Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2603 – 2613
© TÜBİTAK
doi:10.3906/mat-2105-32

On the Betti numbers of the tangent cones for Gorenstein monomial curves

Pınar METE*
Department of Mathematics, Balıkesir University, Balıkesir, 10145 Turkey

Received: 11.05.2021 • **Accepted/Published Online:** 05.10.2021 • **Final Version:** 29.11.2021

Abstract: The aim of the article is to study the Betti numbers of the tangent cone of Gorenstein monomial curves in affine 4-space. If C_S is a noncomplete intersection Gorenstein monomial curve whose tangent cone is Cohen–Macaulay, we show that the possible Betti sequences are (1,5,5,1), (1,5,6,2) and (1,6,8,3).

Key words: Gorenstein monomial curves, tangent cones, Betti numbers

1. Introduction

Let S denote the numerical semigroup generated by the positive integers $n_1 < n_2 < \ldots < n_d$ with $gcd(n_1, \ldots, n_d) = 1$. Consider the polynomial rings $R = k[x_1, \ldots, x_d]$ and $k[t]$ over the field k. The semigroup ring $k[S] = k[t^{n_1}, \ldots, t^{n_d}]$ is the $k-$subalgebra of $k[t]$. $\varphi : R \to k[S] \subset k[t]$ with $\varphi(x_i) = t^{n_i}$ is the $k-$algebra homomorphism for $i = 1, \ldots, d$ and its kernel I_S is called the toric ideal of S.

Let $m = (t^{n_1}, \ldots, t^{n_d})$ be the maximal ideal of the one-dimensional local ring $k[[t^{n_1}, \ldots, t^{n_d}]]$. When k is algebraically closed, the semigroup ring $k[S] = k[t^{n_1}, \ldots, t^{n_d}]$ is isomorphic to the coordinate ring R/I_S of C_S and the coordinate ring $gr_m(k[[t^{n_1}, \ldots, t^{n_d}]])$ of the tangent cone of C_S at the origin is isomorphic to the ring R/I_{S_*}. Here, I_{S_*} is generated by the polynomials f_* which are the homogeneous summands of $f \in I_S$ and is called the defining ideal of the tangent cone of C_S. A monomial curve C_S is Gorenstein if the associated local ring $k[[t^{n_1}, \ldots, t^{n_d}]]$ is Gorenstein. $k[[t^{n_1}, \ldots, t^{n_d}]]$ is Gorenstein if and only if the semigroup S is symmetric [8]. We recall that the numerical semigroup S is symmetric if and only if for all $x \in \mathbb{Z}$ either $x \in S$ or $F(S) - x \in S$, where $F(S)$ denote the Frobenius number of S.

Finding an explicit minimal free resolution of a standard $k-$algebra is one of the core areas in commutative algebra. Since it is very difficult to obtain a description of the differential in the resolution, we can get some information about the numerical invariants of the resolution such as Betti numbers. The i-th Betti number of an R-module M, $\beta_i(M)$, is the rank of the free modules appearing in the minimal free resolution of M where

$$0 \to R^{\beta_{k-1}} \to \ldots \to R^{\beta_1} \to R^{\beta_0}$$

and the Betti sequence of M, $\beta(M)$, is $(\beta_0(M), \beta_1(M), \ldots, \beta_{k-1}(M))$. Stamate [10] gave a broad survey on the Betti numbers of the numerical semigroup rings and stated the problem of describing the Betti numbers and the minimal free resolution for the tangent cone when S is 4-generated semigroup which is symmetric, or equivalently, C_S is a Gorenstein monomial curve in affine 4-space [see [10], Problem 9.9.]. The case has

*Correspondence : pinarm@balikesir.edu.tr

2010 *AMS Mathematics Subject Classification:* primary 13H10; secondary 13P10, 14H20

been addressed for Cohen–Macaulay tangent cone of a monomial curve in $\mathbb{A}^4(k)$ corresponding to a pseudo-symmetric numerical semigroup in [11]. In this paper, we solve the problem for Cohen–Macaulay tangent cone of a monomial curve in $\mathbb{A}^4(k)$ corresponding to a noncomplete intersection symmetric numerical semigroup. All computations have been done using SINGULAR*.

2. The noncomplete intersection Gorenstein monomial curves

For the rest of the paper, we assume that C_S is a Gorenstein noncomplete intersection monomial curve in $\mathbb{A}^4$. Now, we recall Bresinsky's theorem, which gives the explicit description of the defining ideal of C_S.

Theorem 2.1 *[4] Let C_S be a monomial curve having the parametrization*

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3},\ x_4 = t^{n_4}$$

where $S =< n_1, n_2, n_3, n_4 >$ is a numerical semigroup minimally generated by n_1, n_2, n_3, n_4. The semigroup $< n_1, n_2, n_3, n_4 >$ is symmetric and C_S is a noncomplete intersection curve if and only if I_S is generated by the set

$$\{f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}},$$
$$f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

where the polynomials f_i's are unique up to isomorphism with $0 < \alpha_{ij} < \alpha_j$ with $\alpha_i n_i \in < n_1, \ldots, \hat{n}_i, \ldots, n_4 >$ such that α_i's are minimal for $1 \leq i \leq 4$, where $\hat{n}_i$ denotes that $n_i \notin < n_1, \ldots, \hat{n}_i, \ldots, n_4 >$.

