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Sharp inequalities for anti-invariant Riemannian submersions from Sasakian space forms

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ABSTRACT

We obtain sharp inequalities involving the Ricci curvature and the scalar curvature for anti-invariant Riemannian submersions from Sasakian space forms onto Riemannian manifolds.

1. Introduction

To find relationships between the extrinsic and intrinsic invariants of a submanifold has been very popular problems in the last twenty five years. The first study in this direction was started by B.-Y. Chen in 1993. He established some inequalities between the main extrinsic (the squared mean curvature) and main intrinsic invariants (the scalar curvature and the Ricci curvature, or the delta-invariant $\delta(2)$) of a submanifold in a real space form [6]. In 1999, Chen also established a relation between the Ricci curvature and the squared mean curvature for a submanifold [7]. After that, many papers have been published by various authors in different ambient spaces. In 2011, Chen published a book which consists of all studies in these directions [10]. The topic is still very popular and there are many new papers related to the inequalities which are introduced by Chen. For example see [1], [3], [4], [7], [15], [16], [17], [18], [21] and [23].

Let (M, g) and $\left(B, g'\right)$ be m and b-dimensional Riemannian manifolds, respectively. A *Riemannian submersion* $\pi : M \to B$ is a mapping of M onto B such that π has a maximal rank and the differential π_* preserves the lengths of the horizontal vectors [19]. In [8], Chen proved a simple optimal relationship between Riemannian submersions and minimal immersions. In [9], Chen considered the equality case of the inequality obtained in [8]. In [2], Alegre, Chen and Munteanu established a

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sharp relationship between the δ-invariants and Riemannian submersions with totally geodesic fibers. In [22], Şahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. In [13], Küpeli, Murathan and in [14], Lee introduced anti-invariant submersions from Sasakian manifolds. In [12], Gülbahar, Meriç and Kılıç obtained sharp inequalities involving the Ricci curvature for invariant Riemannian submersions.

Motivated by the above studies, in the present study, we consider anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We obtain sharp inequalities involving the Ricci curvature and the scalar curvature.

The paper is organized as follows: In Section 2, we give a brief introduction about Sasakian manifolds and submersions. We give some lemmas which will be used in Section 3 and Section 4. In Section 3, we obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for anti-invariant Riemannian submersions from Sasakian space forms. The equality cases are also discussed. In Section 4, we prove Chen-Ricci inequalities on the vertical and horizontal distributions for anti-invariant Riemannian submersions from Sasakian space forms. We find relationships between the intrinsic and extrinsic invariants using fundamental tensors. The equality cases are also considered.

2. Preliminaries

Let $\pi : M \to B$ be a Riemannian submersion. We put $\dim M = 2m+1$ and $\dim B = b$. For $x \in B$, Riemannian submanifold $\pi^{-1}(x)$ with the induced metric $\overline{g}$ is called a *fiber* and denoted by $\overline{M}$. A vector field on M is called *vertical*, if it is tangent to fibers and *horizontal*, if it is orthogonal to fibers. We notice that the dimension of each fiber is always $(2m+1-b) = r$ and dimension of the horizontal distribution is $b = (2m+1-r)$. In the tangent bundle TM of M, the vertical and horizontal distributions of M are denoted by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. We call a vector field X on M *projectable*, if there exists a vector field X_* on B such that $\pi_*\left(X_p\right) = X_{*\pi(p)}$ for each $p \in M$. In this case, we call that X and X_* are π-related. A vector field X on M is called *basic*, if it is projectable and horizontal ([19] and [20]). For each $p \in M$ the vertical and horizontal spaces in T_pM are denoted by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$, respectively.

The tensor fields T and A of type $(1,2)$ are defined by

$$T_E F = h\nabla_{\upsilon E}\upsilon F + \upsilon\nabla_{\upsilon E} hF$$

and

$$A_E F = h\nabla_{hE}\upsilon F + \upsilon\nabla_{hE} hF,$$

respectively.

Denote by R, R', $\widehat{R}$ and R^* the Riemannian curvature tensors of Riemannian manifolds M, B, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively. Then the Gauss-Codazzi type equations are given by

$$R(U,V,F,W) = \widehat{R}(U,V,F,W) + g(T_U W, T_V F) - g(T_V W, T_U F), \tag{2.1}$$

$$\begin{aligned} R(X,Y,Z,H) = {} & R^*(X,Y,Z,H) - 2g(A_X Y, A_Z H) \\ & + g(A_Y Z, A_X H) - (A_X Z, A_Y H), \end{aligned} \tag{2.2}$$

$$\begin{aligned} R(X,V,Y,W) = {} & g((\nabla_X T)(V,W), Y) + g((\nabla_V A)(X,Y), W) \\ & - g(T_V X, T_W Y) + g(A_Y W, A_X V), \end{aligned} \tag{2.3}$$

where

$$\pi_*\left(R^*(X,Y)Z\right) = R'(\pi_* X, \pi_* Y)\pi_* Z$$

for any $X, Y, Z, H \in \mathcal{H}(M)$ and $U, V, F, W \in \mathcal{V}(M)$ [19].

