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A Note on Killing Calculus on Riemannian Manifolds

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Abstract: In this article, it has been observed that a unit Killing vector field ξ on an n-dimensional Riemannian manifold (M, g), influences its algebra of smooth functions $C^\infty(M)$. For instance, if h is an eigenfunction of the Laplace operator Δ with eigenvalue λ, then $\xi(h)$ is also eigenfunction with same eigenvalue. Additionally, it has been observed that the Hessian $\mathcal{H}^h(\xi, \xi)$ of a smooth function $h \in C^\infty(M)$ defines a self adjoint operator $\square_\xi$ and has properties similar to most of properties of the Laplace operator on a compact Riemannian manifold (M, g). We study several properties of functions associated to the unit Killing vector field ξ. Finally, we find characterizations of the odd dimensional sphere using properties of the operator $\square_\xi$ and the nontrivial solution of Fischer–Marsden differential equation, respectively.

Keywords: Killing vector field; Killing calculus; sphere; isometry

1. Introduction

A smooth vector field ξ on an n-dimensional Riemannian manifold (M, g) is said to be a Killing vector field if its flow consists of isometries of (M, g). We say that a Killing vector field ξ is nontrivial if it is not parallel. It is known that a nontrivial Killing vector field on a compact Riemannian manifold restricts its topology and geometry, for example, it does not allow the Riemannian manifold (M, g) to have negative Ricci curvature and that if (M, g) has positive sectional curvatures, then its fundamental group contains a cyclic subgroup with constant index, depending only on the dimension of M (cf. [1–3]). Riemannian manifolds with Killing vector fields has been subject of interest for many mathematicians (cf. [2,4–13]). There are other important vector fields, such as Jacobi-type vector fields, geodesic vector fields and torqued vector fields, which play important roles in the geometry of a Riemannian manifold (cf. [10,11,14–16]). Moreover, incompressible vector fields have applications in Physics, and as Killing vector fields are incompressible, they have applications in Physics (cf. [17]).

Killing vector fields are found in abundance on Euclidean spaces $\mathbf{E}^n$; for instance all constant vector fields are Killing, though they are trivial Killing vector fields. If $u^1, ..., u^n$ are Euclidean coordinates on $\mathbf{E}^n$, then

$$\xi = u^j \frac{\partial}{\partial u^i} - u^i \frac{\partial}{\partial u^j}, \, i \neq j$$

for fixed i and j, is a nontrivial Killing vector field on $\mathbf{E}^n$. Similarly, the vector field $\xi = J\Psi$ is a Killing vector field on the even dimensional Euclidean space $\mathbf{E}^{2n}$, J being the complex structure and Ψ being the position (Euler) vector field on $\mathbf{E}^{2n}$. However, all these nontrivial Killing vector fields on the Euclidean spaces are of non-constant length. A natural question

arises, whether there exists a nontrivial Killing vector field of constant length on a Euclidean space? The answer is negative.

In this paper, we exhibit several properties of a unit Killing vector field ξ in relation to algebra $C^\infty(M)$ of smooth functions on (M, g). In particular, we show that there is an operator $\Box_\xi : C^\infty(M) \to C^\infty(M)$ that has properties similar to that of the Laplace operator.

On the unit sphere $\mathbf{S}^{2n+1}$, there is a unit Killing vector field ξ provided by the Sasakian structure (cf. [18]). This naturally raises a question of finding necessary and sufficient conditions on a compact (M, g) that admits a unit Killing vector field to be isometric to $\mathbf{S}^{2n+1}$. In this paper we use the properties of the operator $\Box_\xi$ associated to the unit Killing vector field ξ on a compact (M, g) to find a characterization of the sphere $\mathbf{S}^{2n+1}(c)$. Additionally, we use properties of the nontrivial solution h of the Fischer–Marsden equation (cf. [19]) on a compact Riemannian manifold (M, g) with Killing vector field ξ and a suitable lower bound on the Ricci curvature $Ric(gradh, gradh)$ to find a characterization of the unit sphere $\mathbf{S}^{2n+1}(c)$. Note that even dimensional unit spheres $\mathbf{S}^{2n}$ do not admit unit Killing vector fields, owing to the fact that a Killing vector field on a positively curved even dimensional compact Riemannian manifold has a zero. However, other than unit sphere $\mathbf{S}^{2n+1}$, there are ellipsoids admitting unit Killing vector fields (cf. [5]).

2. Preliminaries

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a Killing vector field, if it satisfies

$$\pounds_\xi g = 0,$$

where $\pounds_\xi g$ is the Lie derivative with respect to ξ, or equivalently

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0, \quad X, Y \in \mathfrak{X}(M), \tag{1}$$

where ∇ is the Riemannian connection and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M.

The curvature tensor field R of (M, g) is given by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \mathfrak{X}(M) \tag{2}$$

and Ricci tensor field is

$$Ric(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i),$$

for a local orthonormal frame $\{e_1, ..., e_n\}$ on M.