Theorem 2.1 implies that for any noncomplete intersection Gorenstein monomial curve C_S, the variables can be renamed to obtain generators exactly of the given form, and this means that there are six isomorphic possible permutations which can be considered within three cases:

1. $f_1 = (1, (3, 4))$
   (a) $f_2 = (2, (1, 4)), f_3 = (3, (1, 2)), f_4 = (4, (2, 3)), f_5 = ((1, 3), (2, 4))$
   (b) $f_2 = (2, (1, 3)), f_3 = (3, (2, 4)), f_4 = (4, (1, 2)), f_5 = ((1, 4), (2, 3))$
2. $f_1 = (1, (2, 3))$
   (a) $f_2 = (2, (3, 4)), f_3 = (3, (1, 4)), f_4 = (4, (1, 2)), f_5 = ((2, 4), (1, 3))$
   (b) $f_2 = (2, (1, 4)), f_3 = (3, (2, 4)), f_4 = (4, (1, 3)), f_5 = ((1, 3), (4, 2))$
3. $f_1 = (1, (2, 4))$
   (a) $f_2 = (2, (1, 3)), f_3 = (3, (1, 4)), f_4 = (4, (2, 3)), f_5 = ((1, 2), (3, 4))$
   (b) $f_2 = (2, (3, 4)), f_3 = (3, (1, 2)), f_4 = (4, (1, 3)), f_5 = ((2, 3), (1, 4))$

*SINGULAR 2.0. A Computer Algebra System for Polynomial Computations. Available at http://www.singular.uni-kl.de.

Here, the notations $f_i = (i,(j,k))$ and $f_5 = ((i,j),(k,l))$ denote the generators $f_i = x_i^{\alpha_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}}$ and $f_5 = x_i^{\alpha_{ki}} x_j^{\alpha_{lj}} - x_k^{\alpha_{jk}} x_l^{\alpha_{il}}$. Thus, given a Gorenstein monomial curve C_S, if we have the extra condition $n_1 < n_2 < n_3 < n_4$, then the generator set of its defining ideal I_S is exactly given by one of these six permutations.

The study of the Cohen–Macaulayness of tangent cones of monomial curves constitutes an important problem [1],[2]. In [3], Arslan and Mete determined the common arithmetic conditions satisfied by the generators of the defining ideals of C_S and under these conditions they found the generators of the tangent cone of C_S. In [2], they provided necessary and sufficient conditions for the Cohen–Macaulayness of the tangent cone of C_S in all six permutations and gave the following theorem:

Theorem 2.2 *[2] (1) Suppose that I_S is given as in the Case 1(a). Then R/I_{S_*} is Cohen–Macaulay if and only if $\alpha_2 \le \alpha_{21} + \alpha_{24}$.*

(2) Suppose that I_S is given as in Case 1(b). (i) Assume that $\alpha_{32} < \alpha_{42}$ and $\alpha_{14} \le \alpha_{34}$. Then R/I_{S_*} is Cohen-Macaulay if and only if

1. $\alpha_2 \le \alpha_{21} + \alpha_{23}$

2. $\alpha_{42} + \alpha_{13} \le \alpha_{21} + \alpha_{34}$ and

3. $\alpha_3 + \alpha_{13} \le \alpha_1 + \alpha_{32} + \alpha_{34} - \alpha_{14}$.

(ii) Assume that $\alpha_{42} \le \alpha_{32}$. Then R/I_{S_*} is Cohen-Macaulay if and only if

1. $\alpha_2 \le \alpha_{21} + \alpha_{23}$

2. $\alpha_{42} + \alpha_{13} \le \alpha_{21} + \alpha_{34}$ and

3. either $\alpha_{34} < \alpha_{14}$ and $\alpha_3 + \alpha_{13} \le \alpha_{21} + \alpha_{32} - \alpha_{42} + 2\alpha_{34}$
or $\alpha_{14} \le \alpha_{34}$ and $\alpha_3 + \alpha_{13} \le \alpha_1 + \alpha_{32} + \alpha_{34} - \alpha_{14}$.

(3) Suppose that I_S is given as in Case 2(a). (i) Assume that $\alpha_{24} < \alpha_{34}$ and $\alpha_{13} \le \alpha_{23}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_3 \le \alpha_{31} + \alpha_{34}$

2. $\alpha_{12} + \alpha_{34} \le \alpha_{41} + \alpha_{23}$ and

3. $\alpha_2 + \alpha_{12} \le \alpha_1 + \alpha_{23} - \alpha_{13} + \alpha_{24}$.

