}$ in the $L_\omega^{p(\cdot)}(\mathbb{T})$, where supremum is taken over all subinterval $B \subset \mathbb{T}$ that contain x (see, [6]). So we have that if $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, then the maximal operator $M(f)$ is bounded in $L_\omega^{p(\cdot)}(\mathbb{T})$. In this case there exists a positive constant $c(p)$ such that the inequality

$$\|M(f)\|_{p(\cdot),\omega} \le c(p) \|f\|_{p(\cdot),\omega} \tag{2}$$

holds for every $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$. In $L_{\omega}^{p(\cdot)}(\mathbb{T})$ the boundedness of $M(f)$ provides us the boundedness of $\Omega_r(f,\delta)_{p(\cdot),\omega}$. Namely, if $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p)$ such that

$$\Omega_r(f,\delta)_{p(\cdot),\omega} \leq c(p)\left\|f\right\|_{p(\cdot),\omega} . \tag{3}$$

Moreover, it can be shown that if $f, g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then

$$\Omega_r(f+g,\delta)_{p(\cdot),\omega} \leq \Omega_r(f,\delta)_{p(\cdot),\omega} + \Omega_r(g,\delta)_{p(\cdot),\omega} \text{ and } \lim_{\delta\to 0}\Omega_r(f,\delta)_{p(\cdot),\omega} = 0.$$

Lebesgue spaces with variable exponent were introduced by Orlicz in [23]. Lebesgue spaces with variable exponent enable us important tools for explanations of different applications in mechanic, like modelling of electrorheological fluids. As a result of this fact, investigations on the fundamental problems of these spaces have been studied recently, in view of potential theory, maximal and singular integral operator theory and others. The corresponding results can be found in the monographs [5, 9]. If we focus on the approximation theory in Lebesgue spaces with variable exponent, we can found some of the pioneering results in [13] and [10]. Under the condition $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, by defining the first order modulus of smoothness for $f \in L^{p(\cdot)}(\mathbb{T})$ as

$$\Omega_{p(\cdot)}(f,\delta) := \left\|\frac{1}{h}\int_0^h \left|f(x+t) - f(x)\right| dt\right\|_{p(\cdot)},$$

the direct theorem in $L^{p(\cdot)}(\mathbb{T})$ was proved in [10]. Some problems of approximation theory in $L^{p(\cdot)}(\mathbb{T})$ were investigated in [1] where $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. If the exponent function $p(\cdot)$ satisfies that $1 \leq p_- \leq p < \infty$ and the condition (1), then we say that $p(\cdot) \in \mathcal{P}(\mathbb{T})$. Clearly, $\mathcal{P}(\mathbb{T})$ is more generalized than $\mathcal{P}_0(\mathbb{T})$. In case $p(\cdot) \in \mathcal{P}(\mathbb{T})$, the first order modulus of smoothness for $f \in L^{p(\cdot)}(\mathbb{T})$, which is more sensitive than $\Omega_{p(\cdot)}(f,\delta)$, was defined as

$$\Omega(f,\delta)_{p(\cdot)} := \left\|\frac{1}{h}\int_0^h (f(x) - f(x+t))\, dt\right\|_{p(\cdot)}$$

in [25]. Also, under the condition $p(\cdot) \in \mathcal{P}(\mathbb{T})$, the direct and inverse theorems of approximation theory in $L^{p(\cdot)}(\mathbb{T})$ were proved in [25] and [12], respectively. Later, these results extended to weighted Lebesgue spaces with variable exponent in [14, 15] where the $\omega \in A_{p(\cdot)}(\mathbb{T})$ and $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$.

Approximation properties of de la Vallée Poussin means of given function $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$ were investigated by means of $\Omega(f,\delta)_{p(\cdot)}$ in [26, 27]. Later, these results were extended to $L_{\omega}^{p(\cdot)}(\mathbb{T})$ in [29] under the conditions $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega \in A_{p(\cdot)}(\mathbb{T})$. In this work, one of our aims is to generalize the results proved in [29] by using $\Omega_r(f,1/n)_{p(\cdot),\omega}$, $r = 1,2,...$, and $n = 1,2,...$, which equivalent to $K_r(f,1/n)_{p(\cdot),\omega}$. For this purpose, direct and inverse theorems of approximation theory are proved in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega \in A_{p(\cdot)}(\mathbb{T})$. The similar theorems in nonweighted case, were proved in [16] and [31]. Another one of our aims is to generalize some approximation theorems investigated in [21] for classical Lebesgue space. Previously, the approximation properties of different summation methods were investigated in [11, 19, 20]. On the other hand, by different type modulus of smoothness appropriate approximation theorems in $L_{\omega}^{p(\cdot)}(\mathbb{T})$ were proved in [2, 3], where $\omega^{-p_0} \in A_{q^*(\cdot)}(\mathbb{T})$, $\frac{1}{(p(\cdot)/p_0)} + \frac{1}{q^*(\cdot)} = 1$, for some $p_0 \in (1, p_-)$. Nevertheless, the $A_{p(\cdot)}(\mathbb{T})$ class is more intelligible than the class of weights considered in these work.

The best approximation number of $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ is defined as$E_n(f)_{p(\cdot),\omega} := \inf\left\{\left\|f - T_n\right\|_{p(\cdot),\omega} : T_n \in \Pi_n\right\}$, $n = 0,1,2,..$, where Π_n is the class of trigonometric polynomials of degree not exceeding n.

Also, $T_n^0 := T_n^0(f) \in \Pi_n$ is called best approximating polynomial to $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, if T_n^0 satisfies that $\left\|f - T_n^0\right\|_{p(\cdot),\omega} = E_n(f)_{p(\cdot),\omega}$.

Throughout this work the constants can vary line by line but are independent of n. In addition, we use the notation $f = O(g)$ which means that $f \leq cg$ for some positiveconstant c. We will also use the notation $f \approx g$ for the sake of reader's convenience if there exist two constants $c_1(p,r)$ and $c_2(p,r)$ such that $c_1(p,r) f \leq g \leq c_2(p,r) f$.

The our main results are follows.

Theorem 1.3. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $r = 1,2,...$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p,r)$ such that the inequality*

$$E_n(f)_{p(\cdot),\omega} \leq c(p,r)\, \Omega_r(f, 1/n)_{p(\cdot),\omega}, \quad n = 1,2,...,$$

holds.

Theorem 1.4. *Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Then for $n = 1,2,...$, and $r = 1,2,...$, the equivalent*

$$\Omega_r(f, 1/n)_{p(\cdot),\omega} \approx K_r(f, 1/n)_{p(\cdot),\omega}$$

holds.

Theorem 1.5. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $r = 1,2,...$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p,r)$ such that the inequality*

$$\left\|f - V_m^n(f)\right\|_{p(\cdot),\omega} \leq \frac{c(p,r)}{m+1} \sum_{k=0}^{m} \Omega_r\left(f, \frac{1}{n+k}\right)_{p(\cdot),\omega}, \quad n = 1,2,...,$$

holds.

