Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (1) (2021), 199 – 215
DOI : 10.15672/hujms.683997

Research Article

Weighted variable exponent grand Lebesgue spaces and inequalities of approximation

İsmail Aydın*[1], Ramazan Akgün[2]

[1] *Sinop University, Faculty of Arts and Sciences, Department of Mathematics, Sinop, Turkey*
[2] *Balikesir University, Faculty of Arts and Sciences, Department of Mathematics, Balıkesir, Turkey*

Abstract

In this paper we discuss and investigate trigonometric approximation in weighted grand variable exponent Lebesgue spaces. We also prove the direct and inverse theorems in these spaces.

Mathematics Subject Classification (2020). 46E35, 43A15, 46E30

Keywords. weighted grand variable exponent Lebesgue, Sobolev and Lipschitz space, maximal operator, modulus of smoothness, best approximation, Jackson and inverse theorems, K-functional

1. Introduction

In 1992, T. Iwaniec and C. Sbordone [22] introduced the grand Lebesgue spaces $L^{p)}(\Omega)$, $1 < p < \infty$, on bounded sets $\Omega \subset \mathbb{R}^d$, with applications to differential equations. A generalized version $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [18]. During last years these spaces were intensively studied for various applications (see, e.g., [1, 16–18, 20, 22, 23]). The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(.)}$ appeared in literature for the first time in 1931 with an article written by Orlicz [25]. Kováčik and Rákosník [24] introduced the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^d)$ and Sobolev space $\mathcal{W}^{k,p(.)}(\mathbb{R}^d)$ in higher dimensional Euclidean spaces. There are several applications of these spaces, such as, elastic mechanics, electrorheological fluids, image restoration and nonlinear degenerated partial differential equations (see [10, 11, 14]). The spaces $L^{p(.)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ have many common properties, such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings. A crucial difference between $L^{p(.)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ is that the variable exponent Lebesgue space is not invariant under translation in general, see [13, Lemma 2.3] and [24, Example 2.9]. For more information see [10, 14]. The grand variable exponent Lebesgue space $L^{p(.),\theta}(\Omega)$ was introduced and studied by Kokilasvili and Meski [23]. In their studies they established the boundedness of maximal and Calderon operators in these spaces. The space $L^{p(.),\theta}(\Omega)$ is not reflexive, separable, rearrangement invariant and translation invariant. There are several published papers about direct and inverse theorems of approximation theory in some function spaces weighted, variable or non-weighted, see, [2–8, 12, 19, 21].

*Corresponding Author.
Email addresses: iaydin@sinop.edu.tr (İ. Aydın), rakgun@balikesir.edu.tr (R. Akgün)
Received: 03.02.2020; Accepted: 29.05.2020

In this study we obtain some inequalities involving trigonometric polynomial approximation in a certain subspace of the weighted variable exponent grand Lebesgue space $L_w^{p(.),\theta}$. Also we give some basic properties of these spaces. Finally, we prove some direct and inverse theorems of approximation in $L_w^{p(.),\theta}$.

2. Notations and preliminaries

In this section, we give some essential definitions, theorems and remarks for weighted grand variable exponent Lebesgue spaces.

Definition 2.1. Let $\mathbb{T} := [0, 2\pi]$ and let $p(.) : \mathbb{T} \longrightarrow [1, \infty)$ be a measurable 2π-periodic function such that

$$1 \leq p^- = \operatorname*{ess\,inf}_{x\in\mathbb{T}} p(x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{T}} p(x) := p^+ < \infty.$$

Assume that $p(.)$ satisfies the local log-continuity condition, i.e., there exists a constant $C > 0$ such that the inequality