(ii) Assume that $\alpha_{34} \le \alpha_{24}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_3 \le \alpha_{31} + \alpha_{34}$

2. $\alpha_{12} + \alpha_{34} \le \alpha_{41} + \alpha_{23}$ and

3. either $\alpha_{23} < \alpha_{13}$ and $\alpha_2 + \alpha_{12} \le \alpha_{41} + 2\alpha_{23} + \alpha_{24} - \alpha_{34}$
or $\alpha_{13} \le \alpha_{23}$ and $\alpha_2 + \alpha_{12} \le \alpha_1 + \alpha_{23} - \alpha_{13} + \alpha_{24}$.

(4) Suppose that I_S is given as in Case 2(b). (i) Assume that $\alpha_{34} < \alpha_{24}$ and $\alpha_{12} \le \alpha_{32}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_2 \le \alpha_{21} + \alpha_{24}$ and

2. $\alpha_3 + \alpha_{13} \le \alpha_1 + \alpha_{32} - \alpha_{12} + \alpha_{34}$.

(ii) Assume that $\alpha_{24} \le \alpha_{34}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_2 \le \alpha_{21} + \alpha_{24}$ and

2. either $\alpha_{32} < \alpha_{12}$ and $\alpha_3 + \alpha_{13} \le \alpha_{41} + 2\alpha_{32} + \alpha_{34} - \alpha_{24}$
or $\alpha_{12} \le \alpha_{32}$ and $\alpha_3 + \alpha_{13} \le \alpha_1 + \alpha_{32} - \alpha_{12} + \alpha_{34}$.

(5) Suppose that I_S is given as in the Case 3(a). Then R/I_{S_*} is Cohen–Macaulay tangent cone if and only if $\alpha_2 \le \alpha_{21} + \alpha_{23}$ and $\alpha_3 \le \alpha_{31} + \alpha_{34}$.

(6) Suppose that I_S is given as in Case 3(b). (i) Assume that $\alpha_{23} < \alpha_{43}$ and $\alpha_{14} \leq \alpha_{24}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_{12} + \alpha_{43} \leq \alpha_{31} + \alpha_{24}$ *and*

2. $\alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_{23} + \alpha_{24} - \alpha_{14}$.

(ii) Assume that $\alpha_{43} \leq \alpha_{23}$. Then R/I_{S_*} is Cohen–Macaulay if and only if

1. $\alpha_{12} + \alpha_{43} \leq \alpha_{31} + \alpha_{24}$ *and*

2. either $\alpha_{24} < \alpha_{14}$ *and* $\alpha_2 + \alpha_{12} \leq \alpha_{31} + 2\alpha_{24} + \alpha_{23} - \alpha_{43}$
or $\alpha_{14} \leq \alpha_{24}$ *and* $\alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_{23} + \alpha_{24} - \alpha_{14}$.

Recently, Katsabekis gave the remaining standard bases for I_S in [7] using above conditions for the Cohen-Macaulayness of the tangent cone of C_S.

3. Betti sequences of Cohen–Macaulay tangent cones

In this section, we determine the Betti sequences of the tangent cone of C_S whose tangent cone is Cohen–Macaulay. Buchsbaum–Eisenbud criterion [5] and the following Lemma (see also [6]) will be used in the all proofs.

Lemma 3.1 *[11] Assume that the multiplicity of the monomial curve is n_i. Suppose that the k-algebra homomorphism $\pi_i : R \to \overline{R} = k[x_1, \dots, \overline{x_i}, \dots, x_d]$ is defined by $\pi_i(x_i) = \overline{x_i} = 0$ and $\pi_j(x_j) = x_j$ for $i \neq j$, and set $\overline{I} = \pi_i(I_{S_*})$. If the tangent cone $gr_m(k[[t^{n_1}, \dots, t^{n_d}]])$ is Cohen–Macaulay, then the Betti sequences of $gr_m(k[[t^{n_1}, \dots, t^{n_d}]])$ and of $\overline{R}/\overline{I}$ are the same.*

This is a very effective result to reduce the number of cases for finding the Betti numbers of the tangent cones.

Case 1(a): Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, \qquad f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, \qquad f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}},$$

$$f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, \qquad f_5 = x_1^{\alpha_{21}} x_3^{\alpha_{43}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{24}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{13} + \alpha_{14}$, $\alpha_4 < \alpha_{42} + \alpha_{43}$ and $\alpha_3 < \alpha_{31} + \alpha_{32}$.

C_S has Cohen–Macaulay tangent cone only under the condition $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ by [2]. Mete and Zengin [9] gave the explicit minimal free resolution for the tangent cone of C_S and showed that the Betti sequence of its tangent cone is $(1, 5, 6, 2)$.

Case 1(b) : Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, \qquad f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, \qquad f_3 = x_3^{\alpha_3} - x_2^{\alpha_{32}} x_4^{\alpha_{34}},$$

$$f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}} x_2^{\alpha_{42}}, \qquad f_5 = x_1^{\alpha_{21}} x_4^{\alpha_{34}} - x_2^{\alpha_{42}} x_3^{\alpha_{13}}$$

where $\alpha_1 = \alpha_{21} + \alpha_{41}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$, $\alpha_4 = \alpha_{14} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{13} + \alpha_{14}$ and $\alpha_4 < \alpha_{41} + \alpha_{42}$. Under the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{23}$ by Theorem 2.2 and one possible condition $\alpha_3 \leq \alpha_{32} + \alpha_{34}$, the Betti sequences for the Cohen–Macaulay tangent cone of C_S are $(1,5,5,1)$ and $(1,5,6,2)$, as given in [9]. Here, the other possible condition $\alpha_3 > \alpha_{32} + \alpha_{34}$ will be considered.