Moreover, the mean curvature vector field H of any fiber of Riemannian submersion π is given by

$$H = rN, \quad N = \sum_{j=1}^{r} T_{U_j} U_j,$$

where $\{U_1, \ldots, U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Furthermore, π *has totally geodesic fibers* if T vanishes on $\mathcal{H}(M)$ and $\mathcal{V}(M)$ [19].

Now we give the following lemmas:

Lemma 2.1. *[11] Let (M,g) and $\left(B,g'\right)$ be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. For $E, F, G \in TM$, we have*

$$g(T_E F, G) = -g(F, T_E G),$$

$$g(A_E F, G) = -g(F, A_E G).$$

That is, A_E and T_E are anti-symmetric with respect to g.

Lemma 2.2. [11] *Let (M,g) and (B,g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$.*
(i) *For $U, V \in \mathcal{V}(M)$,*

$$T_U V = T_V U;$$

(ii) *For $X, Y \in \mathcal{H}(M)$, $A_X Y = -A_Y X$.*

For more details for Riemannian submersions see also [24].

Let (M,ϕ,ξ,η,g) be a $(2m+1)$-dimensional contact metric manifold. If in a contact metric manifold,

$$\nabla_X \xi = -\phi X, \qquad (\nabla_X \phi) Y = g(X,Y)\xi - \eta(Y)X,$$

then $(M,\nabla,g,\phi,\xi,\eta)$ is called a *Sasakian manifold* [5], where ∇ denotes the Levi-Civita connection of g. A plane section π in TM is called a *ϕ-section*, if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* [5] and is denoted by $M(c)$. The curvature tensor R of $M(c)$ is expressed by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X \\ & - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z]. \end{aligned} \tag{2.4}$$

Definition 2.1. [13] Let $(M,\nabla,g,\phi,\xi,\eta)$ be a Sasakian manifold and (B,g') a Riemannian manifold. A Riemannian submersion $\pi : M \to B$ is called anti-invariant, if $\mathcal{V}(M)$ is anti-invariant with respect to ϕ, i.e. $\phi(\mathcal{V}(M)) \subseteq \mathcal{H}(M)$.

Let $\pi : (M,\nabla,g,\phi,\xi,\eta) \to (B,g')$ be an anti-invariant Riemannian submersion from a Sasakian manifold $(M,\nabla,g,\phi,\xi,\eta)$ to a Riemannian manifold (B,g'). From Definition 2.1, we have $\phi(\mathcal{V}(M)) \cap \mathcal{H}(M) \neq \{0\}$. We denote the complementary orthogonal distribution to $\phi(\mathcal{V}(M))$ in $\mathcal{H}(M)$ by μ. Then we have

$$\mathcal{H}(M) = \phi(\mathcal{V}(M)) \oplus \mu.$$

Suppose that ξ is vertical. It is easy to see that μ is an invariant distribution of $\mathcal{H}(M)$ under the endomorphism ϕ. Thus for $X \in \mathcal{H}(M)$, we write

$$\phi X = BX + CX,$$

where $BX \in \mathcal{V}(M)$ and $CX \in \chi(\mu)$ [13].

Suppose that ξ is horizontal. It is easy to see that $\mu = \phi\mu \oplus \{\xi\}$. Thus for $X \in \mathcal{H}(M)$, we write

$$\phi X = BX + CX,$$

where $BX \in \mathcal{V}(M)$ and $CX \in \chi(\mu)$ [13].

Lemma 2.3. [13] *Let $\pi : M \to B$ be an anti-invariant Riemannian submersion from a Sasakian manifold $(M,\nabla,g,\phi,\xi,\eta)$ to a Riemannian manifold (B,g').*
(i) *If ξ is vertical, then $C^2X = -X - \phi BX$;*
(ii) *If ξ is horizontal, then $C^2X = -X + \eta(X)\xi - \phi BX$.*

Example 2.1. [5] Let us take $M = \mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1,...,x_m,y_1,...,y_m,z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^m y_i dx_i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field ϕ given by

$$\phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^m \left((dx_i)^2 + (dy_i)^2\right)$. Then $\left(M^{2m+1},\phi,\xi,\eta,g\right)$ is a Sasakian space form with constant ϕ-sectional curvature $c=-3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$. The vector fields

$$E_i = 2\frac{\partial}{\partial y_i},\ E_{i+m} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}),\ 1 \le i \le m,\ \xi = 2\frac{\partial}{\partial z},$$

form a g-orthonormal basis for the contact metric structure.

Example 2.2. [13] We consider $M=\mathbb{R}^5(-3)$ with the structure given in Example 2.1. The Riemannian metric $g_{\mathbb{R}^2}$ is given by

$$g_{\mathbb{R}^2}=\frac{1}{8}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

on $\mathbb{R}^2$. Let $\pi:\mathbb{R}^5(-3)\to\mathbb{R}^2$ be a map defined by

$$\pi\left(x_1,x_2,y_1,y_2,z\right)=\left(x_1+y_1,x_2+y_2\right).$$

Then

$$\mathcal{V}(M)=sp\left\{V_1=E_1-E_3,V_2=E_2-E_4,V_3=E_5=\xi\right\}$$

and

$$\mathcal{H}(M)=sp\left\{H_1=E_1+E_3,H_2=E_2+E_4\right\}.$$

So π is a Riemannian submersion. Moreover, $\phi V_1=H_1,\phi V_2=H_2,\phi V_3=0$ imply that $\phi(\mathcal{V}(M))=\mathcal{H}(M)$. Hence π is an anti-invariant Riemannian submersion such that ξ is vertical.