Applying Theorem 1.5 and Lemma 2.5 proved in next section we have

Corollary 1.6. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in W_\omega^{p(\cdot),r}(\mathbb{T})$, $r = 1,2,...$, then there exists a positive constant $c(p,r)$ such that the inequality*

$$\left\|f - V_m^n(f)\right\|_{p(\cdot),\omega} \leq \frac{c(p,r)}{m+1} \sum_{k=0}^{m} \frac{1}{(n+k)^r} \left\|f^{(r)}\right\|_{p(\cdot),\omega}, \quad n = 1,2,...,$$

holds.

Theorem 1.7. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $r = 1,2,...$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p,r)$ such that the inequality*

$$\Omega_r(f, 1/n)_{p(\cdot),\omega} \leq \frac{c(p,r)}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{p(\cdot),\omega}, \quad n = 1,2,...,$$

holds.

Firstly, Theorems 1.3, 1.4 and 1.7 in weighted Lebesgue space $L_\omega^p(\mathbb{T})$, $1 < p < \infty$ were proved in [22]. Under the condition $p(\cdot) \in \mathcal{P}(\mathbb{T})$ in nonweighted case, Theorems 1.3 and 1.7 were proved in [16] and [31] independently of each other. Theorem 1.5 for $r = 1$ was proved in [27]. In case $r = 1$, Theorem 1.3 was proved in [14], Theorem 1.7 was proved in [15], Theorems 1.4 and 1.5 were proved in [29]. In case $r \in \mathbb{R}^+$, Theorems 1.3 and 1.7 were proved in [2] by different type modulus of smoothness where $\omega^{-p_0} \in A_{q^*(\cdot)}(\mathbb{T})$, $\frac{1}{(p(\cdot)/p_0)} + \frac{1}{q^*(\cdot)} = 1$, for some $p_0 \in (1, p_-)$.

Let $\alpha > 0$ and $[\alpha]$ be is the integer part of α. If we define a generalized Lipschitz class $Lip_{\alpha,r}^{p(\cdot),\omega}(\mathbb{T})$ for $r := [\alpha] + 1$ as

$$Lip_{\alpha,r}^{p(\cdot),\omega}(\mathbb{T}) := \left\{ f \in L_{\omega}^{p(\cdot)}(\mathbb{T}) : \Omega_r(f,\delta)_{p(\cdot),\omega} = O(\delta),\ \delta > 0 \right\},$$

then taking into account Theorem 1.3 and 1.7 by simple computations we have

Corollary 1.8. *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and $\alpha > 0$. Then the following statements are equivalent:*

$$i) E_n(f)_{p(\cdot),\omega} = O(n^{-\alpha}),\ n = 1,2,...,\ \text{and}\ \ ii)\ f \in Lip_{\alpha,r}^{p(\cdot)}(\mathbb{T}).$$

Theorem 1.9. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in W_{\omega}^{p(\cdot),k}(\mathbb{T})$, $k = 1,2,...$, then there exists a positive constant $c(p,k)$ such that the inequality*

$$\left\| f - V_m^n(f) \right\|_{p(\cdot),\omega} \le \frac{c(p,k)}{(n+1)^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega}$$

holds.

Theorem 1.10. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in W_{\omega}^{p(\cdot),r}(\mathbb{T})$, $r = 1,2,...$, then there exists a positive constant $c(p,r)$ such that the inequality*

$$\left\| f - V_m^n(f) \right\|_{p(\cdot),\omega} \le \frac{c(p,r)}{m+1} \sum_{k=n}^{n+m} \frac{1}{(k+1)^r} E_k\left(f^{(r)}\right)_{p(\cdot),\omega}$$

holds.

In classical Lebesgue space, Theorem 1.9 was proved in [24] where $m \in \{n, n-1\}$. In nonweighted case, Theorem 1.9 was proved under the condition $p(\cdot) \in \mathcal{P}(\mathbb{T})$ in [26] and [27] where $m \in \{n, n-1\}$, $n = O(m)$ respectively. In case $r = 1$, Theorem 1.10 was proved in [29].

If trigonometric polynomial $T_n^* := T_n^*(f) \in \Pi_n$ satisfies the inequality $\left\| f - T_n^* \right\|_{p(\cdot),\omega} \le cE_n(f)_{p(\cdot),\omega}$, $n = 0,1,2,..$, for some positive constant c is independent of n, then T_n^* is called near-best approximating polynomail to $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$.

Theorem 1.11. *Let $f \in W_{\omega}^{p(\cdot),k}(\mathbb{T})$, $k = 1,2,...$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $T_n^* \in \Pi_n$ is a near-best approximating polynomial to f, then there exists a positive constant $c(p,k)$ such that the inequality*

$$\left\| f^{(k)} - (T_n^*)^{(k)} \right\|_{p(\cdot),\omega} \le c(p,k) E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \ ,\ n = 1,2,..,$$

holds.

This theorem in Lebesgue space $L^p(\mathbb{T})$, $1 \le p < \infty$ was proved in [7]. In nonweighted case, Theorem 1.11 was proved in [17] when $p(\cdot) \in \mathcal{P}(\mathbb{T})$. In case $k = 1$, Theorem 1.11 was proved in [15]. Moreover, in the case $k \in \mathbb{R}^+$, Theorem 1.11 was proved in [2] where $\omega^{-p_0} \in A_{q^*(\cdot)}(\mathbb{T})$, $\frac{1}{(p(\cdot)/p_0)} + \frac{1}{q^*(\cdot)} = 1$ for some $p_0 \in (1, p_-)$ and it was proved in weighted Lebesgue and weighted Orlicz spaces in [32] and [4], respectively.

The inequality

$$\left\| f - S_n(f) \right\|_{p(\cdot),\omega} \le c(p) E_n(f)_{p(\cdot),\omega},\ n = 1,2,..., \tag{4}$$

proved in [29] implies that $S_n(f)$ is near-best approximating polynomail to $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$. Hence, applying Theorem 1.11 we have immediately

Corollary 1.12. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in W_{\omega}^{p(\cdot),k}(\mathbb{T})$, $k = 1, 2, ...$, then there exists a positive constant $c(p,k)$ such that the inequality*

$$\left\| f^{(k)} - S_n^{(k)}(f) \right\|_{p(\cdot),\omega} \leq c(p,k) E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \ , n = 1, 2, ..,$$

holds.

Theorem 1.13. *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Let $\sum\limits_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} < \infty$ where $\alpha \in \mathbb{R}$. Then the series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} k^{\alpha} (a_k \cos kx + b_k \sin kx) \tag{5}$$

is Fourier series of the function $g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ and for every $g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ the inequalities

$$E_n(g)_{p(\cdot),\omega} \leq c(p) \left\{ n^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\}, \ n = 1, 2, ..., \tag{6}$$

and

$$E_0(g)_{p(\cdot),\omega} \leq c(p) \left\{ E_0(f)_{p(\cdot),\omega} + \sum_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\} \tag{7}$$

hold with positive constants independent of n.

Theorem 1.14. *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $r = 1, 2, ...$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Let $\sum\limits_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} < \infty$ where $\alpha \in \mathbb{R}$ and let the series (5) be Fourier series of $g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$. Then the inequality*

$$\Omega_r\left(g, \frac{1}{n}\right)_{p(\cdot),\omega} \leq c(p) \left\{ \frac{1}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_{\nu}(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{p(\cdot),\omega} \right\}$$

for $n = 1, 2, ...$, holds with positive constants independent of n.