Theorem 3.2 *The Betti sequence of the tangent cone R/I_{S_*} of C_S is $(1,6,8,3)$, if I_S is given as in Case 1(b) when $\alpha_3 > \alpha_{32} + \alpha_{34}$.*

Proof Suppose that I_S is given as in the Case 1(b) when $\alpha_3 > \alpha_{32} + \alpha_{34}$.

(i) By Proposition 2.7 in [7], if $\alpha_{32} < \alpha_{42}$ and $\alpha_{14} \leq \alpha_{34}$, then,

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_1} x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{14}}\},$$

(ii) By Proposition 2.9 in [7],

(1) if $\alpha_{42} \leq \alpha_{32}$ and $\alpha_{34} < \alpha_{14}$, then

$$G = \{f_1, f_2, f_3 f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_{21}} x_2^{\alpha_{32}-\alpha_{42}} x_4^{2\alpha_{34}}\},$$

(2) if $\alpha_{42} \leq \alpha_{32}$ and $\alpha_{14} \leq \alpha_{34}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_1} x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{14}}\}$$

are standard bases for I_S. Since $\overline{I} = \pi_i(I_{S_*})$ which sends x_1 to 0, the generators of the defining ideal of $\overline{I}$ is generated by

$$G_* = (x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_2^{\alpha_{42}} x_3^{\alpha_{13}}, x_3^{\alpha_3+\alpha_{13}}).$$

Now, consider the case (i). Since the Betti sequences of R/I_{S_*} and $\overline{R}/\overline{I}$ are the same which follows from Lemma 3.1, we will show that the sequence,

$$0 \to R^3 \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R^1 \to 0$$

is a minimal free resolution of $\overline{R}/\overline{I}$, where

$$\varphi_1 = \begin{pmatrix} x_3^{\alpha_{13}} x_4^{\alpha_{14}} & x_2^{\alpha_2} & x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_2^{\alpha_{42}} x_3^{\alpha_{13}} & x_3^{\alpha_3+\alpha_{13}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} 0 & -x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{14}} & 0 & -x_4^{\alpha_{34}} & x_2^{\alpha_{42}} & x_3^{\alpha_3} & 0 & 0 \\ 0 & 0 & -x_4^{\alpha_{34}} & 0 & 0 & 0 & x_3^{\alpha_{13}} & 0 \\ x_4^{\alpha_{14}} & x_3^{\alpha_{13}} & x_2^{\alpha_{42}} & 0 & 0 & 0 & 0 & 0 \\ -x_2^{\alpha_{32}} & 0 & 0 & x_3^{\alpha_{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_4^{\alpha_{14}} & 0 & -x_2^{\alpha_{32}} & x_3^{\alpha_3} \\ 0 & 0 & 0 & 0 & 0 & -x_4^{\alpha_{14}} & 0 & -x_2^{\alpha_{42}} \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} x_3^{\alpha_{13}} & 0 & 0 \\ -x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 \\ 0 & -x_3^{\alpha_{13}} & 0 \\ x_2^{\alpha_{32}} & 0 & 0 \\ 0 & x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{14}} & x_3^{\alpha_3} \\ 0 & 0 & -x_2^{\alpha_{42}} \\ 0 & x_4^{\alpha_{34}} & 0 \\ 0 & 0 & x_4^{\alpha_{14}} \end{pmatrix}.$$

Since $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$, the sequence above is a complex. One can easily check that $rank(\varphi_1) = 1$, $rank(\varphi_2) = 5$ and $rank(\varphi_3) = 3$. As the 5-minors of φ_2, we get $x_3^{2\alpha_3+3\alpha_{13}}$ by deleting the row 6, and the columns 1, 3, 5, and $-x_4^{2\alpha_4+\alpha_{14}}$ by deleting the row 4 and the columns 2, 7, 8. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. As the 3-minors of φ_3, we have $x_2^{\alpha_2+\alpha_{42}}$ by deleting the rows 1, 3, 5, 7, 8, and $-x_3^{\alpha_3+2\alpha_{13}}$ by deleting the rows 2, 4, 6, 7, 8, and finally, $-x_4^{\alpha_4+\alpha_{14}}$ by deleting the rows 1, 3, 4, 5, 6 Since these are relatively prime, $I(\varphi_3)$ contains a regular sequence of length 3.