Example 2.3. [13] We consider $M=\mathbb{R}^5(-3)$ with the structure given in Example 2.1. Let $N=\mathbb{R}^3-\left\{(y_1,y_2,z)\in\mathbb{R}^3 \mid y_1^2+y_2^2\leq 2\right\}$. The Riemannian metric tensor g_N is given by

$$g_N=\frac{1}{4}\begin{bmatrix}\frac{1}{2} & \frac{y_1y_2}{2} & -\frac{y_1}{2}\\ \frac{y_1y_2}{2} & \frac{1}{2} & -\frac{y_2}{2}\\ -\frac{y_1}{2} & -\frac{y_2}{2} & 1\end{bmatrix}$$

on N. Let $\pi:\mathbb{R}^5(-3)\to N$ be a map defined by

$$\pi\left(x_1,x_2,y_1,y_2,z\right)=\left(x_1+y_1,x_2+y_2,\frac{y_1^2}{2}+\frac{y_2^2}{2}+z\right).$$

Then

$$\mathcal{V}(M)=sp\left\{V_1=E_1-E_3,V_2=E_2-E_4\right\}$$

and

$$\mathcal{H}(M)=sp\left\{H_1=E_1+E_3,H_2=E_2+E_4,H_3=E_5=\xi\right\}.$$

So π is a Riemannian submersion. Moreover, $\phi V_1=H_1,\phi V_2=H_2$ imply that $\phi(\mathcal{V}(M))\subset\mathcal{H}(M)=\phi(\mathcal{V}(M))\oplus\{\xi\}$. Hence π is an anti-invariant Riemannian submersion such that ξ is horizontal.

3. Inequalities for anti-invariant Riemannian submersions

In the present section, we aim to obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for anti-invariant Riemannian submersions from Sasakian space forms. We shall also consider the equality cases of these inequalities.

Using (2.4) and (2.1), we have

$$\begin{aligned}\widehat{R}(U,V,F,W)=&\frac{c+3}{4}\left\{g(V,F)g(U,W)-g(U,F)g(V,W)\right\}\\&+\frac{c-1}{4}\left\{\eta(U)\eta(F)g(V,W)-\eta(V)\eta(F)g(U,W)\right.\\&+\eta(V)\eta(W)g(U,F)-\eta(U)\eta(W)g(V,F)+g(\phi V,F)g(\phi U,W)\\&\left.-g(\phi V,W)g(\phi U,F)-2g(W,\phi F)g(\phi U,V)\right\}\\&-g(T_UW,T_VF)+g(T_VW,T_UF).\end{aligned}\tag{3.1}$$

Similarly, from (2.4) and (2.2), we get

$$R^{*}(X,Y,Z,H)=\frac{c+3}{4}\left\{g(Y,Z)g(X,H)-g(X,Z)g(Y,H)\right\}$$
$$+\frac{c-1}{4}\{\eta(X)\eta(Z)g(Y,H)-\eta(Y)\eta(Z)g(X,H)$$
$$+\eta(Y)\eta(H)g(X,Z)-\eta(X)\eta(H)g(Y,Z)+g(\phi Y,Z)g(\phi X,H)$$
$$-g(\phi Y,H)g(\phi X,Z)-2g(H,\phi Z)g(\phi X,Y)\}$$
$$+2g(A_XY,A_ZH)-g(A_YZ,A_XH)+(A_XZ,A_YH). \tag{3.2}$$

Let $(M(c),g)$, (B,g') be a Sasakian space form and a Riemannian manifold, respectively and $\pi:M(c)\to B$ an anti-invariant Riemannian submersion. Furthermore, for each point $p\in M$, let $\{U_1,...,U_r,X_1,...,X_n\}$ be an orthonormal basis of $T_pM(c)$ such that $\mathcal{V}_p(M)=span\{U_1,...,U_r\}$, $\mathcal{H}_p(M)=span\{X_1,...,X_n\}$.

Case I: Assume that ξ is vertical.

For the vertical distribution, in view of (3.1), since π is anti-invariant and ξ is vertical, with the use of $U_1=U$, we find

$$\widehat{Ric}(U)=\frac{c+3}{4}(r-1)g(U,U)+\frac{c-1}{4}\left\{(2-r)\eta(U)^2-g(U,U)\right\}$$
$$-rg(T_UU,H)+\sum_{j=1}^{r}g\left(T_{U_j}U,T_UU_j\right).$$

Hence we obtain the following proposition:

Proposition 3.1. *Let $\pi:M(c)\to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c),g)$ onto a Riemannian manifold (B,g') such that ξ is vertical. Then*

$$\widehat{Ric}(U)\geq\frac{c+3}{4}(r-1)-\frac{c-1}{4}\left\{(r-2)\eta(U)^2+1\right\}-rg(T_UU,H).$$

The equality case of the inequality holds for a unit vertical vector $U\in\mathcal{V}_p(M(c))$ if and only if each fiber is totally geodesic.