Similar results were proved in [2] and [3] where $\omega^{-p_0} \in A_{q^*(\cdot)}(\mathbb{T})$, $\frac{1}{(p(\cdot)/p_0)} + \frac{1}{q^*(\cdot)} = 1$ for some $p_0 \in (1, p_-)$. In weighted generalized grand Lebesgue spaces, Theorem 1.13 and 1.14 for conjugate functions were proved in [18]. Similar theorems in Lebesgue spaces were proved in [21] and [28].

2. Auxiliary Results

Lemma 2.1 ([14]). *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. If $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then the operator $S_n(f) : f(\cdot) \to S_n(f, \cdot)$ is bounded in $L_{\omega}^{p(\cdot)}(\mathbb{T})$ and there exists a positive constant $c(p)$ such that $\left\| S_n(f) \right\|_{p(\cdot),\omega} \leq c(p) \left\| f \right\|_{p(\cdot),\omega}$, $n = 1, 2, ...$, holds.*

If $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ then applying *Lemma 2.1* we have

$$\left\| V_m^n(f) \right\|_{p(\cdot),\omega} \leq \frac{1}{m+1} \sum_{k=n}^{n+m} \left\| S_k(f) \right\|_{p(\cdot),\omega} \leq c(p) \left\| f \right\|_{p(\cdot),\omega}$$

and

$$E_n(f)_{p(\cdot),\omega} \leq \left\| f - S_n(f) \right\|_{p(\cdot),\omega} \leq c(p) \left\| f \right\|_{p(\cdot),\omega} \ . \tag{8}$$

Let $f \in L^1(\mathbb{T})$ and let $\widetilde{f}$ be its conjugate function defined as

$$\widetilde{f}(x) := \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2\tan\frac{t-x}{2}}dt.$$

It is well known that if $\widetilde{f}$ is integrable, then its Fourier series coincides with the conjugate Fourier series of f [30, p. 155] and hence

$$\widetilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx).$$

Lemma 2.2 ([14]). *Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. If $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then the operator $\widetilde{f} : f(\cdot) \to \widetilde{f}(\cdot)$ is bounded in $L_\omega^{p(\cdot)}(\mathbb{T})$ and there exists a positive constant $c(p)$ such that $\left\|\widetilde{f}\right\|_{p(\cdot),\omega} \leq c(p)\left\|f\right\|_{p(\cdot),\omega}$ holds.*

Since $\widetilde{S_n}(f)$ is a trigonometric polynomials with degree n, by *Lemma 2.2* and (4) we can obtain

$$E_n\left(\widetilde{f}\right)_{p(\cdot),\omega} \leq c(p)E_n(f)_{p(\cdot),\omega}, \ n = 1, 2, ..., . \tag{9}$$

Lemma 2.3 ([15]). *Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If T_n is a trigonometric polynomial with degree n, then there exists a positive constant $c(p)$ such that for $r = 1, 2, 3, ...$ the inequality $\left\|T_n^{(r)}\right\|_{p(\cdot),\omega} \leq c(p)\, n^r \left\|T_n\right\|_{p(\cdot),\omega}$, $n = 1, 2, ...,$ holds.*

Lemma 2.4. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in W_\omega^{p(\cdot),k}(\mathbb{T})$, $k = 1, 2, ...,$ then there exists a positive constant $c(p, k)$ such that the inequality*

$$E_n(f)_{p(\cdot),\omega} \leq \frac{c(p,k)}{(n+1)^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega}$$

holds.

Proof. Let $f \in W_\omega^{p(\cdot),k}(\mathbb{T})$, $k = 1, 2, ...,$. For the Fourier coefficients of f, denoted by a_ν and b_ν, $\nu = 1, 2, ...,$ we set

$$A_0(f, x) = \frac{a_0}{2};\ A_\nu(f, x) = a_\nu \cos \nu x + b_\nu \sin \nu x \text{ and } A_\nu\left(\widetilde{f}, x\right) = a_\nu \sin \nu x - b_\nu \cos \nu x.$$

Since

$$A_\nu(f, x) = A_\nu\left(f, x + \frac{k\pi}{2\nu}\right)\cos\frac{k\pi}{2} + A_\nu\left(\widetilde{f}, x + \frac{k\pi}{2\nu}\right)\sin\frac{k\pi}{2}$$

$$A_\nu\left(f^{(k)}, x\right) = \nu^k A_k\left(f, x + \frac{k\pi}{2\nu}\right) \text{ and } A_\nu\left(\widetilde{f^{(k)}}, x\right) = \nu^k A_\nu\left(\widetilde{f}, x + \frac{k\pi}{2\nu}\right),$$

we have

$$\sum_{\nu=n+1}^{\infty} \frac{1}{\nu^k} A_\nu\left(f^{(k)}, x\right) = \sum_{\nu=n+1}^{\infty} \frac{1}{\nu^k}\left(\left[S_\nu\left(f^{(k)}, x\right) - f^{(k)}(x)\right] - \left[S_{\nu-1}\left(f^{(k)}, x\right) - f^{(k)}(x)\right]\right)$$

$$= \sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k} - \frac{1}{(\nu+1)^k}\right)\left[S_\nu\left(f^{(k)}, x\right) - f^{(k)}(x)\right] - \frac{1}{(n+1)^k}\left[S_n\left(f^{(k)}, x\right) - f^{(k)}(x)\right].$$

Similar relation holds also for $\sum\limits_{\nu=n+1}^{\infty}\frac{1}{\nu^k}A_\nu\left(\widetilde{f^{(k)}},x\right)$. By (4) and (9) we have

$$\left\|f(x)-S_n(f,x)\right\|_{p(\cdot),\omega}=\left\|\sum_{\nu=n+1}^{\infty}A_\nu(f,x)\right\|_{p(\cdot),\omega}=\left\|\cos\frac{k\pi}{2}\sum_{\nu=n+1}^{\infty}\frac{1}{\nu^k}A_\nu\left(f^{(k)},x\right)+\sin\frac{k\pi}{2}\sum_{\nu=n+1}^{\infty}\frac{1}{\nu^k}A_\nu\left(\widetilde{f^{(k)}},x\right)\right\|_{p(\cdot),\omega}$$

$$\leq\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}\right)\left\|S_\nu\left(f^{(k)},x\right)-f^{(k)}(x)\right\|_{p(\cdot),\omega}+\frac{1}{(n+1)^k}\left\|S_n\left(f^{(k)},x\right)-f^{(k)}(x)\right\|_{p(\cdot),\omega}$$

$$+\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}\right)\left\|S_\nu\left(\widetilde{f^{(k)}},x\right)-\widetilde{f^{(k)}}(x)\right\|_{p(\cdot),\omega}+\frac{1}{(n+1)^k}\left\|S_n\left(\widetilde{f^{(k)}},x\right)-\widetilde{f^{(k)}}(x)\right\|_{p(\cdot),\omega}$$

$$\leq c(p)\left\{\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}\right)E_\nu\left(f^{(k)}\right)_{p(\cdot),\omega}+\frac{1}{(n+1)^k}E_n\left(f^{(k)}\right)_{p(\cdot),\omega}\right\}$$

$$+c(p)\left\{\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}\right)E_\nu\left(\widetilde{f^{(k)}}\right)_{p(\cdot),\omega}+\frac{1}{(n+1)^k}E_n\left(\widetilde{f^{(k)}}\right)_{p(\cdot),\omega}\right\}$$

$$\leq c(p)\left\{\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}\right)E_\nu\left(f^{(k)}\right)_{p(\cdot),\omega}+\frac{1}{(n+1)^k}E_n\left(f^{(k)}\right)_{p(\cdot),\omega}\right\}$$

$$\leq c(p)\sum_{\nu=n+1}^{\infty}\left(\frac{1}{\nu^k}-\frac{1}{(\nu+1)^k}+\frac{1}{(n+1)^k}\right)E_n\left(f^{(k)}\right)_{p(\cdot),\omega}\leq\frac{c(p,k)}{(n+1)^k}E_n\left(f^{(k)}\right)_{p(\cdot),\omega}.$$