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}$$

holds for all $x, y \in \mathbb{T}$ with $|x - y| \leq \frac{1}{2}$ (briefly $p(.) \in P(\mathbb{T})$). We also define a subclass

$$P_0(\mathbb{T}) = \left\{p(.) \in P(\mathbb{T}) : 1 < p^-\right\}.$$

Definition 2.2. Let $p(.) \in P(\mathbb{T})$. Variable exponent Lebesgue space $L^{p(.)} := L^{p(.)}(\mathbb{T})$ is defined as the set of all measurable, 2π-periodic functions f on $\mathbb{T}$ such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where $\varrho_{p(.)}(f) = \int_{\mathbb{T}} |f(x)|^{p(x)}\, dx$. The space $L^{p(.)}$ is a Banach space with the norm $\|.\|_{p(.)}$. Moreover, the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ whenever $p(.) = p$ is a constant function. If $p^+ < \infty$, then $f \in L^{p(.)}$ if and only if $\varrho_{p(.)}(f) < \infty$.

Definition 2.3. A Lebesgue measurable and locally integrable function $w : \mathbb{T} \longrightarrow (0, \infty)$ is called a weight function. Suppose that $p(.) \in P(\mathbb{T})$. The weighted modular is defined by

$$\varrho_{p(.),w}(f) = \int_{\mathbb{T}} |f(x)|^{p(x)}\, w(x)dx.$$

The weighted variable exponent Lebesgue space $L_w^{p(.)} := L_w^{p(.)}(\mathbb{T})$ consists of all measurable functions f on $\mathbb{T}$ for which $\|f\|_{p(.),w} = \left\|f w^{\frac{1}{p(.)}}\right\|_{p(.)} < \infty$. Also, $L_w^{p(.)}$ is a uniformly convex Banach space, thus reflexive.

Remark 2.4. Let w be a weight on $\mathbb{T}$ and $p(.) \in P(\mathbb{T})$.

(i) Relations between the modular $\varrho_{p(.),w}(.)$ and $\|.\|_{p(.),w}$ are as follows:

$$\min\left\{\varrho_{p(.),w}(f)^{\frac{1}{p^-}}, \varrho_{p(.),w}(f)^{\frac{1}{p^+}}\right\} \leq \|f\|_{p(.),w} \leq \max\left\{\varrho_{p(.),w}(f)^{\frac{1}{p^-}}, \varrho_{p(.),w}(f)^{\frac{1}{p^+}}\right\},$$

$$\min\left\{\|f\|_{p(.),w}^{p^+}, \|f\|_{p(.),w}^{p^-}\right\} \leq \varrho_{p(.),w}(f) \leq \max\left\{\|f\|_{p(.),w}^{p^+}, \|f\|_{p(.),w}^{p^-}\right\}.$$

(ii) If $0 < C \leq w$, then we have $L_w^{p(.)} \hookrightarrow L^{p(.)}$, since one gets easily that

$$C\int_{\mathbb{T}} |f(x)|^{p(x)}\, dx \leq \int_{\mathbb{T}} |f(x)|^{p(x)}\, w(x)dx$$

and $C\left\|f\right\|_{p(.)} \leq \left\|f\right\|_{p(.),w}$ (see [9]). Moreover, due to $|\mathbb{T}| < \infty$ and $1 \leq p(.)$ we have $L_w^{p(.)}(\mathbb{T}) \hookrightarrow L^{p(.)}(\mathbb{T}) \hookrightarrow L^1(\mathbb{T})$.

Definition 2.5. Let $\theta > 0$ and $p(.) \in P(\mathbb{T})$. The grand variable exponent Lebesgue space, $L^{p(.),\theta}$, is the class of all measurable functions f for which

$$\left\|f\right\|_{p(.),\theta} := \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|f\right\|_{p(.)-\varepsilon} < \infty.$$

When $p(.) = p$ is a constant function, these spaces coincide with the grand Lebesgue spaces $L^{p),\theta}(\mathbb{T})$.