The Ricci operator S is a symmetric operator associated to the Ricci tensor, defined by

$$g(SX, Y) = Ric(X, Y).$$

The trace $\mathbf{r} = TrS$ is the scalar curvature of M. Note that $grad\mathbf{r}$, the gradient of the scalar curvature $\mathbf{r}$, satisfies

$$\frac{1}{2} grad\mathbf{r} = \sum_{i=1}^{n} (\nabla S)(e_i, e_i), \tag{3}$$

where ∇S is given by

$$(\nabla S)(X, Y) = \nabla_X SY - S\nabla_X Y.$$

We denote by η, 1-form dual to the Killing vector field ξ on (M, g) and define an operator $\mathbf{F}$ on (M, g) by

$$2g(\mathbf{F}X, Y) = d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{4}$$

We use bold faced letters for scalar curvature and specific vectors on a Euclidean space and some specific tensors. The operator $\mathbf{F}$ is skew-symmetric and using

$$d\eta(X, Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)$$

together with Equations (1) and (4), we conclude

$$\nabla_X \xi = \mathbf{F} X, \quad X \in \mathfrak{X}(M). \tag{5}$$

If the length of the Killing vector field ξ is a constant, then on taking the inner product with ξ in above Equation (5), we conclude $g(\nabla_X \xi, \xi) = g(\mathbf{F} X, \xi) = 0$, and as $\mathbf{F}$ is skew-symmetric operator, we get

$$\mathbf{F}(\xi) = 0. \tag{6}$$

Additionally, using Equation (5), we have

$$(\nabla \mathbf{F})(X, Y) - (\nabla \mathbf{F})(Y, X) = R(X, Y)\xi, \quad X, Y \in \mathfrak{X}(M), \tag{7}$$

where $(\nabla \mathbf{F})(X, Y) = \nabla_X \mathbf{F} Y - \mathbf{F}(\nabla_X Y)$. Using the fact that the 2-form $d\eta$ is closed, $\mathbf{F}$ is skew-symmetric and, from Equation (7), we conclude

$$R(X, \xi)Y = (\nabla \mathbf{F})(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{8}$$

We denote by $C^\infty(M)$ the algebra of smooth functions on the Riemannian manifold (M, g) and for a $h \in C^\infty(M)$, we denote its gradient by $grad h$. Then the Hessian operator A^h of h is defined by

$$A^h(X) = \nabla_X grad h, \quad X \in \mathfrak{X}(M) \tag{9}$$

and it is a symmetric operator. Moreover the Hessian $\mathcal{H}^h$ of h is defined by

$$\mathcal{H}^h(X, Y) = g\left(A^h(X), Y\right), \quad X, Y \in \mathfrak{X}(M). \tag{10}$$

The Laplace operator $\Delta : C^\infty(M) \to C^\infty(M)$ on a Riemannian manifold (M, g) is defined by $\Delta h = div(grad h)$ and we also have

$$\Delta h = Tr A^h. \tag{11}$$

If M is compact and $h \in C^\infty(M)$ is, such that

$$\int_M h = 0,$$

then by minimum principle, we have

$$\int_M \|grad h\|^2 \geq \lambda_1 \int_M h^2 \tag{12}$$

λ_1 being first nonzero eigenvalue of Δ.

3. Killing Calculus

Let ξ be a unit Killing vector field on an n-dimensional Riemannian manifold (M, g). For each $h \in C^\infty(M)$, we define $\overline{h} = \xi(h)$, $\overline{\overline{h}} = \xi\left(\overline{h}\right)$. We are interested in studying the properties of these functions $\overline{h}$, $\overline{\overline{h}}$. From Equation (5), it follows that if ξ is a nontrivial Killing vector field, then the skew-symmetric operator $\mathbf{F}$ is non-vanishing. If the Euclidean space $\mathbf{E}^n$ admits a Killing vector field ξ that has constant length, then Equation (8) implies

$$(\nabla \mathbf{F})(X, Y) = 0, \quad X, Y \in \mathfrak{X}(\mathbf{E}^n).$$

Choosing $Y = \xi$ in above equation and using Equation (6), we get $-\mathbf{F}^2 X = 0$, that is, $\|\mathbf{F}\|^2 = 0$, where

$$\|\mathbf{F}\|^2 = \sum_{i=1}^{n} g(\mathbf{F}e_i, \mathbf{F}e_i),$$

$\{e_1, ..., e_n\}$ being an orthonormal frame on the Euclidean space $\mathbf{E}^n$. Thus, we have $\mathbf{F} = 0$ and the Killing vector field ξ is trivial. Thus, we have the following:

Proposition 1. *There does not exist a nontrivial Killing vector field of constant length on the Euclidean space* $\mathbf{E}^n$.