Similarly, in view of (3.1), using the symmetry of T, we have

$$2\widehat{\tau}=\frac{c+3}{4}r(r-1)+\frac{c-1}{4}(2-2r)-r^2\|H\|^2+\sum_{i,j=1}^{r}g\left(T_{U_i}U_j,T_{U_i}U_j\right),$$

where $\widehat{\tau}=\sum_{1\leq i<j\leq r}\widehat{R}\left(U_i,U_j,U_j,U_i\right)$. Then we can write

$$2\widehat{\tau}\geq\frac{c+3}{4}r(r-1)-\frac{c-1}{2}(r-1)-r^2\|H\|^2.$$

The equality case of the inequality holds if and only if $T=0$, which means that each fiber is totally geodesic. Thus we can state the following proposition:

Proposition 3.2. *Let $\pi:M(c)\to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c),g)$ onto a Riemannian manifold (B,g') such that ξ is vertical. Then*

$$2\widehat{\tau}\geq\frac{c+3}{4}r(r-1)-\frac{c-1}{2}(r-1)-r^2\|H\|^2.$$

The equality case of the inequality holds if and only if each fiber is totally geodesic.

For the horizontal distribution, in view of (3.2), since π is anti-invariant and ξ is vertical, using the anti-symmetry of A, we find

$$2\tau^{*}=\frac{c+3}{4}n(n-1)$$
$$+\sum_{i,j=1}^{n}\left[\frac{3(c-1)}{4}g\left(CX_i,X_j\right)g\left(CX_i,X_j\right)-3g\left(A_{X_i}X_j,A_{X_i}X_j\right)\right]. \tag{3.3}$$

By the use of Lemma 2.3, we obtain

$$2\tau^* = \frac{c+3}{4} n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)) - \sum_{i,j=1}^{n} 3g\left(A_{X_i}X_j, A_{X_i}X_j\right).$$

Then we can write

$$2\tau^* \leq \frac{c+3}{4} n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)), \tag{3.4}$$

where $\tau^* = \sum_{1 \leq i < j \leq n} R^*\left(X_i, X_j, X_j, X_i\right)$. The equality case of (3.4) holds if and only if $A = 0$, which means that the horizontal distribution is integrable. So we can state the following result:

Proposition 3.3. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $\left(B, g'\right)$ such that ξ is vertical. Then*

$$2\tau^* \leq \frac{c+3}{4} n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)).$$

The equality case of (3.4) holds if and only if $\mathcal{H}(M)$ is integrable.

Case II: Assume that ξ is horizontal.

From (3.1), since π is anti-invariant submersion, after some computations, we have

$$2\widehat{\tau} = \frac{c+3}{4} r(r-1) - r^2 \|H\|^2 + \sum_{i,j=1}^{r} g\left(T_{U_i}U_j, T_{U_i}U_j\right).$$

Hence we can state the following proposition:

Proposition 3.4. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $\left(B, g'\right)$ such that ξ is horizontal. Then*

$$2\widehat{\tau} \geq \frac{c+3}{4} r(r-1) - r^2 \|H\|^2.$$

The equality case of the inequality holds if and only if each fiber is totally geodesic.

For the horizontal distribution, from (3.2), since ξ is horizontal and A is anti-symmetric, after some computations, we have

$$2\tau^* = \frac{c+3}{4} n(n-1) + \sum_{i,j=1}^{n} \left[\frac{c-1}{4}\left\{2 - 2n + 3g\left(CX_i, X_j\right) g\left(CX_i, X_j\right)\right\} - 3g\left(A_{X_i}X_j, A_{X_i}X_j\right)\right].$$

Then using Lemma 2.3, we obtain

$$2\tau^* = \frac{c+3}{4} n(n-1) + \frac{c-1}{4}(3tr\phi B + n - 1) - \sum_{i,j=1}^{n} 3g\left(A_{X_i}X_j, A_{X_i}X_j\right),$$

where $\tau^* = \sum_{1 \leq i < j \leq n} R^*\left(X_i, X_j, X_j, X_i\right)$.

So we can state the following result:

Proposition 3.5. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $\left(B, g'\right)$ such that ξ is horizontal. Then*

$$2\tau^* \leq \frac{c+3}{4} n(n-1) + \frac{(c-1)}{4}(3tr(\phi B) + n - 1).$$

The equality case of the inequality holds if and only if $\mathcal{H}(M)$ is integrable.

4. Chen-Ricci inequalities for anti-invariant Riemannian submersions

In this section, we aim to obtain Chen-Ricci inequality on the vertical and horizontal distributions for anti-invariant Riemannian submersions from a Sasakian space forms onto a Riemannian manifold. The equality cases will be also considered.