The other cases (ii)-(1) and (2) can be done similarly and are hence omitted. □

Case 2(a): Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}, \qquad f_2 = x_2^{\alpha_2} - x_3^{\alpha_{23}}x_4^{\alpha_{24}}, \qquad f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}}x_4^{\alpha_{34}},$$

$$f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}}x_2^{\alpha_{42}}, \qquad f_5 = x_1^{\alpha_{41}}x_3^{\alpha_{23}} - x_2^{\alpha_{12}}x_4^{\alpha_{34}}$$

where $\alpha_1 = \alpha_{31} + \alpha_{41}$, $\alpha_2 = \alpha_{12} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$, $\alpha_4 = \alpha_{24} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{12} + \alpha_{13}$, $\alpha_2 > \alpha_{23} + \alpha_{24}$ and $\alpha_4 < \alpha_{41} + \alpha_{42}$. Note that the assumption $\alpha_3 \leq \alpha_{31} + \alpha_{34}$ is one of the necessary and sufficient conditions for the Cohen–Macaulayness of the tangent cone by Theorem 2.2.

Theorem 3.3 *The Betti sequence of the tangent cone R/I_{S_*} of C_S is $(1, 6, 8, 3)$, if I_S is given as in Case 2(a).*

Proof Suppose that I_S is given as in the Case 2(a).
(i) By Proposition 2.11 in [7], if $\alpha_{24} < \alpha_{34}$ and $\alpha_{13} \leq \alpha_{23}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2+\alpha_{12}} - x_1^{\alpha_1}x_3^{\alpha_{23}-\alpha_{13}}x_4^{\alpha_{24}}\}$$

(ii) By Proposition 2.13 in [7],

(1) if $\alpha_{34} \leq \alpha_{24}$ and $\alpha_{23} < \alpha_{13}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2+\alpha_{12}} - x_1^{\alpha_{41}}x_3^{2\alpha_{23}}x_4^{\alpha_{24}-\alpha_{34}}\}$$

(2) if $\alpha_{34} \leq \alpha_{24}$ and $\alpha_{13} \leq \alpha_{23}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2+\alpha_{12}} - x_1^{\alpha_1}x_3^{\alpha_{23}-\alpha_{13}}x_4^{\alpha_{24}}\}$$

are standard bases for I_S. Since $\overline{I} = \pi_i(I_{S_*})$ which sends x_1 to 0, the generators of the defining ideal of $\overline{I}$ is generated by

$$G_* = (x_2^{\alpha_{12}}x_3^{\alpha_{13}}, x_3^{\alpha_{23}}x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{12}}x_4^{\alpha_{34}}, x_2^{\alpha_2+\alpha_{12}}).$$

Now, consider the case (i). Since the Betti sequences of R/I_{S_*} and $\overline{R}/\overline{I}$ are the same which follows from Lemma 3.1, we will show that the sequence,

$$0 \to R^3 \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R^1 \to 0$$

is a minimal free resolution of $\overline{R}/\overline{I}$, where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}}x_3^{\alpha_{13}} & x_3^{\alpha_{23}}x_4^{\alpha_{24}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{12}}x_4^{\alpha_{34}} & x_2^{\alpha_2+\alpha_{12}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_3^{\alpha_{23}} & x_2^{\alpha_{23}-\alpha_{13}}x_4^{\alpha_{24}} & x_4^{\alpha_{34}} & x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & -x_2^{\alpha_{12}} & 0 & 0 & -x_3^{\alpha_{13}} & x_4^{\alpha_{34}} & 0 & 0 \\ -x_2^{\alpha_{12}} & 0 & 0 & 0 & x_4^{\alpha_{24}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_3^{\alpha_{23}} & -x_2^{\alpha_{12}} & 0 \\ 0 & 0 & -x_3^{\alpha_{13}} & 0 & 0 & 0 & x_4^{\alpha_{24}} & x_2^{\alpha_2} \\ 0 & 0 & 0 & -x_3^{\alpha_{13}} & 0 & 0 & 0 & -x_4^{\alpha_{34}} \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} x_4^{\alpha_{24}} & 0 & 0 \\ -x_3^{\alpha_{13}} & x_4^{\alpha_{34}} & 0 \\ 0 & -x_3^{\alpha_{23}-\alpha_{13}}x_4^{\alpha_{24}} & x_2^{\alpha_2} \\ 0 & 0 & -x_4^{\alpha_{34}} \\ x_2^{\alpha_{12}} & 0 & 0 \\ 0 & x_2^{\alpha_{12}} & 0 \\ 0 & -x_3^{\alpha_{23}} & 0 \\ 0 & 0 & x_3^{\alpha_{13}} \end{pmatrix}.$$

Since $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$, the sequence above is a complex. One can easily check that $rank(\varphi_1) = 1$, $rank(\varphi_2) = 1$ and $rank(\varphi_3) = 3$.