Now, suppose ξ is a Killing vector field on a Riemannian manifold (M, g). Then as **F** is skew-symmetric, using Equation (5), we have $div\xi = 0$, and for each $h \in C^\infty(M)$, we have $div(h\xi) = \bar{h}$. Thus, we get

Lemma 1. *Let ξ be a Killing vector field on a compact Riemannian manifold (M, g). Then for each* $h \in C^\infty(M)$

$$\int_M \bar{h} = 0.$$

As $\bar{h} = \xi(h)$, we find $X\left(\bar{h}\right) = Xg(\xi, gradh) = g(\mathbf{F}X, gradh) + g\left(\xi, A^h(X)\right), X \in \mathfrak{X}(M)$ and get the following expression

$$grad\bar{h} = -\mathbf{F}(gradh) + A^h(\xi). \tag{13}$$

Lemma 2. *Let ξ be a Killing vector field on a Riemannian manifold (M, g). Then for each* $h \in C^\infty(M)$, $\Delta\bar{h} = \xi(\Delta h)$.

Proof. Using Equation (9), we have

$$R(X, Y)gradh = \left(\nabla A^h\right)(X, Y) - \left(\nabla A^h\right)(Y, X), \quad X, Y \in \mathfrak{X}(M),$$

and using a local orthonormal frame $\{e_1, ..., e_n\}$ on M, $n = dimM$ in above equation, we conclude

$$Ric(Y, gradh) = g\left(Y, \sum_{i=1}^{n}\left(\nabla A^h\right)(e_i, e_i)\right) - Y(\Delta h),$$

where we have used the symmetry of the Hessian operator A^h and $TrA^h = \Delta h$. Thus, above equation implies

$$S(gradh) = -grad\Delta h + \sum_{i=1}^{n}\left(\nabla A^h\right)(e_i, e_i). \tag{14}$$

Additionally, note that

$$Ric(\xi, X) = \sum_{i=1}^{n} R(e_i, \xi; X, e_i) = -\sum_{i=1}^{n} R(e_i, \xi; e_i, X), \quad X \in \mathfrak{X}(M),$$

which implies

$$S(\xi) = -\sum_{i=1}^{n} R(e_i, \xi)e_i.$$

Using the above equation with Equation (8), we conclude

$$\sum_{i=1}^{n} (\nabla\mathbf{F})(e_i, e_i) = -S(\xi). \tag{15}$$

Now, using Equations (9) and (15), we get

$$div(\mathbf{F}(gradh)) = -\sum_{i=1}^{n} g\left(A^h e_i, \mathbf{F}e_i\right) + Ric(gradh, \xi) = Ric(gradh, \xi), \qquad (16)$$

where the first term is zero, owing to the symmetry of A^h and the skew-symmetry of $\mathbf{F}$. Similarly, using Equations (5) and (14), we compute

$$
\begin{aligned}
div A^h(\xi) &= \sum_{i=1}^{n} g\left(A^h e_i, \mathbf{F}e_i\right) + g\left(\xi, \sum_{i=1}^{n}\left(\nabla A^h\right)(e_i, e_i)\right) \\
&= Ric(gradh, \xi) + \xi(\Delta h).
\end{aligned}
\qquad (17)
$$

Thus, using Equations (13), (16) and (17), we get $\Delta \bar{h} = \xi(\Delta h)$. □

Lemma 3. *Let ξ be a Killing vector field on a compact Riemannian manifold (M, g). Then for each $h \in C^\infty(M)$,*

$$\int_M \bar{\bar{h}} \Delta h = \int_M \left\|grad\bar{h}\right\|^2.$$

Proof. On using above Lemma, we have $\bar{\bar{h}} \Delta h = \xi\left(\bar{h}\right)\Delta h = \xi\left(\bar{h}\Delta h\right) - \bar{h}\xi(\Delta h) = \xi\left(\bar{h}\Delta h\right) - \bar{h}\Delta\bar{h}$. Now, using

$$\frac{1}{2}\Delta\bar{h}^2 = \bar{h}\Delta\bar{h} + \left\|grad\bar{h}\right\|^2,$$

we get

$$\bar{\bar{h}} \Delta h = \xi\left(\bar{h}\Delta h\right) + \left\|grad\bar{h}\right\|^2 - \frac{1}{2}\Delta\bar{h}^2.$$

Integrating the above equation while using Lemma 1, we get the desired result. □

It is interesting, as the following Lemma suggests, to note that for each $h \in C^\infty(M)$ on a compact (M, g), functions h and $\bar{h}$ are orthogonal functions.

Lemma 4. *Let ξ be a Killing vector field on a compact Riemannian manifold (M, g). Then for each $h \in C^\infty(M)$*

$$(i)\ \int_M h\bar{h} = 0, \quad (ii)\ \int_M h\bar{\bar{h}} = -\int_M \bar{h}^2.$$

Proof. Note that $h\bar{h} = h\xi(h) = \frac{1}{2}\xi(h^2)$. Integrating this equation and using Lemma 1, we get (i). Additionally, we have

$$h\bar{\bar{h}} = h\xi\left(\bar{h}\right) = \xi\left(h\bar{h}\right) - \bar{h}\xi(h) = \xi\left(h\bar{h}\right) - \bar{h}^2.$$

Integrating the above equation and using Lemma 1, we get (ii). □

Proposition 2. *Let ξ be a unit Killing vector field on a compact Riemannian manifold (M, g). Then for each $h \in C^\infty(M)$, the volume $V(M)$ of M satisfies*

$$V(M) \leq \int_M \left(\left\|grad\bar{h} - \xi\right\|^2 - \lambda_1 \bar{h}^2\right),$$

where λ_1 is the first nonzero eigenvalue of the Laplace operator Δ and the equality holds for h satisfying $\Delta h = -\lambda_1 h$.