Let $(M(c), g)$ be a Sasakian space form and $\left(B, g'\right)$ a Riemannian manifold. Assume that $\pi : M(c) \to B$ is an anti-invariant Riemannian submersion and $\{U_1, ..., U_r, X_1, ..., X_n\}$ is an orthonormal basis of $T_pM(c)$ such that $\mathcal{V}_p(M) = span\{U_1, ..., U_r\}$, $\mathcal{H}_p(M) = span\{X_1, ..., X_n\}$. Now we denote T_{ij}^s by

$$T_{ij}^s = g\left(T_{Ui}U_j, X_s\right), \tag{4.1}$$

where $1 \le i, j \le r$ and $1 \le s \le n$ (see [12]).

Similarly, we denote A_{ij}^α by

$$A_{ij}^\alpha = g\left(A_{Xi}X_j, U_\alpha\right), \tag{4.2}$$

where $1 \le i, j \le n$ and $1 \le \alpha \le r$. From [12], we use

$$\delta(N) = \sum_{i=1}^{n}\sum_{k=1}^{r} g\left(\left(\nabla_{X_i}T\right)_{U_k}U_k, X_i\right). \tag{4.3}$$

Case I: Assume that ξ is vertical.

Then from (3.1), we have

$$2\widehat{\tau} = \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) - r^2\|H\|^2 + \sum_{i,j=1}^{r} g\left(T_{U_i}U_j, T_{U_i}U_j\right).$$

Using (4.1) in the last equality and the symmetry of T, we can write

$$2\widehat{\tau} = \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) - r^2\|H\|^2 + \sum_{s=1}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2. \tag{4.4}$$

We know from [12] that

$$\begin{aligned}\sum_{s=1}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2 &= \frac{1}{2}r^2\|H\|^2 + \frac{1}{2}\sum_{s=1}^{n}\left[T_{11}^s - T_{22}^s - \dots - T_{rr}^s\right]^2 \\ &\quad + 2\sum_{s=1}^{n}\sum_{j=2}^{r}\left(T_{1j}^s\right)^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}\left[T_{ii}^sT_{jj}^s - \left(T_{ij}^s\right)^2\right].\end{aligned} \tag{4.5}$$

So using (4.5) in (4.4), we get

$$\begin{aligned}2\widehat{\tau} &= \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) \\ &\quad - \frac{1}{2}r^2\|H\|^2 + \frac{1}{2}\sum_{s=1}^{n}\left[T_{11}^s - T_{22}^s - \dots - T_{rr}^s\right]^2 \\ &\quad + 2\sum_{s=1}^{n}\sum_{j=2}^{r}\left(T_{1j}^s\right)^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}\left[T_{ii}^sT_{jj}^s - \left(T_{ij}^s\right)^2\right].\end{aligned}$$

Then from the last equality, we have

$$\begin{aligned}2\widehat{\tau} &\ge \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) \\ &\quad - \frac{1}{2}r^2\|H\|^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}\left[T_{ii}^sT_{jj}^s - \left(T_{ij}^s\right)^2\right].\end{aligned} \tag{4.6}$$

Furthermore, from (2.1), taking $U = W = U_i$, $V = F = U_j$ and using (4.1), we can write

$$2\sum_{2\leq i<j\leq r} R\left(U_i, U_j, U_j, U_i\right) = 2\sum_{2\leq i<j\leq r} \widehat{R}\left(U_i, U_j, U_j, U_i\right) + 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}\left[T_{ii}^{s}T_{jj}^{s} - \left(T_{ij}^{s}\right)^2\right].$$

In view of the last equality, (4.6) can be written as

$$2\widehat{\tau} \geq \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) - \frac{1}{2}r^2\|H\|^2 + 2\sum_{2\leq i<j\leq r}\widehat{R}\left(U_i, U_j, U_j, U_i\right) - 2\sum_{2\leq i<j\leq r} R\left(U_i, U_j, U_j, U_i\right). \tag{4.7}$$

Then using the equality

$$2\widehat{\tau} = 2\sum_{2\leq i<j\leq r}\widehat{R}\left(U_i, U_j, U_j, U_i\right) + 2\sum_{j=1}^{r}\widehat{R}\left(U_1, U_j, U_j, U_1\right), \tag{4.8}$$

in view of (4.7), we have

$$2\widehat{Ric}(U_1) \geq \frac{c+3}{4}r(r-1) - \frac{c-1}{2}(r-1) - \frac{1}{2}r^2\|H\|^2 - 2\sum_{2\leq i<j\leq r} R\left(U_i, U_j, U_j, U_i\right).$$

Since M is a Sasakian space form, its curvature tensor R satisfies the equality (2.4). So we obtain

$$\widehat{Ric}(U_1) \geq \frac{c+3}{4}(r-1) + \frac{c-1}{4}\left\{(2-r)\eta(U_1)^2 - 1\right\} - \frac{1}{4}r^2\|H\|^2.$$

Hence we state the following theorem:

Theorem 4.1. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold (B, g') such that ξ is vertical. Then*

$$\widehat{Ric}(U_1) \geq \frac{c+3}{4}(r-1) - \frac{c-1}{4}\left\{(r-2)\eta(U_1)^2 + 1\right\} - \frac{1}{4}r^2\|H\|^2.$$