Definition 2.6. Let w be a weight on $\mathbb{T}$ and $p(.) \in P(\mathbb{T})$. The weighted grand variable exponent Lebesgue spaces $L_w^{p(.),\theta} := L_w^{p(.),\theta}(\mathbb{T})$ is the class of all measurable functions f for which

$$\left\|f\right\|_{p(.),w,\theta} := \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|f\right\|_{p(.)-\varepsilon,w} < \infty.$$

Remark 2.7. Let w be a weight on $\mathbb{T}$ and $p(.) \in P(\mathbb{T})$.

(i) It is easy to see that the following continuous embeddings hold

$$L^{p(.)} \hookrightarrow L^{p(.),\theta} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^1,\ 0 < \varepsilon < p^- - 1$$

due to $|\mathbb{T}| < \infty$ (see [12, 23]).

(ii) For $f \in L_w^{p(.),\theta}(\mathbb{T})$ the norm equality $\left\|f\right\|_{p(.),w,\theta} = \left\|fw^{\frac{1}{p(.)}}\right\|_{p(.),\theta}$ is not valid in $L_w^{p(.),\theta}(\mathbb{T})$ (see [17]).

Example 2.8. Let $\alpha > 0$, $\theta = 1$, $p(.) = p$ =constant and choose a weight $w(x) = x^\alpha$. If we take $f(x) = x^\beta$ for $\beta > -\alpha - 1$, then we have $f \in L_w^{p)}(0,1)$. But, $\left(fw^{\frac{1}{p}}\right)^{p-\varepsilon}$ is not integrable in $(0,1)$ for any $0 < \varepsilon < p - 1$ and so $fw^{\frac{1}{p}} \notin L^{p)}(0,1)$ (see [16]).

Proposition 2.9 (Nesting Property)**.** *If $0 < C \leq w$, $p(.) \in \mathrm{P}(\mathbb{T})$ and $\theta_1 < \theta_2$, then we have the following continuous embeddings*

$$L_w^{p(.)} \hookrightarrow L_w^{p(.),\theta_1} \hookrightarrow L_w^{p(.),\theta_2} \hookrightarrow L_w^{p(.)-\varepsilon} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^1,\ 0 < \varepsilon < p^- - 1$$

due to $|\mathbb{T}| < \infty$ (see [12, 23]).

Remark 2.10. Let w be a weight on $\mathbb{T}$ and $p(.) \in P(\mathbb{T})$. There are several differences between $L_w^{p(.)}$ and $L_w^{p(.),\theta}$. For instance, the set of the bounded functions is not dense in $L_w^{p(.),\theta}$, and the closure of $L^\infty(\mathbb{T})$ in the norm of $L_w^{p(.),\theta}$ can be characterized by the functions f such that

$$\lim_{\varepsilon\to 0} \sup \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|f\right\|_{p(.)-\varepsilon,w} = 0$$

(see [1]). Moreover, the closure of simple functions is not dense in $L_w^{p(.),\theta}$. Also, the space $L_w^{p(.),\theta}$ is not reflexive, not separable and not rearrangement invariant. Since the closure of $L_w^{p(.)}$ in $L_w^{p(.),\theta}$ does not coincide with the latter space, that is, $L_w^{p(.)}$ is not dense in $L_w^{p(.),\theta}$, then we redefine this set in the following theorem as a subspace of $L_w^{p(.),\theta}$ (see [12, 23]).

Theorem 2.11. *Let w be a weight on $\mathbb{T}$ and $p(.) \in \mathrm{P}(\mathbb{T})$. The following statements hold:*

(i) *The space $L_w^{p(.),\theta}$ is complete.*

(ii) *The closure of $L_w^{p(.)}$ in $L_w^{p(.),\theta}$ consists of functions f, which belong to $L_w^{p(.),\theta}$, for which $\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \left\|f\right\|_{p(.)-\varepsilon,w} = 0$.*