Proof. Using Lemma 1, and Equation (12), for any $h \in C^\infty(M)$, we get

$$\int_M \left\|grad\bar{h}\right\|^2 \geq \lambda_1 \int_M \bar{h}^2. \qquad (18)$$

We have

$$\left\| grad\bar{h} - \xi \right\|^2 = \left\| grad\bar{h} \right\|^2 + 1 - 2\xi\left(\bar{h}\right).$$

Integrating the above equation and using Lemma 1 and Equation (18), we get the result. Moreover, if $\Delta h = -\lambda_1 h$, then by Lemma 2, we have $\Delta \bar{h} = -\lambda_1 \bar{h}$ and that the equality in inequality (18) holds and consequently, the equality holds in the statement. $\square$

Next, given a unit Killing vector field ξ on a compact Riemannian manifold (M, g), we define an operator $\square_\xi : C^\infty(M) \to C^\infty(M)$ by $\square_\xi(h) = \mathcal{H}^h(\xi, \xi)$, where $\mathcal{H}^h$ is the Hessian of the function h. We shall show that this operator $\square_\xi$ is self adjoint operator with respect to the inner product

$$(f, h) = \int_M fh, \quad f, h \in C^\infty(M).$$

Proposition 3. *Let ξ be a unit Killing vector field on a compact Riemannian manifold (M, g). Then the operator $\square_\xi$ is a self adjoint operator on $C^\infty(M)$. Consequently,*

$$\int_M \square_\xi h = 0, \quad h \in C^\infty(M).$$

Proof. For $f, h \in C^\infty(M)$, we have

$$(\square_\xi f, h) = \int_M (\square_\xi f)h = \int_M h\mathcal{H}^f(\xi, \xi). \tag{19}$$

In view of Equations (5) and (6), we have $\nabla_\xi \xi = 0$, and, therefore,

$$\mathcal{H}^f(\xi, \xi) = \xi\xi(f).$$

Thus,

$$\begin{aligned}
h\mathcal{H}^f(\xi, \xi) &= h\xi\xi(f) = \xi(h\xi(f)) - \xi(h)\xi(f) \\
&= \xi(h\xi(f)) - \xi(f\xi(h)) + f\xi\xi h \\
&= \xi(h\xi(f)) - \xi(f\xi(h)) + f\mathcal{H}^h(\xi, \xi).
\end{aligned}$$

Integrating the above equation and using Lemma 1 and Equation (19), we conclude

$$(\square_\xi f, h) = \int_M h\mathcal{H}^f(\xi, \xi) = (f, \square_\xi h).$$

Hence, the operator $\square_\xi$ is self adjoint operator on $C^\infty(M)$. Note for a constant c, we have $\square_\xi c = \mathcal{H}^c(\xi, \xi) = 0$ and, therefore,

$$\int_M \square_\xi h = \int_M h \square_\xi 1 = 0.$$

$\square$

Note that the Laplace operator satisfies $\frac{1}{2}\Delta h^2 = h\Delta h + \|grad h\|^2$, $h \in C^\infty(M)$ and we will show that the operator $\square_\xi$ has a similar property. Indeed, we have

$$\mathcal{H}^{\frac{1}{2}h^2}(\xi, \xi) = \xi\xi\left(\frac{1}{2}h^2\right) = \xi(h\xi(h)) = \xi(h)^2 + h\mathcal{H}^h(\xi, \xi),$$

that is, the operator $\square_\xi$ satisfies

$$\frac{1}{2}\square_\xi h^2 = h \square_\xi h + \bar{h}^2. \tag{20}$$

Using Stokes's Theorem, we know that, on a compact (M, g), $\Delta h = 0$ implies h is a constant. We have a similar result for the operator $\square_\xi$ as a consequence of Proposition 3, as seen in the following result.