The equality case of the inequality holds if and only if

$$T_{11}^{s} = T_{22}^{s} + \ldots + T_{rr}^{s},$$
$$T_{1j} = 0, \quad j = 2, \ldots, r.$$

On the other hand, using (4.2) and Lemma 2.3, the equation (3.3) can be rewritten as

$$2\tau^* = \frac{c+3}{4}n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)) - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}\left(A_{ij}^{\alpha}\right)^2.$$

Since A is anti-symmetric on $\mathcal{H}(M(c))$, the above equality turns into

$$2\tau^* = \frac{c+3}{4}n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}\left(A_{1j}^{\alpha}\right)^2 - 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}\left(A_{ij}^{\alpha}\right)^2. \tag{4.9}$$

Furthermore, from (2.2), taking $X = H = X_i$, $Y = Z = X_j$ and using (4.2), we have

$$2\sum_{2\leq i<j\leq n} R\left(X_i, X_j, X_j, X_i\right) = 2\sum_{2\leq i<j\leq n} R^*\left(X_i, X_j, X_j, X_i\right) + 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}\left(A_{ij}^{\alpha}\right)^2. \tag{4.10}$$

If we consider the last equality in (4.9), then we get

$$2\tau^* = \frac{c+3}{4} n(n-1) + \frac{3}{4}(c-1)(n + tr(\phi B)) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}\left(A_{1j}^{\alpha}\right)^2 + 2\sum_{2\le i<j\le n} R^*\left(X_i, X_j, X_j, X_i\right) - 2\sum_{2\le i<j\le n} R\left(X_i, X_j, X_j, X_i\right).$$

Since M is a Sasakian space form, its curvature tensor R satisfies the equality (2.4). Then we have

$$2Ric^*(X_1) = \frac{c+3}{2}(n-1) + \frac{3}{4}(c-1)\|CX_1\|^2 - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}\left(A_{1j}^{\alpha}\right)^2.$$

So we can write

$$Ric^*(X_1) \le \frac{c+3}{2}(n-1) + \frac{3}{4}(c-1)\|CX_1\|^2.$$

Hence we obtain the following theorem:

Theorem 4.2. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold (B, g') such that ξ is vertical. Then*

$$Ric^*(X_1) \le \frac{c+3}{4}(n-1) + \frac{3}{4}(c-1)\|CX_1\|^2.$$

The equality case of the inequality holds if and only if

$$A_{1j} = 0, \quad j = 2, ..., n.$$

Now, we compute the Chen-Ricci inequality between the vertical and horizontal distributions for the case of ξ is vertical. For the scalar curvature τ of $M(c)$, we obtain

$$2\tau = \sum_{s=1}^{n} Ric(X_s, X_s) + \sum_{k=1}^{r} Ric(U_k, X_k),$$

$$2\tau = \sum_{j,k=1}^{r} R\left(U_j, U_k, U_k, U_j\right) + \sum_{i=1}^{n}\sum_{k=1}^{r} R(X_i, U_k, U_k, X_i) + \sum_{i,s=1}^{n} R(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n}\sum_{j=1}^{r} R\left(U_j, X_s, X_s, U_j\right). \tag{4.11}$$

Since $M(c)$ is a Sasakian space form, using (4.11) and (2.4), we find

$$2\tau = \frac{c+3}{4}(r(r-1) + n(n-1) + 2nr) + \frac{c-1}{4}(4(r-1) + n + 3tr\phi B). \tag{4.12}$$

On the other hand, from the Gauss-Codazzi type equations (2.1), (2.2) and (2.3), we have

$$2\tau = 2\hat{\tau} + 2\tau^* + r^2\|H\|^2 + \sum_{k,j=1}^{r} g\left(T_{U_k}U_j, T_{U_k}U_j\right) + 3\sum_{i,s=1}^{n} g\left(A_{X_i}X_s, A_{X_i}X_s\right) - \sum_{i=1}^{n}\sum_{k=1}^{r} g\left(\left(\nabla_{X_i}T\right)_{U_k}U_k, X_i\right) + \sum_{i=1}^{n}\sum_{k=1}^{r}\left\{g\left(T_{U_k}X_i, T_{U_k}X_i\right) - g\left(A_{X_i}U_k, A_{X_i}U_k\right)\right\} - \sum_{s=1}^{n}\sum_{j=1}^{r} g\left(\left(\nabla_{X_s}T\right)_{U_j}U_j, X_s\right) + \sum_{s=1}^{n}\sum_{j=1}^{r}\left\{g\left(T_{U_j}X_s, T_{U_j}X_s\right) - g\left(A_{X_s}U_j, A_{X_s}U_j\right)\right\}. \tag{4.13}$$