As the 5-minors of φ_2 , we have $-x_3^{2\alpha_3+\alpha_{13}}$ by removing the columns 2, 7, 8 and the row 3, and $-x_4^{2\alpha_4+\alpha_{34}}$ by removing the columns 1, 2, 4 and the row 4. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. As the 3-minors of φ_3, we get $x_2^{\alpha_2+2\alpha_{12}}$ by removing the rows 1, 2, 4, 7, 8, and $x_3^{\alpha_3+\alpha_{13}}$ by removing the rows 1, 3, 4, 5, 6, and finally $-x_4^{\alpha_4+\alpha_{34}}$ by removing the rows 3, 5, 6, 7, 8. These three determinants constitute a regular sequence. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

The other cases (ii)-(1) and (2) can be done similarly and are hence omitted. □

Case 2(b) : Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}, \qquad f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}}, \qquad f_3 = x_3^{\alpha_3} - x_2^{\alpha_{32}}x_4^{\alpha_{34}},$$

$$f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}}x_3^{\alpha_{43}}, \qquad f_5 = x_1^{\alpha_{41}}x_2^{\alpha_{32}} - x_3^{\alpha_{13}}x_4^{\alpha_{24}}.$$

where $\alpha_1 = \alpha_{21} + \alpha_{41}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{24} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{12} + \alpha_{13}$ and $\alpha_4 < \alpha_{41} + \alpha_{43}$.

Under the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ by Theorem 2.2 and one possible condition $\alpha_3 \leq \alpha_{32} + \alpha_{34}$, the Betti sequences for the Cohen–Macaulay tangent cone of C_S are $(1, 5, 5, 1)$ and $(1, 5, 6, 2)$, as given in [9]. Here, the other possible condition $\alpha_3 > \alpha_{32} + \alpha_{34}$ will be considered.

Theorem 3.4 *The Betti sequence of the tangent cone R/I_{S_*} of C_S is $(1, 6, 8, 3)$, if I_S is given as in Case 2(b) when $\alpha_3 > \alpha_{32} + \alpha_{34}$.*

Proof Suppose that I_S is given as in the Case 2(b).

(i) By Proposition 2.16 in [7], if $\alpha_{34} < \alpha_{24}$ and $\alpha_{12} \leq \alpha_{32}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_1}x_2^{\alpha_{32}-\alpha_{12}}x_4^{\alpha_{34}}\}$$

(ii) By Proposition 2.18 in [7],

(1) if $\alpha_{24} \leq \alpha_{34}$ and $\alpha_{32} < \alpha_{12}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_{41}} x_2^{2\alpha_{32}} x_4^{\alpha_{34}-\alpha_{24}}\},$$

(2) if $\alpha_{24} \leq \alpha_{34}$ and $\alpha_{12} \leq \alpha_{32}$, then

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6 = x_3^{\alpha_3+\alpha_{13}} - x_1^{\alpha_1} x_2^{\alpha_{32}-\alpha_{12}} x_4^{\alpha_{34}}\}$$

are standard bases for I_S. Since $\overline{I} = \pi_i(I_{S_*})$ which sends x_1 to 0, then the generators of the defining ideal of $\overline{I}$ is generated by

$$G_* = (x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2}, x_2^{\alpha_{32}} x_4^{\alpha_{34}}, x_4^{\alpha_4}, x_3^{\alpha_{13}} x_4^{\alpha_{24}}, x_3^{\alpha_3+\alpha_{13}}).$$

Now, consider the case (i). Since the Betti sequences of R/I_{S_*} and $\overline{R}/\overline{I}$ are the same which follows from Lemma 3.1, we will show that the sequence,

$$0 \to R^3 \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R^1 \to 0$$

is a minimal free resolution of $\overline{R}/\overline{I}$, where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}} x_3^{\alpha_{13}} & x_2^{\alpha_2} & x_2^{\alpha_{32}} x_4^{\alpha_{34}} & x_4^{\alpha_4} & x_3^{\alpha_{13}} x_4^{\alpha_{24}} & x_3^{\alpha_3+\alpha_{13}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_4^{\alpha_{24}} & x_2^{\alpha_{32}} & x_3^{\alpha_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3^{\alpha_{13}} & 0 & 0 & 0 & 0 & x_4^{\alpha_{34}} & 0 \\ 0 & 0 & 0 & 0 & -x_3^{\alpha_{13}} & 0 & -x_2^{\alpha_{12}} & -x_4^{\alpha_{24}} \\ 0 & 0 & 0 & -x_3^{\alpha_{13}} & 0 & 0 & 0 & x_2^{\alpha_{32}} \\ -x_2^{\alpha_{12}} & 0 & 0 & x_4^{\alpha_{34}} & x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{24}} & x_3^{\alpha_3} & 0 & 0 \\ 0 & 0 & -x_2^{\alpha_{12}} & 0 & 0 & -x_4^{\alpha_{24}} & 0 & 0 \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} x_3^{\alpha_3} & -x_2^{\alpha_{32}} x_4^{\alpha_{34}-\alpha_{24}} & 0 \\ 0 & x_4^{\alpha_{34}} & 0 \\ -x_4^{\alpha_{24}} & 0 & 0 \\ 0 & 0 & x_2^{\alpha_{32}} \\ 0 & -x_2^{\alpha_{12}} & -x_4^{\alpha_{24}} \\ x_2^{\alpha_{12}} & 0 & 0 \\ 0 & x_3^{\alpha_{13}} & 0 \\ 0 & 0 & x_3^{\alpha_{13}} \end{pmatrix}.$$