Corollary 1. *Let ξ be a unit Killing vector field on a compact Riemannian manifold (M, g). Then $\square_\xi h = 0$, if and only if, h is a constant on the integral curves of ξ.*

Proof. Let $h \in C^\infty(M)$ be such that $\square_\xi h = 0$. Then, Equation (20) implies

$$\frac{1}{2} \square_\xi h^2 = \overline{h}^2.$$

Integrating the above equation and using Proposition 3, we conclude

$$\int_M \overline{h}^2 = 0.$$

Thus, $\overline{h} = 0$, that is, h is a constant on the integral curves of ξ. The converse is obvious. $\square$

Recall that the unit sphere $\mathbf{S}^{2n+1}$ possesses a unit Killing vector field ξ provided by the Sasakian structure (cf. [18]). Additionally, there is a $h \in C^\infty(\mathbf{S}^{2n+1})$ satisfying

$$\nabla_X grad\, h = -hX, \quad X \in \mathfrak{X}\left(\mathbf{S}^{2n+1}\right),$$

that is, $A^h(\xi) = -h\xi$. Note that h is the eigenfunction of the Laplace operator Δ corresponding to the first nonzero eigenvalue $2n + 1$ and, also, we see that $\mathcal{H}^h(\xi, \xi) = -h$. Thus, $\square_\xi h = -h$, that is, h is an eigenfunction of the operator $\square_\xi$ corresponding to eigenvalue 1.

Let $h \in C^\infty(M)$ be a non-constant function, satisfying $\square_\xi h = \lambda h$ for a nonzero constant λ and M be compact. Then in view of Proposition 3, Equation (20) implies

$$\int_M \left(\lambda h^2 + \overline{h}^2 \right) = 0.$$

As the constant λ is nonzero and h is non-constant function, the above equation proves $\lambda < 0$. Hence, if h is a non-constant eigenfunction, we have $\square_\xi h = -\mu h$ for $\mu > 0$ and we say μ is the eigenvalue of the operator $\square_\xi$ and conclude that nonzero eigenvalues of the operator $\square_\xi$ are positive.

Recall that, owing to Lemma 1, on a compact Riemannian manifold (M, g) that admits a unit Killing vector field ξ, we have the Poisson equation $\Delta u = \overline{h}$ and this is known to have unique solution up to a constant. Additionally, we consider an analogue of the Poisson equation involving the operator $\square_\xi$, the differential equation of the form

$$\square_\xi u = \overline{h},$$

for a $h \in C^\infty(M)$. We have the following result:

Proposition 4. *Let ξ be a unit Killing vector field on a compact Riemannian manifold (M, g). Then $h \in C^\infty(M)$ is a solution of the differential equation $\square_\xi u = \overline{h}$, if and only if, h is a constant on the integral curves of ξ.*

Proof. Suppose h is a solution of $\square_\xi u = \overline{h}$. Then, using Equations (5) and (6), we have $\nabla_\xi \xi = 0$ and we get $\mathcal{H}^h(\xi, \xi) = \xi\xi(h) = \overline{\overline{h}}$. Thus, using $\square_\xi h = \overline{h}$, we get $\overline{\overline{h}} = \overline{h}$, that is,

$$h\overline{\overline{h}} = h\overline{h}.$$

Integrating the above equation and using (i) of Lemma 4, we get

$$\int_M h\overline{\overline{h}} = 0,$$

which in view of (ii) of Lemma 4, we conclude

$$\int_M \overline{h}^2 = 0.$$

Hence, $\overline{h} = 0$ and h is a constant on integral curves of ξ. The converse is trivial and it follows from Corollary 1. $\square$

4. Characterizations of Odd Dimensional Spheres

In this section, we use the Killing calculus developed in the previous section to find a characterization of the odd dimensional sphere $\mathbf{S}^{2m+1}(c)$. We prove the following:

Theorem 1. *Let ξ be a unit Killing vector field on an n-dimensional compact Riemannian manifold (M, g), $h \in C^\infty(M)$ be such that $\overline{h}$ is a non-constant function, and λ_1 be the first nonzero eigenvalue of the Laplace operator Δ. Then $\Box_\xi \Delta h = -nc\overline{\overline{h}}$, for a constant $c > 0$ and the Ricci curvature in the direction of the vector field $grad\overline{h}$, is bounded below by $c\left(\frac{nc}{\lambda_1} + n - 2\right)$, if and only if, n is odd ($n = 2m + 1$) and (M, g) is isometric to the sphere $\mathbf{S}^{2m+1}(c)$.*

Proof. Suppose $h \in C^\infty(M)$ is such that $\overline{h}$ is a non-constant function, satisfying

$$\Box_\xi \Delta h = -nc\overline{\overline{h}} \tag{21}$$

for a positive constant c, and the Ricci curvature satisfies

$$Ric\left(grad\overline{h}, grad\overline{h}\right) \geq c\left(\frac{nc}{\lambda_1} + n - 2\right)\left\|grad\overline{h}\right\|^2. \tag{22}$$

Note that, using Lemma 2, we have $\left(\Delta\overline{h}\right)^2 = (\xi(\Delta h))^2 = \left(\overline{\Delta h}\right)^2$, and combining it with Equation (20), we have

$$\left(\overline{\Delta h}\right)^2 = \frac{1}{2}\Box_\xi (\Delta h)^2 - \Delta h \Box_\xi \Delta h.$$