Thus lemma is proved. □

Lemma 2.5. *Let $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. Then there exists a positive constant $c(p)$, such that for any $r=1,2,...,$ $\delta>0$ and for any function $f\in W_\omega^{p(\cdot),r}(\mathbb{T})$ the inequality*

$$\Omega_r(f,\delta)_{p(\cdot),\omega}\leq c(p,r)\delta^r\left\|f^{(r)}\right\|_{p(\cdot),\omega}$$

holds.

Proof. Since

$$\Delta_t^r f(x)=\int_0^t\int_0^t...\int_0^t f^{(r)}(x+t_1+...+t_r)\,dt_1...dt_r,$$

applying r times the generalized Minkowski inequality and (2) we have

$$\begin{aligned}
\left\| \frac{1}{h}\int_0^h \Delta_t^r f dt \right\|_{p(\cdot),\omega} &\leq c(p)\frac{1}{h}\int_0^h \left\|\Delta_t^r f\right\|_{p(\cdot),\omega} dt \\
&\leq c(p)h^r \frac{1}{h^{r+1}}\int_0^h \left\| \int_0^t \dots \int_0^t f^{(r)}\left(\cdot + t_1 + \dots + t_r\right) dt_1 \dots dt_r \right\|_{p(\cdot),\omega} dt \\
&\leq c(p)h^r \frac{1}{h}\int_0^h \left\| \frac{1}{h}\int_0^t \left| \frac{1}{h^{r-1}}\int_0^t \dots \int_0^t f^{(r)}\left(\cdot + t_1 + \dots + t_r\right) dt_1 \dots dt_{r-1} \right| dt_r \right\|_{p(\cdot),\omega} dt \\
&= c(p)h^r \frac{1}{h}\int_0^h \left\| \frac{1}{h}\int_0^h \left| \frac{1}{h^{r-1}}\int_0^t \dots \int_0^t f^{(r)}\left(\cdot + t_1 + \dots + t_r\right) dt_1 \dots dt_{r-1} \right| dt_r \right\|_{p(\cdot),\omega} dt \\
&\leq c(p)h^r \frac{1}{h}\int_0^h \left\| \frac{1}{h^{r-1}}\int_0^t \dots \int_0^t f^{(r)}\left(\cdot + t_1 + \dots + t_{r-1}\right) dt_1 \dots dt_{r-1} \right\|_{p(\cdot),\omega} dt \\
&\leq \dots \leq c(p)h^r \frac{1}{h}\int_0^h \left\| \left\{ \frac{1}{h}\int_0^h \left| f^{(r)}\left(\cdot + t_1\right) \right| dt_1 \right\} \right\|_{p(\cdot),\omega} dt \\
&\leq c(p)h^r \left\|f^{(r)}\right\|_{p(\cdot),\omega} \frac{1}{h}\int_0^h dt = c(p)h^r \left\|f^{(r)}\right\|_{p(\cdot),\omega}
\end{aligned}$$

and taking here the supremum we obtain the inequality $\Omega_r(f,\delta)_{p(\cdot),\omega} \leq c(p,r)\delta^r \left\|f^{(r)}\right\|_{p(\cdot),\omega}$. □

For $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ and $\delta > 0$ we define the Steklov mean value function

$$f_{r,\delta}(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\int_0^h \dots \int_0^h f\left(x + \frac{r-s}{r}\left[t_1 + \dots + t_r\right]\right) dt_1 \dots dt_r dh.$$

Lemma 2.6. *If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$, then $f_{r,\delta} \in W_\omega^{p(\cdot),r}(\mathbb{T})$ for $r = 1, 2, \dots$, and $\delta > 0$.*

Proof. The equation

$$f_{r,\delta}^{(r-1)}(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\left\{ \sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\int_0^{\frac{r-s}{r}h}\left(\frac{r}{r-s}\right)^r \Delta_{\frac{r-s}{r}h}^{r-1} f(x+t)\, dt \right\} dh$$

$$= \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\left\{ \sum_{s=0}^{r-1}\int_x^{x+\frac{r-s}{r}h}(-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r \Delta_{\frac{r-s}{r}h}^{r-1} f(t)\, dt \right\} dh \tag{10}$$

were proved in [16]. The Steklov mean value function $f_{r,\delta}^{(r-1)}$ is absolute continuity on $[0, 2\pi]$, this fact can be showed by standard way. It remains to prove the imbedding $f_{r,\delta}^{(r)} \in L_\omega^{p(\cdot)}(\mathbb{T})$. Differentiating the relation

(10) we obtain

$$f_{r,\delta}^{(r)}(x)=\frac{2}{\delta}\int_{\delta/2}^{\delta}\frac{1}{h^{r}}\left(\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\Delta_{\frac{r-s}{r}h}^{r}f(x)\right)dh$$

and denoting $t:=\frac{r-s}{r}h$ we have

$$\begin{aligned}\left|f_{r,\delta}^{(r)}(x)\right| &\leq \frac{2^{r+1}}{\delta^{r}}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left|\frac{1}{\delta}\int_{\delta/2}^{\delta}\Delta_{\frac{r-s}{r}h}^{r}f(x)\,dh\right|=\frac{2^{r+1}}{\delta^{r}}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left|\frac{1}{\frac{r-s}{r}\delta}\int_{\frac{r-s}{r}(\delta/2)}^{\frac{r-s}{r}\delta}\Delta_{t}^{r}f(x)\,dt\right|\\ &\leq \frac{2^{r+1}}{\delta^{r}}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left\{\left|\frac{1}{\frac{r-s}{r}\delta}\int_{0}^{\frac{r-s}{r}\delta}\Delta_{t}^{r}f(x)\,dt\right|+\left|\frac{1}{\frac{r-s}{r}\delta}\int_{0}^{\frac{r-s}{r}(\delta/2)}\Delta_{t}^{r}f(x)\,dt\right|\right\},\end{aligned}$$

which by (3) implies the inequality

$$\left\|f_{r,\delta}^{(r)}\right\|_{p(\cdot),\omega}\leq 2c(r)\delta^{-r}\Omega_{r}(f,\delta)_{p(\cdot),\omega}\leq c(p,r)\left\|f\right\|_{p(\cdot),\omega}. \tag{11}$$