Since $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$, the sequence above is a complex. One can easily check that $rank(\varphi_1) = 1$, $rank(\varphi_2) = 5$ and $rank(\varphi_3) = 3$. As the 5-minors of φ_2, we get $-x_3^{2\alpha_3+3\alpha_{13}}$ by removing the row 6 and the columns 1, 7, 8, and $x_4^{2\alpha_4+\alpha_{24}}$ by removing the row 4 and the columns 2, 3, 5. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. As the 3-minors of φ_3, we have $-x_2^{\alpha_2+\alpha_{12}}$ by removing the rows 1, 2, 3, 7, 8, and $x_3^{\alpha_3+2\alpha_{13}}$ by removing the rows 2, 3, 4, 5, 6, and finally, $-x_4^{\alpha_4+\alpha_{24}}$ by removing the rows 1, 4, 6, 7, 8. These three determinants constitute a regular sequence. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

The other cases (ii)-(1) and (2) can be done similarly and are hence omitted. □

Case 3(a): Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_4^{\alpha_{14}}, \qquad f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, \qquad f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_4^{\alpha_{34}},$$

$$f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, \qquad f_5 = x_1^{\alpha_{31}} x_2^{\alpha_{42}} - x_3^{\alpha_{23}} x_4^{\alpha_{14}}$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{42}$, $\alpha_3 = \alpha_{23} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{34}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{12} + \alpha_{14}$ and $\alpha_4 < \alpha_{42} + \alpha_{43}$.

C_S has Cohen–Macaulay tangent cone only under the conditions $\alpha_2 \le \alpha_{21} + \alpha_{23}$ and $\alpha_3 \le \alpha_{31} + \alpha_{34}$ by [2]. Mete and Zengin [9] gave the explicit minimal free resolution for the tangent cone of C_S and showed that the Betti sequence of its tangent cone is $(1, 5, 6, 2)$.

Case 3(b): Suppose that I_S is minimally generated by the polynomials

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_4^{\alpha_{14}}, \qquad f_2 = x_2^{\alpha_2} - x_3^{\alpha_{23}} x_4^{\alpha_{24}}, \qquad f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}},$$

$$f_4 = x_4^{\alpha_4} - x_1^{\alpha_{41}} x_3^{\alpha_{43}}, \qquad f_5 = x_2^{\alpha_{12}} x_3^{\alpha_{43}} - x_1^{\alpha_{31}} x_4^{\alpha_{24}}.$$

Here, $\alpha_1 = \alpha_{31} + \alpha_{41}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{23} + \alpha_{43}$, $\alpha_4 = \alpha_{14} + \alpha_{24}$. The condition $n_1 < n_2 < n_3 < n_4$ implies $\alpha_1 > \alpha_{12} + \alpha_{14}$, $\alpha_2 > \alpha_{23} + \alpha_{24}$, $\alpha_3 < \alpha_{31} + \alpha_{32}$ and $\alpha_4 < \alpha_{41} + \alpha_{43}$.

Theorem 3.5 *The Betti sequence of the tangent cone R/I_{S_*} of C_S is $(1, 6, 8, 3)$, if I_S is given as in the Case 3(b).*

Proof Suppose that I_S is given as in the Case 3(b).
(i) By Proposition 2.21 in [7], if $\alpha_{23} < \alpha_{43}$ and $\alpha_{14} \le \alpha_{24}$, then

$$\{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2 + \alpha_{12}} - x_1^{\alpha_1} x_3^{\alpha_{23}} x_4^{\alpha_{24} - \alpha_{14}}\},$$

(ii) By Proposition 2.23 in [7],

(1) if $\alpha_{43} \le \alpha_{23}$ and $\alpha_{24} < \alpha_{14}$ then

$$\{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2 + \alpha_{12}} - x_1^{\alpha_{31}} x_3^{\alpha_{23} - \alpha_{43}} x_4^{2\alpha_{24}}\},$$

(2) if $\alpha_{43} \le \alpha_{23}$ and $\alpha_{14} \le \alpha_{24}$ then

$$\{f_1, f_2, f_3, f_4, f_5, f_6 = x_2^{\alpha_2 + \alpha_{12}} - x_1^{\alpha_1} x_3^{\alpha_{23}} x_4^{\alpha_{24} - \alpha_{14}}\}$$

are standard bases for I_S. Since $\overline{I} = \pi_i(I_{S_*})$ which sends x_1 to 0, then the generators of the defining ideal of $\overline{I}$ is generated by

$$G_* = (x_2^{\alpha_{12}} x_4^{\alpha_{14}}, x_3^{\alpha_{23}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{12}} x_3^{\alpha_{43}}, x_2^{\alpha_2 + \alpha_{12}}).$$