Integrating the above equation and using Equation (21), we get

$$\int_M \left(\overline{\Delta h}\right)^2 = nc \int_M \overline{\overline{h}}\Delta h$$

and the above equation in view of Lemma 3 implies

$$\int_M \left(\overline{\Delta h}\right)^2 = nc \int_M \left\|grad\overline{h}\right\|^2. \tag{23}$$

Using Bochner's formula (cf. [20]), we have

$$\int_M \left(Ric\left(grad\overline{h}, grad\overline{h}\right) + \left\|A^{\overline{h}}\right\|^2 - \left(\Delta\overline{h}\right)^2\right) = 0. \tag{24}$$

Additionally, we have

$$\left\|A^{\overline{h}} + c\overline{h}I\right\|^2 = \left\|A^{\overline{h}}\right\|^2 + nc^2\overline{h}^2 + 2c\overline{h}\Delta\overline{h}$$

and integrating the above equation while using Equation (24), we get

$$\int_M \left\| A^{\bar{h}} + c\bar{h}I \right\|^2 = \int_M \left(\left(\Delta\bar{h} \right)^2 - Ric\left(grad\bar{h}, grad\bar{h} \right) + nc^2\bar{h}^2 + 2c\bar{h}\Delta\bar{h} \right).$$

Using Lemma 1 and inequality (12), in the above equation, we get

$$\int_M \left\| A^{\bar{h}} + c\bar{h}I \right\|^2 \leq \int_M \left(\left(\Delta\bar{h} \right)^2 - Ric\left(grad\bar{h}, grad\bar{h} \right) + \frac{nc^2}{\lambda_1}\left\| grad\bar{h} \right\|^2 + 2c\bar{h}\Delta\bar{h} \right)$$

and using $\frac{1}{2}\Delta\bar{h}^2 = \bar{h}\Delta\bar{h} + \left\| grad\bar{h} \right\|^2$ in last term of above inequality, we conclude

$$\int_M \left\| A^{\bar{h}} + c\bar{h}I \right\|^2 \leq \int_M \left(\left(\Delta\bar{h} \right)^2 - Ric\left(grad\bar{h}, grad\bar{h} \right) + \frac{nc^2}{\lambda_1}\left\| grad\bar{h} \right\|^2 - 2c\left\| grad\bar{h} \right\|^2 \right).$$

Now, using Equation (23) in the above inequality, we arrive at

$$\int_M \left\| A^{\bar{h}} + c\bar{h}I \right\|^2 \leq \int_M \left(c\left(\frac{nc}{\lambda_1} + n - 2 \right)\left\| grad\bar{h} \right\|^2 - Ric\left(grad\bar{h}, grad\bar{h} \right) \right).$$

Finally, inequality (22) and the above inequality implies $A^{\bar{h}} + c\bar{h}I = 0$ and we have

$$\nabla_X grad\bar{h} = -c\bar{h}X, \quad X \in \mathfrak{X}(M).$$

Thus, the function $\bar{h}$ satisfies the Obata's differential equation (cf. [21,22]) and, therefore, (M, g) is isometric to $\mathbf{S}^n(c)$. However, if n is even, it is known that on an even dimensional Riemannian manifold of positive sectional curvature, a Killing vector field has a zero (cf. [12]) and we get a contradiction to the fact that ξ is a unit Killing vector field. Hence, n must be odd, and $2m + 1$ and (M, g) are isometric to the sphere $\mathbf{S}^{2m+1}(c)$.

Conversely, suppose (M, g) is isometric to $\mathbf{S}^{2m+1}(c)$. Treating $\mathbf{S}^{2m+1}(c)$ as a hypersurface of the complex space $C^{(m+1)}$ with unit normal vector field ς and shape operator $A = -\sqrt{c}I$. Using complex structure J on $C^{(m+1)}$, we get unit vector field $\xi = -J\varsigma$ on $\mathbf{S}^{2m+1}(c)$. Denote the Euclidean connection on $C^{(m+1)}$ by D and the Hermitian Euclidean metric by $\langle,\rangle$, we have

$$D_X\xi = \sqrt{c}JX, \quad X \in \mathfrak{X}(\mathbf{S}^{2m+1}(c)).$$

Denoting the induced Riemannian connection on $\mathbf{S}^{2m+1}(c)$ by ∇ and defining $\mathbf{F}X = (JX)^T$, the tangential component of JX, in the above equation gives

$$\nabla_X\xi - \sqrt{c}g(X, \xi)\varsigma = \sqrt{c}\mathbf{F}X + \sqrt{c}(JX)^\perp,$$

where $(JX)^\perp$ is the normal component of JX and g is the induced metric on $\mathbf{S}^{2m+1}(c)$. Equating tangential components, we have

$$\nabla_X\xi = \sqrt{c}\mathbf{F}X, \quad X \in \mathfrak{X}(M),$$

and since, by definition of $\mathbf{F}$, it is skew-symmetric and we conclude that ξ is a unit Killing vector field on $\mathbf{S}^{2m+1}(c)$ and that