Using (4.5) and (4.3), we get

$$2\tau = 2\widehat{\tau} + 2\tau^* + \frac{1}{2}r^2 \left\|H\right\|^2 - \frac{1}{2}\sum_{s=1}^{n}\left[T_{11}^s - T_{22}^s - \ldots - T_{rr}^s\right]^2$$
$$-2\sum_{s=1}^{n}\sum_{j=2}^{r}\left(T_{1j}^s\right)^2 + 2\sum_{s=1}^{n}\sum_{2\leq j<k\leq r}\left[T_{jj}^sT_{kk}^s - \left(T_{jk}^s\right)^2\right] + 6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}\left(A_{1s}^\alpha\right)^2$$
$$+6\sum_{\alpha=1}^{r}\sum_{2\leq i<s\leq n}\left(A_{is}^\alpha\right)^2 + \sum_{i=1}^{n}\sum_{k=1}^{r}\left\{g\left(T_{U_k}X_i, T_{U_k}X_i\right) - g\left(A_{X_i}U_k, A_{X_i}U_k\right)\right\}$$
$$-2\delta\left(N\right) + \sum_{s=1}^{n}\sum_{j=1}^{r}\left\{g\left(T_{U_j}X_s, T_{U_j}X_s\right) - g\left(A_{X_s}U_j, A_{X_s}U_j\right)\right\}.$$

By making use of (4.8), (4.10) and (4.12) in the last equality, we obtain

$$\frac{c+3}{2}nr + \frac{c-1}{2}\left(3\left(r-1\right)-n\right)$$
$$+2\sum_{k=1}^{r}R\left(U_1, U_k, U_k, U_1\right) + 2\sum_{s=1}^{n}R\left(X_1, X_s, X_s, X_1\right)$$
$$= 2\widehat{Ric}\left(U_1\right) + 2Ric^*\left(X_1\right) + \frac{1}{2}r^2\left\|H\right\|^2 - \frac{1}{2}\sum_{s=1}^{n}\left[T_{11}^s - T_{22}^s - \ldots - T_{rr}^s\right]^2$$
$$-2\sum_{s=1}^{n}\sum_{j=2}^{r}\left(T_{1j}^s\right)^2 + 6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}\left(A_{1s}^\alpha\right)^2 + \sum_{i=1}^{n}\sum_{k=1}^{r}\left\{g\left(T_{U_k}X_i, T_{U_k}X_i\right) - g\left(A_{X_i}U_k, A_{X_i}U_k\right)\right\}$$
$$-2\delta\left(N\right) + \sum_{s=1}^{n}\sum_{j=1}^{r}\left\{g\left(T_{U_j}X_s, T_{U_j}X_s\right) - g\left(A_{X_s}U_j, A_{X_s}U_j\right)\right\}.$$

We denote

$$\left\|T^V\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r}g\left(T_{U_k}X_i, T_{U_k}X_i\right)$$

and

$$\left\|A^H\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r}g\left(A_{X_i}U_k, A_{X_i}U_k\right),$$

(see [12]).

Since $(M(c), g)$ is a Sasakian space form, from (2.4), we obtain the following theorem:

Theorem 4.3. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $\left(B, g'\right)$ such that ξ is vertical. Then*

$$\frac{c+3}{4}\left\{nr+n+r-2\right\} + \frac{c-1}{4}\left\{3r-4-n-\left(r-2\right)\eta\left(U_1\right)^2 + 3\left\|CX_1\right\|^2\right\} \leq \widehat{Ric}\left(U_1\right) + Ric^*\left(X_1\right) + \frac{1}{4}r^2\left\|H\right\|^2$$
$$+3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}\left(A_{1s}^\alpha\right)^2 - \delta\left(N\right) + \left\|T^V\right\|^2 - \left\|A^H\right\|^2.$$

The equality case of the inequality holds if and only if

$$T_{11}^s = T_{22}^s + \ldots + T_{rr}^s,$$
$$T_{1j} = 0, \quad j = 2, \ldots, r.$$

Case II: Assume that ξ is horizontal.

From (3.1), similar to Theorem 4.1, we can state the following theorem:

Theorem 4.4. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold (B, g') such that ξ is horizontal. Then*

$$\widehat{Ric}(U_1) \geq \frac{c+3}{4}(r-1) - \frac{1}{4}r^2 \|H\|^2 .$$

The equality case of the inequality holds if and only if

$$T^s_{11} = T^s_{22} + ... + T^s_{rr},$$
$$T_{1j} = 0, \quad j = 2, ..., r.$$

From (3.2), similar to Theorem 4.2, we have the following theorem:

Theorem 4.5. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold (B, g') such that ξ is horizontal. Then*

$$Ric^*(X_1) \leq \frac{c+3}{4}(n-1) + \frac{c-1}{4}\left\{(2-n)\eta(X_1)^2 - 1 + 3\|CX_1\|^2\right\}.$$

The equality case of the inequality holds if and only if

$$A_{1j} = 0, \ j = 2, ..., n.$$

Now, we compute the Chen-Ricci inequality between the vertical and horizontal distributions for the case of ξ is horizontal. Since ξ is horizontal, from (4.11), we find

$$2\tau = \frac{c+3}{4}\left[r(r-1) + n(n-1) + 2nr\right] + \frac{c-1}{4}\left[n + 3tr\phi B + 4r - 7\right].$$