Proof. (i) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_w^{p(.),\theta}$. Then for all $\eta>0$ there exists $N(\eta)>0$ such that, whenever $n,m>N(\eta)$ we have

$$\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n-f_m\right\|_{p(.)-\varepsilon,w}<\frac{\eta}{3} \tag{2.1}$$

for any $\varepsilon\in(0,p^- -1)$. Therefore $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_w^{p(.)-\varepsilon}$ for arbitrary $\varepsilon\in(0,p^- -1)$. Then there is an f in $L_w^{p(.)-\varepsilon}$ such that

$$\left\|f-f_n\right\|_{p(.)-\varepsilon,w}\to 0 \tag{2.2}$$

for every $\varepsilon\in(0,p^- -1)$ (note that the function f is unique for all $\varepsilon\in(0,p^- -1)$, see [23]). For $n>N(\eta)$, there is an $\varepsilon_0(n)\in(0,p^- -1)$ such that

$$\left\|f-f_n\right\|_{p(.),w,\theta}\le\varepsilon_0(n)^{\frac{\theta}{p^- - \varepsilon}}\left\|f-f_n\right\|_{p(.)-\varepsilon_0(n),w}+\frac{\eta}{3} \tag{2.3}$$

by using the definition of the supremum. Moreover, there exists $N_1\in\mathbb{N}$ such that for $m>N_1$ we have

$$\varepsilon^{\frac{\theta}{p^- - \varepsilon_0(n)}}\left\|f-f_m\right\|_{p(.)-\varepsilon_0(n),w}\le\frac{\eta}{3} \tag{2.4}$$

due to (2.2). If we combine (2.3), (2.4) and (2.1), then we get

$$\left\|f-f_n\right\|_{p(.),w,\theta}\le\varepsilon_0(n)^{\frac{\theta}{p^- - \varepsilon}}\left\|f-f_n\right\|_{p(.)-\varepsilon_0(n),w}+\frac{\eta}{3}$$

$$\begin{aligned}&\le \varepsilon_0(n)^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n-f_m\right\|_{p(.)-\varepsilon_0(n),w}+\varepsilon_0(n)^{\frac{\theta}{p^- - \varepsilon}}\left\|f-f_m\right\|_{p(.)-\varepsilon_0(n),w}+\frac{\eta}{3}\\ &\le \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta\end{aligned}$$

for $n>N(\eta)$ and $m>N_1$. This completes the proof of (i).

(ii) Denote by $\left[L_w^{p(.)}\right]_{p(.),w,\theta}$ the closure of $L_w^{p(.)}$ in $L_w^{p(.),\theta}$. For $f\in\left[L_w^{p(.)}\right]_{p(.),w,\theta}$ we can obtain that there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $L_w^{p(.)}$ such that $\left\|f-f_n\right\|_{p(.),w,\theta}\to 0$ by the definition of the closure set. Then, for fixed $\delta>0$, there exists $N=N(\delta)>0$ such that, whenever $n>N(\delta)$ we obtain

$$\left\|f-f_n\right\|_{p(.),w,\theta}<\frac{\delta}{2}. \tag{2.5}$$

It is well-known that the continuous embedding $L_w^{q(.)}(\mathbb{T})\hookrightarrow L_w^{p(.)}(\mathbb{T})$ holds if and only if $q(.)\ge p(.)$ because of $|\mathbb{T}|<\infty$ [24]. Hence we get

$$\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n\right\|_{p(.)-\varepsilon,w}\le(1+|\mathbb{T}|)\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n\right\|_{p(.),w}\to 0 \tag{2.6}$$

as $\varepsilon\to 0$. If we take $\varepsilon_0>0$ such that $0<\varepsilon<\varepsilon_0$, then we can write

$$\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n\right\|_{p(.)-\varepsilon,w}<\frac{\delta}{2}. \tag{2.7}$$