$$\mathbf{F}(\xi) = 0. \tag{25}$$

Now, choose a nonzero constant vector field $\mathbf{u}$ on the complex space $C^{(m+1)}$ and define smooth function h on $\mathbf{S}^{2m+1}(c)$ by $h = \langle\mathbf{u}, \varsigma\rangle$ and define a vector field $\mathbf{v}$ on $\mathbf{S}^{2m+1}(c)$ by $\mathbf{v} = \mathbf{u}^T$, the tangential component of $\mathbf{u}$ to $\mathbf{S}^{2m+1}(c)$. Then, we have $\mathbf{u} = \mathbf{v} + h\varsigma$. Differentiating this equation with respect to $X \in \mathfrak{X}(\mathbf{S}^{2m+1}(c))$, we get

$$0 = \nabla_X\mathbf{v} - \sqrt{c}g(X, \mathbf{v})\varsigma + X(h)\varsigma + \sqrt{c}hX$$

and we conclude

$$\nabla_X \mathbf{v} = -\sqrt{c}hX, \quad X(h) = \sqrt{c}g(X, \mathbf{v}), \quad X \in \mathfrak{X}(\mathbf{S}^{2m+1}(c)),$$

that is, $\mathbf{v} = \frac{1}{\sqrt{c}} grad h$ and

$$\nabla_X grad h = -chX, \quad X \in \mathfrak{X}(\mathbf{S}^{2m+1}(c)). \tag{26}$$

We claim that h is not a constant, if h is a constant, Equation (26) implies $h = 0$ and then $\mathbf{v} = 0$, which means constant vector field $\mathbf{u} = 0$ on $\mathbf{S}^{2m+1}(c)$. However, $\mathbf{u}$ being a constant vector field, it will be zero on $C^{(m+1)}$, a contradiction to our assumption that $\mathbf{u} \neq 0$. Hence, h is a non-constant function. Additionally, we have $\bar{h} = \xi(h) = g(\xi, grad h) = \sqrt{c}g(\xi, \mathbf{v})$, and it implies

$$X\left(\bar{h}\right) = cg(\mathbf{F}X, \mathbf{v}) - chg(\xi, X),$$

that is,

$$grad\bar{h} = -c\mathbf{F}\mathbf{v} - ch\xi.$$

If $\bar{h}$ is a constant, it will imply $\mathbf{F}\mathbf{v} = -h\xi$, that is, $h = g(\mathbf{v}, \mathbf{F}\xi) = 0$ (in view of Equation (25)) and is a contradiction as h is non-constant. Hence, $\bar{h}$ is non-constant. Additionally, Equation (26) implies $\Delta h = -(2m+1)ch$ and, therefore, we have

$$\Box_\xi \Delta h = -(2m+1)c \Box_\xi h = -(2m+1)c\mathcal{H}^h(\xi, \xi) = -(2m+1)c\bar{\bar{h}}.$$

Using the expression for the Ricci curvature of the sphere $\mathbf{S}^{2m+1}(c)$, we have

$$Ric\left(grad\bar{h}, grad\bar{h}\right) = 2mc\left\|grad\bar{h}\right\|^2.$$

Moreover, the first nonzero eigenvalue λ_1 of the sphere $\mathbf{S}^{2m+1}(c)$ is $\lambda_1 = (2m+1)c$ and, therefore, with $n = 2m+1$, we have

$$\left(\frac{nc}{\lambda_1} + n - 2\right) = 2m$$

and, consequently,

$$Ric\left(grad\bar{h}, grad\bar{h}\right) = c\left(\frac{nc}{\lambda_1} + n - 2\right)\left\|grad\bar{h}\right\|^2$$

and all the requirements of the statement are met. □

Recall that, Fischer and Marsden considered a differential equation

$$(\Delta h)g + hRic = \mathcal{H}^h \tag{27}$$

on a Riemannian manifold (M, g) (cf. [19]), and have shown that, if a Riemannian manifold admits a nontrivial solution of this differential equation, then its scalar curvature **r** is a constant.

Definition 1. *We call a Riemannian manifold (M, g) admitting a nontrivial solution of the differential Equation (27) a Fischer–Marsden manifold.*

Observe that, on an n-dimensional Fischer–Marsden manifold (M, g), the nontrivial solution h satisfies

$$\Delta h = -\frac{\mathbf{r}}{n-1}h. \tag{28}$$

Suppose a Fischer–Marsden manifold (M, g) admits a unit Killing vector field ξ, then using Equations (27) and (28), we observe that the nontrivial solution h of differential Equation (27), satisfies

$$\Box_\xi h = \left(Ric(\xi, \xi) - \frac{\mathbf{r}}{n-1} \right) h.$$

Using Equation (20), we conclude

$$\frac{1}{2} \Box_\xi h^2 = \left(Ric(\xi, \xi) - \frac{\mathbf{r}}{n-1} \right) h^2 + \overline{h}^2.$$

Thus, we have the following.