Since $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$ the relation (11) means that $f_{r,\delta}^{(r)}\in L_{\omega}^{p(\cdot)}(\mathbb{T})$. $\square$

3. Proofs of Main Results

Proof of Theorem 1.3. Let $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_{0}(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. For $\delta>0$ and $r=1,2,...$, after some necessary simplifications we have

$$\left|f_{r,\delta}(x)-f(x)\right|=\frac{2}{\delta}\left|\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r}}\int_{0}^{h}\cdots\int_{0}^{h}\Delta_{\frac{t_{1}+...+t_{r}}{r}}^{r}f(x)\,dt_{1}...dt_{r}\right\}dh\right|$$

and then by the generalized Minkowski inequality

$$\begin{aligned}\left\|f_{r,\delta}-f\right\|_{p(\cdot),\omega} &\leq c(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_{0}^{h}\cdots\int_{0}^{h}\left\|\frac{1}{h}\int_{0}^{h}\Delta_{\frac{t_{1}+...+t_{r}}{r}}^{r}f\,dt_{1}\right\|_{p(\cdot),\omega}dt_{2}...dt_{r}\right\}dh\\ &=c(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_{0}^{h}\cdots\int_{0}^{h}\left\|\frac{1}{h}\int_{t_{2}+...+t_{r}}^{h+t_{2}+...+t_{r}}\Delta_{\frac{t}{r}}^{r}f\,dt\right\|_{p(\cdot),\omega}dt_{2}...dt_{r}\right\}dh.\end{aligned} \tag{12}$$

Since

$$\begin{aligned}\left\|\frac{1}{h}\int_{t_{2}+...+t_{r}}^{h+t_{2}+...+t_{r}}\Delta_{\frac{t}{r}}^{r}f\,dt\right\|_{p(\cdot),\omega} &=\left\|\frac{1}{h}\left(\int_{0}^{h+t_{2}+...+t_{r}}\Delta_{\frac{t}{r}}^{r}f\,dt-\int_{0}^{t_{2}+...+t_{r}}\Delta_{\frac{t}{r}}^{r}f\,dt\right)\right\|_{p(\cdot),\omega}\\ &\leq\left\|\frac{1}{(h+t_{2}+...+t_{r})/r}\int_{0}^{(h+t_{2}+...+t_{r})/r}\Delta_{t}^{r}f\,dt\right\|_{p(\cdot),\omega}+\left\|\frac{1}{(t_{2}+...+t_{r})/r}\int_{0}^{(t_{2}+...+t_{r})/r}\Delta_{t}^{r}f\,dt\right\|_{p(\cdot),\omega}\end{aligned}$$

$$\leq \sup_{(h+t_2+...+t_r)/r\leq\delta}\left\|\frac{1}{(h+t_2+...+t_r)/r}\int_0^{(h+t_2+...+t_r)/r}\Delta_t^r f dt\right\|_{p(\cdot),\omega}+\sup_{(t_2+...+t_r)/r\leq\delta}\left\|\frac{1}{(t_2+...+t_r)/r}\int_0^{(t_2+...+t_r)/r}\Delta_t^r f dt\right\|_{p(\cdot),\omega}$$

$$=\Omega_r(f,\delta)_{p(\cdot),\omega}+\Omega_r(f,\delta)_{p(\cdot),\omega}=2\Omega_r(f,\delta)_{p(\cdot),\omega}, \tag{13}$$

combining (12) and (13) we have

$$\left\|f_{r,\delta}-f\right\|_{p(\cdot),\omega}\leq c(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_0^h...\int_0^h\Omega_r(f,\delta)_{p(\cdot),\omega}dt_2...dt_r\right\}dh$$

$$\leq c(p,r)\Omega_r(f,\delta)_{p(\cdot),\omega}\frac{2}{\delta}\int_{\delta/2}^{\delta}dh=c(p,r)\Omega_r(f,\delta)_{p(\cdot),\omega}. \tag{14}$$

By Lemma 2.4 and (8) we have $E_n(f)_{p(\cdot),\omega}\leq\frac{c(p)}{n^r}\left\|f^{(r)}\right\|_{p(\cdot),\omega}$, later using the relations (8), (14), (11) for $\delta:=1/n$ we conclude that

$$E_n(f)_{p(\cdot),\omega}\leq E_n(f-f_{r,1/n})_{p(\cdot),\omega}+E_n(f_{r,1/n})_{p(\cdot),\omega}\leq\left\|f_{r,1/n}-f\right\|_{p(\cdot),\omega}+\frac{c(p)}{n^r}\left\|f_{r,1/n}^{(r)}\right\|_{p(\cdot),\omega}$$

$$\leq c(p,r)\Omega_r(f,1/n)_{p(\cdot),\omega}+\frac{c(p,r)}{n^r}n^r\Omega_r(f,1/n)_{p(\cdot),\omega}\leq c(p,r)\Omega_r(f,1/n)_{p(\cdot),\omega}.\quad\square$$

Proof of Theorem 1.4. Let $f\in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. By (11), (14) for $\delta:=1/n$ and taking infimum we have

$$K_r(f,1/n)_{p(\cdot),\omega}\leq c(p,r)\Omega_r(f,1/n)_{p(\cdot),\omega}\ ,\ n=1,2,..., \tag{15}$$

for $r=1,2,...,$. On the other hand, by (3) and Lemma 2.5 we obtain

$$\begin{aligned}\Omega_r(f,1/n)_{p(\cdot),\omega} &\leq \Omega_r(f-f_{r,1/n},1/n)_{p(\cdot),\omega}+\Omega_r(f_{r,1/n},1/n)_{p(\cdot),\omega}\\ &\leq c(p,r)\left\{\left\|f_{r,1/n}-f\right\|_{p(\cdot),\omega}+\frac{1}{n^r}\left\|f_{r,1/n}^{(r)}\right\|_{p(\cdot),\omega}\right\}.\end{aligned}$$

Taking infimum at last inequality we have

$$\Omega_r(f,1/n)_{p(\cdot),\omega}\leq c(p,r)K_r(f,1/n)_{p(\cdot),\omega}\ ,\ n=1,2,...,$$

for $r=1,2,...,$ and by (15) we obtain that $\Omega_r(f,1/n)_{p(\cdot),\omega}\approx K_r(f,1/n)_{p(\cdot),\omega}$. $\square$

Proof of Theorem 1.5. Let $f\in W_\omega^{p(\cdot),r}(\mathbb{T})$, $r=1,2,...$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. By (4) and Theorem 1.3 we have

$$\begin{aligned}\left\|f-V_m^n(f)\right\|_{p(\cdot),\omega} &\leq \frac{1}{m+1}\sum_{k=n}^{n+m}\left\|f-S_k(f)\right\|_{p(\cdot),\omega}\leq\frac{c(p)}{m+1}\sum_{k=n}^{n+m}E_k(f)_{p(\cdot),\omega}\\ &= \frac{c(p)}{m+1}\sum_{k=0}^{m}E_{n+k}(f)_{p(\cdot),\omega}\leq\frac{c(p,r)}{m+1}\sum_{k=0}^{m}\Omega_r\left(f,\frac{1}{n+k}\right)_{p(\cdot),\omega}.\quad\square\end{aligned}$$