Finally, if we collect (2.5) and (2.7), then we have

$$\begin{aligned}\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f\right\|_{p(.)-\varepsilon,w}&\le \varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f-f_n\right\|_{p(.)-\varepsilon,w}+\varepsilon^{\frac{\theta}{p^- - \varepsilon}}\left\|f_n\right\|_{p(.)-\varepsilon,w}\\ &\le \left\|f-f_n\right\|_{p(.),w,\theta}+\frac{\delta}{2}\le\delta\end{aligned}$$

as $\varepsilon\to 0$. □

Definition 2.12. We denote the closure of $L_w^{p(.)}$ by $L_{0,w}^{p(.),\theta}$. For $f \in L_{0,w}^{p(.),\theta}(\mathbb{T})$ we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \|f\|_{p(.)-\varepsilon,w} = 0$$

by the last theorem (see [12]).

Proposition 2.13. *Let w be a weight on $\mathbb{T}$ and $p(.) \in \mathrm{P}(\mathbb{T})$. Then, $\left(L_w^{p(.),\theta}(\mathbb{T}), \|.\|_{p(.),w,\theta}\right)$ is a Banach function space (see [1]).*

We denote the Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_I \frac{1}{|I|} \int_I |f(t)|\, dt, \quad t \in \mathbb{T},$$

where the supremum is taken over all intervals I whose length is less than 2π.

The boundedness of the Hardy-Littlewood maximal operator M on the space $L_W^{p(.),\theta}$, $\theta > 0$, $p(.) \in P_0(\mathbb{T})$, was proved in the following theorem for power weights of the form $W(x) = |x - x_0|^\gamma$, where $x_0 \in \mathbb{T}$, $-1 < \gamma < p(x_0) - 1$.

Theorem 2.14. ([17]) *Let $p(.) \in \mathrm{P}_0(\mathbb{T})$, $x_0 \in (-\pi, \pi)$, $\theta > 0$, and $-1 < \gamma < p(x_0) - 1$. Then the operator M is bounded in $L_W^{p(.),\theta}$, i.e. for all $f \in L_W^{p(.),\theta}$ there exists a $C > 0$ such that the inequality*

$$\|Mf\|_{p(.),W,\theta} \le C \|f\|_{p(.),W,\theta}$$

holds with $W(x) = |x - x_0|^\gamma$.

In what follows, all weights W considered will be power weight of the form $W(x) = |x - x_0|^\gamma$ satisfying the hypothesis of the last theorem.

Since $W(x) = |x - x_0|^\gamma$ satisfies the $A_{p(.)}$ condition of Muckenhoupt weights, then we have the continuous embedding $L_W^{p(.),\theta} \hookrightarrow L^1(\mathbb{T})$ [8]. This means that we can consider the corresponding Fourier series of $f \in L_W^{p(.),\theta}$ given by

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f) \cos kx + b_k(f) \sin kx\right), \tag{2.8}$$

where $a_0(f) = \pi^{-1} \int_{\mathbb{T}} f(t)\, dt$ and

$$a_k(f) = \pi^{-1} \int_{\mathbb{T}} f(t) \cos kt dt, \quad b_k(f) = \pi^{-1} \int_{\mathbb{T}} f(t) \sin kt dt, \quad k = 1, 2, \ldots.$$

The n-th partial sums of the series (2.8) is defined by

$$S_n(x, f) := \sum_{k=0}^{n} A_k(f)(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} \left(a_k(f) \cos kx + b_k(f) \sin kx\right).$$

Definition 2.15. Let $W(x) = |x - x_0|^\gamma$, $\theta > 0$, $p(.) \in P_0(\mathbb{T})$, $r = 1, 2, ...$ and $f \in L_{0,W}^{p(.),\theta}$. Then the r-th modulus of smoothness $\Omega_r(f, .)_{p(.),W,\theta} : [0, \infty) \to [0, \infty)$ is defined as

$$\Omega_r(f, \delta)_{p(.),W,\theta} = \sup_{0<h\le\delta} \|\rho_h^r f\|_{p(.),W,\theta}, \ r \in \mathbb{N},$$

where

$$\rho_h^r f(x) := \frac{1}{h} \int_0^h \triangle_t^r f(x) dt,$$

$$\triangle_t^r f(x) := \sum_{s=0}^{r} (-1)^{r+s+1} b_{r,s} f(x + st), \quad t > 0,$$

and $b_{r,s}$ are binomial coefficients.