Corollary 2. *Let ξ be a unit Killing vector field on an n-dimensional compact Fischer–Marsden manifold (M, g) with constant Ricci curvature $Ric(\xi, \xi)$. Then*

$$Ric(\xi, \xi) \leq \frac{\mathbf{r}}{n-1}$$

and the equality holds if, and only if, the nontrivial solution h of the Fischer–Marsden equation is a constant on the integral curves of ξ.

In [19], Fischer and Marsden conjectured that a compact Fischer–Marsden manifold is an Einstein manifold. Recall that a Riemannian manifold (M, g) is said to be an Einstein manifold if $Ric = \lambda g$, where λ is a constant. In the rest of this section, we show that some additional conditions of Fischer–Marsden manifold gives additional outcomes to the Einstein manifold—namely, with additional conditions, we show that a compact Fischer–Marsden manifold is not only Einstein but also a sphere. Note that scalar curvature $\mathbf{r}$ is a constant and on a compact Fischer–Marsden manifold (M, g), Equation (28) implies

$$\int_M \|gradh\|^2 = \frac{\mathbf{r}}{n-1} \int_M h^2, \tag{29}$$

that is, $\mathbf{r} > 0$ (as h is a nontrivial solution of differential Equation (27)). On an n-dimensional compact Fischer–Marsden manifold (M, g), we put $\mathbf{r} = n(n-1)c$, where constant $c > 0$.

Theorem 2. *Let ξ be a unit Killing vector field on an n-dimensional compact Fischer–Marsden manifold (M, g) with scalar curvature $\mathbf{r} = n(n-1)c$. Then, the Ricci curvature in the direction of the vector field $gradh$ is bounded below by $(n-1)c$, if and only if, n is odd ($n = 2m + 1$) and (M, g) is isometric to the sphere $\mathbf{S}^{2m+1}(c)$.*

Proof. Let (M, g) be a compact Fischer–Marsden manifold with scalar curvature $\mathbf{r} = n(n-1)c$ and h be a nontrivial solution of the Equation (27). Now,

$$\left\| A^h + chI \right\|^2 = \left\| A^h \right\|^2 + nc^2 h^2 + 2ch\Delta h = \left\| A^h \right\|^2 + nc^2 h^2 + 2c \left(\frac{1}{2} \Delta h^2 - \|gradh\|^2 \right).$$

Integrating the above equation and using Equation (29), we conclude

$$\int_M \left\| A^h + hI \right\|^2 = \int_M \left(\left\| A^h \right\|^2 - c\|gradh\|^2 \right). \tag{30}$$

Additionally, the Bochner's formula gives

$$\int_M \left\| A^h \right\|^2 = \int_M \left((\Delta h)^2 - Ric(gradh, gradh) \right),$$

and in view of Equations (28) and (29), the above equation takes the form

$$\int_M \left\| A^h \right\|^2 = \int_M \left(nc\|gradh\|^2 - Ric(gradh, gradh) \right).$$

Using above equation in Equation (30), we conclude

$$\int_M \left\| A^h + chI \right\|^2 = \int_M \left((n-1)c\|gradh\|^2 - Ric(gradh, gradh) \right)$$

and using the bound on the Ricci curvature, $Ric(gradh, gradh) \geq (n-1)c\|gradh\|^2$, in the above equation, we get $A^h = -chI$. Thus,

$$\nabla_X gradh = -chX, \quad X \in \mathfrak{X}(M),$$

which is Obata's differential equation (cf. [21,22]). This proves that (M, g) is isometric to the sphere $\mathbf{S}^n(c)$. As seen in the proof of Theorem 1, we see that n is odd, $n = 2m + 1$ and (M, g) is isometric to $\mathbf{S}^{2m+1}(c)$.

Conversely, we have shown in the proof of Theorem 1, that there exists a unit Killing vector field ξ on the sphere $\mathbf{S}^{2m+1}(c)$ and the eigenfunction h of Δ corresponding to first nonzero eigenvalue $(2m + 1)c$. Moreover, using Equation (26), we have

$$\mathcal{H}^h = -chg,$$

that is,

$$(\Delta h)g + hRic = -(2m + 1)chg + 2mchg = -chg.$$

Hence, the Fischer–Marsden differential Equation (27) holds and consequently, $\mathbf{S}^{2m+1}(c)$ is a Fischer–Marsden manifold with Ricci curvature equal to $2mc$. Thus, all the conditions in the statement are met. □

5. Conclusions

We have seen that, given a unit Killing vector field ξ on a compact Riemannian manifold (M, g), there is associated a self adjoint operator $\square_\xi : C^\infty(M) \to C^\infty(M)$ that has similar properties to that of the Laplace operator. As an application of this operator $\square_\xi$ we get a characterization of an odd dimensional sphere (Theorem 1). There are questions related to this operator $\square_\xi$ those could be subject of future research, such as showing that eigenspaces of this operator $\square_\xi$ are finite dimensional and are mutually orthogonal with respect to different eigenvalues, as well as the relation between volume and the first nonzero eigenvalue.

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