Proof of Theorem 1.7. Let T_n be the best approximation trigonometric polynomial to $f\in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. Let also $m=1,2,...,$ be the number, such that $2^m\leq n<2^{m+1}$. Since

$$\Omega_r(f,1/n)_{p(\cdot),\omega}\leq\Omega_r(f-T_{2^{m+1}},1/n)_{p(\cdot),\omega}+\Omega_r(T_{2^{m+1}},1/n)_{p(\cdot),\omega}, \tag{16}$$

using the inequality [8, p. 209]

$$2^{(\nu+1)r}E_{2^{\nu}}(f)_{p(\cdot),\omega} \leq 2^{2r}\sum_{k=2^{\nu-1}+1}^{2^{\nu}} k^{r-1}E_k(f)_{p(\cdot),\omega}, \tag{17}$$

by (3) we have

$$\Omega_r(f-T_{2^{m+1}},1/n)_{p(\cdot),\omega} \leq c(p,r)\left\|f-T_{2^{m+1}}\right\|_{p(\cdot),\omega} = c(p,r)E_{2^{m+1}}(f)_{p(\cdot),\omega}$$

$$\leq c(p,r)\frac{2^{(m+1)r}}{n^r}E_{2^m}(f)_{p(\cdot),\omega} \leq \frac{c(p,r)}{n^r}2^{2r}\sum_{k=2^{m-1}+1}^{2^m} k^{r-1}E_k(f)_{p(\cdot),\omega}. \tag{18}$$

On the other hand, applying Lemma 2.5, Lemma 2.3 and (17) we get

$$\begin{aligned}
\Omega_r(T_{2^{m+1}},1/n)_{p(\cdot),\omega} &\leq \frac{c(p,r)}{n^r}\left\|T^{(r)}_{2^{m+1}}\right\|_{p(\cdot),\omega} = \frac{c(p,r)}{n^r}\left\|T^{(r)}_1+\sum_{\nu=0}^{m}\left(T^{(r)}_{2^{\nu+1}}-T^{(r)}_{2^{\nu}}\right)\right\|_{p(\cdot),\omega} \\
&\leq \frac{c(p,r)}{n^r}\left(\left\|T^{(r)}_1\right\|_{p(\cdot),\omega}+\left\|\sum_{\nu=0}^{m}\left(T^{(r)}_{2^{\nu+1}}-T^{(r)}_{2^{\nu}}\right)\right\|_{p(\cdot),\omega}\right) \\
&\leq \frac{c(p,r)}{n^r}\left(\|T_1\|_{p(\cdot),\omega}+\sum_{\nu=0}^{m}2^{(\nu+1)r}\|(T_{2^{\nu+1}}-T_{2^{\nu}})\|_{p(\cdot),\omega}\right) \\
&\leq \frac{c(p,r)}{n^r}\left(E_0(f)_{p(\cdot),\omega}+\sum_{\nu=0}^{m}2^{(\nu+1)r}E_{2^{\nu}}(f)_{p(\cdot),\omega}\right) \\
&= \frac{c(p,r)}{n^r}\left(E_0(f)_{p(\cdot),\omega}+2^rE_1(f)_{p(\cdot),\omega}+\sum_{\nu=1}^{m}2^{(\nu+1)r}E_{2^{\nu}}(f)_{p(\cdot),\omega}\right) \\
&\leq \frac{c(p,r)}{n^r}\left(E_0(f)_{p(\cdot),\omega}+2^rE_0(f)_{p(\cdot),\omega}+\sum_{\nu=1}^{m}\sum_{k=2^{\nu-1}+1}^{2^{\nu}}k^{r-1}E_k(f)_{p(\cdot),\omega}\right) \\
&\leq \frac{c(p,r)}{n^r}\left(E_0(f)_{p(\cdot),\omega}+\sum_{k=1}^{2^m}k^{r-1}E_k(f)_{p(\cdot),\omega}\right).
\end{aligned} \tag{19}$$

Combining (16), (18) and (19) we conclude that

$$\Omega_r(f,1/n)_{p(\cdot),\omega} \leq \frac{c(p,r)}{n^r}\left(\sum_{k=2^{m-1}+1}^{2^m}k^{r-1}E_k(f)_{p(\cdot),\omega}+E_0(f)_{p(\cdot),\omega}+\sum_{k=1}^{2^m}k^{r-1}E_k(f)_{p(\cdot),\omega}\right)$$

$$\leq \frac{c(p,r)}{n^r}\left(\sum_{k=1}^{2^m}k^{r-1}E_k(f)_{p(\cdot),\omega}+E_0(f)_{p(\cdot),\omega}\right) \leq \frac{c(p,r)}{n^r}\left(\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p(\cdot),\omega}\right). \quad \square$$

Proof of Theorem 1.9. Let $f \in W^{p(\cdot),k}_{\omega}(\mathbb{T})$, $k=1,2,...,p(\cdot) \in \mathcal{P}(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Since $E_n(f)_{p(\cdot),\omega}$ decreasing sequence applying (4) and Lemma 2.4 we have

$$\left\|f-V^n_m(f)\right\|_{p(\cdot),\omega} \leq \frac{1}{m+1}\sum_{k=n}^{n+m}\left\|f-S_k(f)\right\|_{p(\cdot),\omega} \leq \frac{c(p)}{m+1}\sum_{k=n}^{n+m}E_k(f)_{p(\cdot),\omega} \leq c(p)E_n(f)_{p(\cdot),\omega} \leq \frac{c(p,k)}{(n+1)^k}E_n\left(f^{(k)}\right)_{p(\cdot),\omega}.$$

Thus theorem is proved. $\square$

Proof of Theorem 1.10. Let $f \in W_{\omega}^{p(\cdot),r}(\mathbb{T}), r = 1, 2, ..., p(\cdot) \in \mathcal{P}(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Applying (4) and Lemma 2.4 we have

$$\left\| f - V_m^n(f) \right\|_{p(\cdot),\omega} \leq \frac{1}{m+1} \sum_{k=n}^{n+m} \left\| f - S_k(f) \right\|_{p(\cdot),\omega} \leq \frac{c(p)}{m+1} \sum_{k=n}^{n+m} E_k(f)_{p(\cdot),\omega} \leq \frac{c(p,r)}{m+1} \sum_{k=n}^{n+m} \frac{1}{(k+1)^r} E_n\left(f^{(r)}\right)_{p(\cdot),\omega}.$$

Thus theorem is proved. □

Proof of Theorem 1.11. Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Let also $T_n^0(f)$, $T_n^*(f) \in \Pi_n$ be the best and near-best approximating trigonometric polynomials to $f \in W_{\omega}^{p(\cdot),k}(\mathbb{T}), k = 1, 2, ...,$ respectively. Then

$$\left\| f^{(k)} - T_n^{*(k)}(f) \right\|_{p(\cdot),\omega} \leq \left\| f^{(k)} - V_{n-1}^n\left(f^{(k)}\right) \right\|_{p(\cdot),\omega} + \left\| T_n^{*(k)}\left(V_{n-1}^n(f)\right) - T_n^{*(k)}(f) \right\|_{p(\cdot),\omega} +$$

$$+ \left\| V_{n-1}^n\left(f^{(k)}\right) - T_n^{*(k)}\left(V_{n-1}^n(f)\right) \right\|_{p(\cdot),\omega} =: I_1 + I_2 + I_3.$$