Using the above equation, (4.13), (4.5), (4.8), (4.10) and (4.3), we get

$$\begin{aligned}
&\frac{c+3}{2}nr + \frac{c-1}{2}(2r-3) \\
&\quad + 2\sum_{k=1}^{r} R(U_1, U_k, U_k, U_1) + 2\sum_{s=1}^{n} R(X_1, X_s, X_s, X_1) \\
&= 2\widehat{Ric}(U_1) + 2Ric^*(X_1) + \frac{1}{2}r^2\|H\|^2 - \frac{1}{2}\sum_{s=1}^{n}\left[T^s_{11} - T^s_{22} - ... - T^s_{rr}\right]^2 \\
&\quad - 2\sum_{s=1}^{n}\sum_{j=2}^{r}\left(T^s_{1j}\right)^2 + 6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}\left(A^\alpha_{1s}\right)^2 - 2\delta(N) \\
&\quad + \sum_{s=1}^{n}\sum_{j=1}^{r}\left\{g\left(T_{U_j}X_s, T_{U_j}X_s\right) - g\left(A_{X_s}U_j, A_{X_s}U_j\right)\right\} \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{r}\left\{g\left(T_{U_k}X_i, T_{U_k}X_i\right) - g\left(A_{X_i}U_k, A_{X_i}U_k\right)\right\}.
\end{aligned}$$

Hence in view of (2.4), we obtain the following theorem:

Theorem 4.6. *Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold (B, g') such that ξ is horizontal. Then*

$$\begin{aligned}
&\frac{c+3}{4}\{nr + n + r - 2\} + \frac{c-1}{4}\Big\{2r - 4 - (n-2)\eta(X_1)^2 \\
&\quad + 3\|CX_1\|^2\Big\} \leq \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4}r^2\|H\|^2 \\
&\quad + 3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}\left(A^\alpha_{1s}\right)^2 - \delta(N) + \left\|T^V\right\|^2 - \left\|A^H\right\|^2 .
\end{aligned}$$

The equality case of the inequality holds if and only if

$$T^s_{11} = T^s_{22} + \dots + T^s_{rr},$$
$$T_{1j} = 0, \quad j = 2, \dots, r.$$

References

[1] P. Alegre, A. Carriazo, Y.H. Kim, D.W. Yoon, B. Y. Chen's inequality for submanifolds of generalized space forms, Indian J. Pure Appl. Math. 38 (3) (2007) 185–201.
[2] P. Alegre, B.-Y. Chen, M.I. Munteanu, Riemannian submersions, δ-invariants, and optimal inequality, Ann. Glob. Anal. Geom. 42 (3) (2012) 317–331.
[3] M.E. Aydın, A. Mihai, I. Mihai, Some inequalities on submanifolds in statistical manifolds of constant curvature, Filomat 29 (3) (2015) 465–477.
[4] H. Aytimur, C. Özgür, Inequalities for submanifolds in statistical manifolds of quasi-constant curvature, Ann. Pol. Math. 121 (3) (2018) 197–215.
[5] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, second edition, Progress in Mathematics, vol. 203, Birkhäuser Boston, Inc., Boston, MA, 2010.
[6] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math. (Basel) 60 (6) (1993) 568–578.
[7] B.-Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, Glasg. Math. J. 41 (1) (1999) 33–41.
[8] B.-Y. Chen, Riemannian submersions, minimal immersions and cohomology class, Proc. Jpn. Acad., Ser. A, Math. Sci. 81 (10) (2005) 162–167.
[9] B.-Y. Chen, Examples and classification of Riemannian submersions satisfying a basic equality, Bull. Aust. Math. Soc. 72 (3) (2005) 391–402.
[10] B.-Y. Chen, Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[11] M. Falciteli, S. Ianus, A.M. Pastore, Riemannian Submersions and Related Topics, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
[12] M. Gülbahar, Ş. Eken Meriç, E. Kiliç, Sharp inequalities involving the Ricci curvature for Riemannian submersions, Kragujev. J. Math. 41 (2) (2017) 279–293.
[13] İ. Küpeli Erken, C. Murathan, Anti-invariant Riemannian submersions from Sasakian manifolds, arXiv:1302.4906, 2013.
[14] J.W. Lee, Anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact manifolds, Hacet. J. Math. Stat. 42 (3) (2013) 231–241.
[15] A. Mihai, Modern Topics in Submanifold Theory, Editura Universităţii din Bucureşti, Bucharest, 2006.
[16] A. Mihai, I. Mihai, Curvature invariants for statistical submanifolds of Hessian manifolds of constant Hessian curvature, Mathematics 6 (3) (2018) 44.
[17] A. Mihai, C. Özgür, Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection, Taiwan. J. Math. 14 (4) (2010) 1465–1477.
[18] A. Mihai, C. Özgür, Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections, Rocky Mt. J. Math. 41 (5) (2011) 1653–1673.
[19] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J. 13 (1966) 459–469.
[20] B. O'Neill, Semi-Riemannian Geometry with Application to Relativity, Academic Press, New York, 1983.
[21] C. Özgür, B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature, Turk. J. Math. 35 (3) (2011) 501–509.
[22] B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, Cent. Eur. J. Math. 8 (3) (2010) 437–447.
[23] B. Şahin, Chen's first inequality for Riemannian maps, Ann. Pol. Math. 117 (3) (2016) 249–258.
[24] B. Şahin, Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications, Elsevier/Academic Press, London, 2017.