Now, consider the case (i). Since the Betti sequences of R/I_{S_*} and $\overline{R}/\overline{I}$ are the same which follows from Lemma 3.1, we will show that the sequence,

$$0 \to R^3 \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R^1 \to 0$$

is a minimal free resolution of $\overline{R}/\overline{I}$, where

$$\varphi_1 = \begin{pmatrix} x_2^{\alpha_{12}}x_4^{\alpha_{14}} & x_3^{\alpha_{23}}x_4^{\alpha_{24}} & x_3^{\alpha_3} & x_4^{\alpha_4} & x_2^{\alpha_{12}}x_3^{\alpha_{43}} & x_2^{\alpha_2+\alpha_{12}} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x_3^{\alpha_{43}} & x_3^{\alpha_{23}}x_4^{\alpha_{24}-\alpha_{14}} & x_4^{\alpha_{24}} & x_2^{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & -x_2^{\alpha_{12}} & 0 & 0 & 0 & 0 & -x_3^{\alpha_{43}} & x_4^{\alpha_{14}} \\ 0 & 0 & 0 & 0 & -x_2^{\alpha_{12}} & 0 & x_4^{\alpha_{24}} & 0 \\ 0 & 0 & -x_2^{\alpha_{12}} & 0 & 0 & 0 & 0 & -x_3^{\alpha_{23}} \\ -x_4^{\alpha_{14}} & 0 & 0 & 0 & x_3^{\alpha_{23}} & x_2^{\alpha_2} & 0 & 0 \\ 0 & 0 & 0 & -x_4^{\alpha_{14}} & 0 & -x_3^{\alpha_{43}} & 0 & 0 \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} x_2^{\alpha_2} & 0 & x_3^{\alpha_{23}}x_4^{\alpha_{24}-\alpha_{14}} \\ 0 & x_4^{\alpha_{14}} & -x_3^{\alpha_{43}} \\ 0 & -x_3^{\alpha_{23}} & 0 \\ -x_3^{\alpha_{43}} & 0 & 0 \\ 0 & 0 & x_4^{\alpha_{24}} \\ x_4^{\alpha_{14}} & 0 & 0 \\ 0 & 0 & x_2^{\alpha_{12}} \\ 0 & x_2^{\alpha_{12}} & 0 \end{pmatrix}.$$

Since $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$, the sequence above is a complex. It is easy to show that $rank(\varphi_1) = 1$, $rank(\varphi_2) = 5$ and $rank(\varphi_3) = 3$. As the 5-minors of φ_2, we have $-x_3^{2\alpha_3+\alpha_{43}}$ by removing the row 3 and the columns 2, 3, 4, and $x_4^{2\alpha_4+\alpha_{14}}$ by removing the row 4 and the columns 2, 5, 6. These two determinants are relatively prime, so $I(\varphi_2)$ contains a regular sequence of length 2. As the 3-minors of φ_3, we get $-x_2^{\alpha_2+2\alpha_{12}}$ by removing the rows 2, 3, 4, 5, 6, and $x_3^{\alpha_3+\alpha_{43}}$ by removing the rows 1, 5, 6, 7, 8, and finally, $x_4^{\alpha_4+\alpha_{14}}$ by removing the rows 1, 3, 4, 7, 8. These three determinants constitute a regular sequence. Thus, $I(\varphi_3)$ contains a regular sequence of length 3.

The other cases (ii)-(1) and (2) can be done similarly and are hence omitted. □

References

[1] Arslan F. Cohen-Macaulayness of tangent cones. Proceedings of the American Mathematical Society 2000; 128: 2243-2251.

[2] Arslan F, Katsabekis A, Nalbandiyan M. On the Cohen-Macaulayness of tangent cones of monomial curves in $\mathbb{A}^4(K)$. Turkish Journal of Mathematics 2019; 43: 1425-1446.

[3] Arslan F, Mete P. Hilbert functions of Gorenstein monomial curves. Proceedings of the American Mathematical Society 2007; 135: 1993-2002.

[4] Bresinsky H. Symmetric semigroups of integers generated by four elements. Manuscripta Mathematica 1975; 17: 205-219.

[5] Buchsbaum D, Eisenbud D. What makes a complex exact? Journal of Algebra 1973; 323: 259-268.

[6] Herzog J, Stamate DI. On the defining equations of the tangent cone of a numerical semigroup ring. Journal of Algebra 2014; 418: 8-28.

[7] Katsabekis A. Hilbert series of tangent cones for Gorenstein monomial curves in $\mathbb{A}^4(K)$. Turkish Journal of Mathematics 2021; 45: 597-616.

[8] Kunz E. The value-semigroup of a one-dimensional Gorenstein ring. Proceedings of the American Mathematical Society 1970; 25: 748-751.

[9] Mete P, Zengin EE. Minimal free resolutions of the tangent cones for Gorenstein monomial curves. Turkish Journal of Mathematics 2019; 43: 2782-2793.

[10] Stamate DI. Betti numbers for numerical semigroup rings. Multigraded Algebra and Applications, Springer Proceedings in Mathematics and Statistics 2018; 238: 133-157.

[11] Şahin M, Şahin N. Betti numbers for certain Cohen-Macaulay tangent cones. Bulletin of the Australian Mathematical Society 2019; 99: 68-77.