By the boundedness of $V_{n-1}^n(f)$

$$\begin{aligned} I_1 &= \left\| f^{(k)} - V_{n-1}^n\left(f^{(k)}\right) \right\|_{p(\cdot),\omega} \leq \left\| f^{(k)} - T_n^0\left(f^{(k)}\right) \right\|_{p(\cdot),\omega} + \left\| T_n^0\left(f^{(k)}\right) - V_{n-1}^n\left(f^{(k)}\right) \right\|_{p(\cdot),\omega} = \\ &= E_n\left(f^{(k)}\right)_{p(\cdot),\omega} + \left\| V_{n-1}^n\left(T_n^0\left(f^{(k)}\right)\right) - V_{n-1}^n\left(f^{(k)}\right) \right\|_{p(\cdot),\omega} \leq E_n\left(f^{(k)}\right)_{p(\cdot),\omega} + c(p) \left\| T_n^0\left(f^{(k)}\right) - f^{(k)} \right\|_{p(\cdot),\omega} \leq \\ &\leq c(p) E_n\left(f^{(k)}\right)_{p(\cdot),\omega}. \end{aligned}$$

By Lemma 2.3, Theorem 1.9 and Lemma 2.4

$$\begin{aligned} I_2 &= \left\| T_n^{*(k)}\left(V_{n-1}^n(f)\right) - T_n^{*(k)}(f) \right\|_{p(\cdot),\omega} \leq c(p) n^k \left\| T_n^*\left(V_{n-1}^n(f)\right) - T_n^*(f) \right\|_{p(\cdot),\omega} \leq \\ &\leq c(p) n^k \left\{ \left\| T_n^*\left(V_{n-1}^n(f)\right) - V_{n-1}^n(f) \right\|_{p(\cdot),\omega} + \left\| V_{n-1}^n(f) - f \right\|_{p(\cdot),\omega} + \left\| f - T_n^*(f) \right\|_{p(\cdot),\omega} \right\} \leq \\ &\leq c(p) n^k \left\{ c(p) E_n\left(V_{n-1}^n(f)\right)_{p(\cdot),\omega} + \left\| V_{n-1}^n(f) - f \right\|_{p(\cdot),\omega} + c(p) E_n(f)_{p(\cdot),\omega} \right\} \leq \\ &\leq c(p) n^k \left\{ c(p) E_n\left(V_{n-1}^n(f)\right)_{p(\cdot),\omega} + \frac{c(p)}{n^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega} + \frac{c(p)}{n^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \right\}. \end{aligned}$$

Since by Theorem 1.9 and Lemma 2.4

$$\begin{aligned} E_n\left(V_{n-1}^n(f)\right)_{p(\cdot),\omega} &\leq \left\| V_{n-1}^n(f) - T_n^0(f) \right\|_{p(\cdot),\omega} \leq \\ &\leq \left\| V_{n-1}^n(f) - f \right\|_{p(\cdot),\omega} + \left\| f - T_n^0(f) \right\|_{p(\cdot),\omega} \leq \frac{c(p,k)}{n^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega}, \end{aligned} \tag{20}$$

we obtain that $I_2 \leq c(p,k) E_n\left(f^{(k)}\right)_{p(\cdot),\omega}$.

Ultimately applying Lemma 2.3 to I_3, and (20) we get

$$\begin{aligned} I_3 &= \left\| V_{n-1}^n\left(f^{(k)}\right) - T_n^{*(k)}\left(V_{n-1}^n(f)\right) \right\|_{p(\cdot),\omega} \leq c(p)(2n-1)^k \left\| V_{n-1}^n(f) - T_n^*\left(V_{n-1}^n(f)\right) \right\|_{p(\cdot),\omega} \leq \\ &\leq c(p)(2n-1)^k E_n\left(V_{n-1}^n(f)\right)_{p(\cdot),\omega} \leq c(p,k) \frac{(2n-1)^k}{n^k} E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \leq c(p,k) E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \end{aligned}$$

and therefore

$$\left\| f^{(k)} - T_n^{*(k)}(f) \right\|_{p(\cdot),\omega} \leq I_1 + I_2 + I_3 \leq c(p,k) E_n\left(f^{(k)}\right)_{p(\cdot),\omega}. \quad \square$$

Proof of Theorem 1.13. Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Let S_n^* be nth partial sum of the series (5) and

$$\mu_k = \begin{cases} 0 & , k = 0 \\ k^{\alpha} & , k = 1, 2, ..., \end{cases} .$$

By Abel transform

$$S_m^* - f = \sum_{j=1}^{m-1} \left(S_j(f) - f \right) \Delta\mu_j + \left(S_m(f) - f \right) \mu_m ,$$

where $m = 1, 2, ...,$ and $\Delta\mu_j = \mu_j - \mu_{j+1}$. Then for a fixed $n = 1, 2, ...,$ and for every $k = 0, 1, 2, ...,$ we obtain

$$S_{2^{k+1}n}^* - S_{2^k n}^* = \sum_{j=2^k n}^{2^{k+1}n-1} \left(S_j(f) - f \right) \Delta\mu_j + \left(S_{2^{k+1}n}(f) - f \right) \mu_{2^{k+1}n} - \left(S_{2^k n}(f) - f \right) \mu_{2^k n}. \tag{21}$$

It can be shown that $\left| \Delta\mu_j \right| < c j^{\alpha-1}$. Since the sequence $\left\{ E_n(f)_{p(\cdot),\omega} \right\}_{n=1}^{\infty}$ is decreasing, combining (4) and (21) we get

$$\begin{aligned} \left\| S_{2^{k+1}n}^* - S_{2^k n}^* \right\|_{p(\cdot),\omega} &\leq c(p) \sum_{j=2^k n}^{2^{k+1}n-1} j^{\alpha-1} E_j(f)_{p(\cdot),\omega} + c(p) 2^{\alpha} \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} \leq \\ &\leq c(p) 2^k n 2^{\alpha-1} \left(2^k n \right)^{\alpha-1} E_{2^k n}(f)_{p(\cdot),\omega} + c(p) 2^{\alpha} \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} \leq \\ &= c(p) \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} , \end{aligned}$$

and hence,

$$\sum_{k=0}^{\infty} \left\| S_{2^{k+1}n}^* - S_{2^k n}^* \right\|_{p(\cdot),\omega} \leq c(p) \sum_{k=0}^{\infty} \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} .$$

It is clear that

$$\begin{aligned} \sum_{k=1}^{\infty} \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} &= 2^{\alpha} \sum_{k=1}^{\infty} \left(2^{k-1} n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} = 2^{\alpha} \sum_{k=0}^{\infty} \left(2^k n \right)^{\alpha-1} 2^k n E_{2^{k+1}n}(f)_{p(\cdot),\omega} \leq \\ &\leq 2^{\alpha} \sum_{k=0}^{\infty} \sum_{j=2^k n+1}^{2^{k+1}n} j^{\alpha-1} E_j(f)_{p(\cdot),\omega} \leq c \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} . \end{aligned} \tag{22}$$

By (22) we have

$$\begin{aligned} \sum_{k=0}^{\infty} \left\| S_{2^{k+1}n}^* - S_{2^k n}^* \right\|_{p(\cdot),\omega} &\leq c(p) \left\{ n^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{k=1}^{\infty} \left(2^k n \right)^{\alpha} E_{2^k n}(f)_{p(\cdot),\omega} \right\} \leq \\ &\leq c(p) \left\{ n^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\} . \end{aligned} \tag{23}$$

Since $\sum_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} < \infty$, the series $S_n^* + \sum_{k=0}^{\infty} \left(S_{2^{k+1}n}^* - S_{2^k n}^* \right)$ convergences to some function $g \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ in $L_{\omega}^{p(\cdot)}(\mathbb{T})$. Hence, it is obtained that the series (5) is the Fourier series of the function g. Since $S_n^* = S_n(g)$, applying Minkowski inequality and (23) we get

$$E_n(g)_{p(\cdot),\omega} \leq \left\| g - S_n^* \right\|_{p(\cdot),\omega} \leq \sum_{k=0}^{\infty} \left\| S_{2^{k+1}n}^* - S_{2^k n}^* \right\|_{p(\cdot),\omega} \leq c(p) \left\{ n^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\} . \tag{24}$$

By (4) and (24) for $n = 1$ we have

$$E_0(g)_{p(\cdot),\omega} \le \left\| g - \frac{a_0}{2} \right\|_{p(\cdot),\omega} \le \left\| g - S_1^* \right\|_{p(\cdot),\omega} + \left\| S_1^* - \frac{a_0}{2} \right\|_{p(\cdot),\omega} \le E_1(g)_{p(\cdot),\omega} + \left\| S_1^* - \frac{a_0}{2} \right\|_{p(\cdot),\omega}$$

$$\le c(p) \left\{ E_1(f)_{p(\cdot),\omega} + \sum_{k=2}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\} + \left\| S_1^* - \frac{a_0}{2} \right\|_{p(\cdot),\omega}$$

$$= c(p) \sum_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} + \left\| S_1^* - \frac{a_0}{2} \right\|_{p(\cdot),\omega}. \tag{25}$$

Let T_0 be number such that $\|g - T_0\|_{p(\cdot),\omega} = E_0(g)_{p(\cdot),\omega}$. Then

$$|a_1| = \frac{1}{\pi} \left| \int_{\mathbb{T}} g(x) \cos x dx \right| = \frac{1}{\pi} \left| \int_{\mathbb{T}} [g(x) - T_0(x)] \cos x dx \right| \le c(p) \|g - T_0\|_{p(\cdot),\omega} = c(p) E_0(g)_{p(\cdot),\omega},$$

and similarly it can be shown that $|b_1| \le c(p) E_0(g)_{p(\cdot),\omega}$. This facts implies that

$$\left\| S_1^* - \frac{a_0}{2} \right\|_{p(\cdot),\omega} = \|a_1 \cos x + b_1 \sin x\|_{p(\cdot),\omega} \le 2\pi (|a_1| + |b_1|) \le c(p) E_0(g)_{p(\cdot),\omega}. \tag{26}$$

Since there exists a constant such that $E_0(g)_{p(\cdot),\omega} \le c(p) E_0(f)_{p(\cdot),\omega}$, by (25) and (26) we obtain

$$E_0(g)_{p(\cdot),\omega} \le c(p) \left\{ E_0(f)_{p(\cdot),\omega} + \sum_{k=1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\}.$$

This inequality and (24) complete the proof. □

Proof of Theorem 1.14. Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $r = 1, 2, ...$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. Applying Theorem 1.7 and using inequalities (6) and (7) we have

$$\Omega_r(g, 1/n)_{p(\cdot),\omega} \le \frac{c(p,r)}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(g)_{p(\cdot),\omega} = \frac{c(p,r)}{n^r} \left\{ E_0(g)_{p(\cdot),\omega} + \sum_{\nu=1}^{n} (\nu+1)^{r-1} E_\nu(g)_{p(\cdot),\omega} \right\} \le$$

$$\le \frac{c(p,r)}{n^r} \left[E_0(f)_{p(\cdot),\omega} + \sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} \right] + \frac{c(p,r)}{n^r} \sum_{\nu=1}^{n} (\nu+1)^{r-1} \left[\nu^{\alpha} E_\nu(f)_{p(\cdot),\omega} + \sum_{k=\nu+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right] \le$$

$$\le \frac{c(p,r)}{n^r} \left\{ \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} \right\}$$

$$+ \frac{c(p,r)}{n^r} \left\{ \sum_{\nu=1}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p(\cdot),\omega} + \sum_{\nu=1}^{n} (\nu+1)^{r-1} \sum_{k=\nu+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right\} \le$$

$$\le \frac{2c(p,r)}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p(\cdot),\omega} + \frac{2c(p,r)}{n^r} \sum_{\nu=1}^{n} (\nu+1)^{r-1} \sum_{k=\nu}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \le$$

$$\le \frac{c(p,r)}{n^r} \sum_{\nu=0}^{n} (\nu+1)^{r+\alpha-1} E_\nu(f)_{p(\cdot),\omega} +$$

$$+ \frac{c(p,r)}{n^r} \sum_{\nu=1}^{n} (\nu+1)^{r-1} \left[\sum_{k=\nu}^{n} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot),\omega} \right] \le$$

$$\leq \frac{c(p,r)}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r+\alpha-1}E_\nu(f)_{p(\cdot),\omega}+$$

$$+\frac{c(p,r)}{n^r}\left\{\sum_{\nu=1}^{n}(\nu+1)^{r-1}\left[\sum_{k=\nu}^{n}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\right]+\sum_{\nu=1}^{n}(\nu+1)^{r-1}\left[\sum_{k=n+1}^{\infty}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\right]\right\}\leq$$

$$\leq \frac{c(p,r)}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r+\alpha-1}E_\nu(f)_{p(\cdot),\omega}+\frac{c(p,r)}{n^r}\sum_{k=1}^{n}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\left[\sum_{\nu=1}^{k}(\nu+1)^{r-1}\right]+$$

$$+\frac{c(p,r)}{n^r}\sum_{k=n+1}^{\infty}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\left[\sum_{\nu=1}^{n}(\nu+1)^{r-1}\right]\leq$$

$$\leq \frac{c(p,r)}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r+\alpha-1}E_\nu(f)_{p(\cdot),\omega}+\frac{c(p,r)}{n^r}\sum_{k=1}^{n}(k+1)^{r+\alpha-1}E_k(f)_{p(\cdot),\omega}+$$

$$+c(p,r)\frac{(n+1)^r}{n^r}\sum_{k=n+1}^{\infty}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\leq$$

$$\leq c(p,r)\left\{\frac{2}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r+\alpha-1}E_\nu(f)_{p(\cdot),\omega}+2^r\sum_{k=n+1}^{\infty}k^{\alpha-1}E_k(f)_{p(\cdot),\omega}\right\}\leq$$

$$\leq c(p,r)\left\{\frac{1}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{r+\alpha-1}E_\nu(f)_{p(\cdot),\omega}+\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{p(\cdot),\omega}\right\},$$

and this compete the proof. □

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