Remark 2.16. Using Theorem 2.14 we get

$$\sup_{0<h\leq\delta}\|\rho_h^r f\|_{p(.),W,\theta}\leq C\|f\|_{p(.),W,\theta}<\infty.$$

This shows that the function $\Omega_r(f,\delta)_{p(.),W,\theta}$ is well defined.

Remark 2.17. The modulus of smoothness $\Omega_r(f,\delta)_{p(.),W,\theta}$ has the following properties:

(i) $\Omega_r(f,\delta)_{p(.),W,\theta}$ is a non-negative, non-decreasing function of $\delta>0$.
(ii) $\Omega_r(f_1+f_2,.)_{p(.),W,\theta}\leq\Omega_r(f_1,.)_{p(.),W,\theta}+\Omega_r(f_2,.)_{p(.),W,\theta}$.
(iii) $\lim_{\delta\to 0}\Omega_r(f,\delta)_{p(.),W,\theta}=0$.

Definition 2.18. The best approximation error $E_n(f)_{p(.),W,\theta}$ of $f\in L_{0,W}^{p(.),\theta}$ is defined by

$$E_n(f)_{p(.),W,\theta}:=\inf\left\{\|f-T_n\|_{p(.),W,\theta}:T_n\in\Pi_n\right\}$$

where Π_n is the set of trigonometric polynomials of degree at most n.

Definition 2.19. The Sobolev space $\mathcal{W}_{p(.),W,\theta}^r$ is the class of functions $f\in L_W^{p(.),\theta}$ such that $f^{(r)}\in L_W^{p(.),\theta}$ and

$$\|f\|_{p(.),W,\theta}^r=\|f\|_{p(.),W,\theta}+\left\|f^{(r)}\right\|_{p(.),W,\theta}<\infty,$$

for $r=1,2,\ldots$. Also the space $\mathcal{W}_{p(.),W,\theta}^r$ is a Banach space with respect to $\|.\|_{p(.),W,\theta}^r$. We define

$$\mathcal{W}_{0,p(.),W,\theta}^r=\left\{f:f\in L_{0,W}^{p(.),\theta}\cap\mathcal{W}_{p(.),W,\theta}^r\right\}.$$

3. Main results

The main results of this paper are the following theorems.

Theorem 3.1. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in\mathrm{P}_0(\mathbb{T})$ and $r,n\in\mathbb{N}$. If $f\in\mathcal{W}_{0,p(.),W,\theta}^r$, then*

$$E_n(f)_{p(.),W,\theta}\leq\frac{c}{n^r}E_n\left(f^{(r)}\right)_{p(.),W,\theta}$$

with a constant $c>0$ independent of n.

Corollary 3.2. *Under the conditions of Theorem 3.1,*

$$E_n(f)_{p(.),W,\theta}\leq\frac{c}{n^r}\left\|f^{(r)}\right\|_{p(.),W,\theta}$$

with a constant $c>0$ independent of $n=0,1,2,3,\ldots$.

Theorem 3.3. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in\mathrm{P}_0(\mathbb{T})$ and $r,n\in\mathbb{N}$. If $f\in L_{0,W}^{p(.),\theta}$, then*

$$E_n(f)_{p(.),W,\theta}\leq c\Omega_r\left(f,\frac{1}{n}\right)_{p(.),W,\theta}$$

with a constant $c>0$ independent of n.

Theorem 3.4. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in\mathrm{P}_0(\mathbb{T})$ and $r,n\in\mathbb{N}$. If $f\in L_{0,W}^{p(.),\theta}$, then*

$$\Omega_r\left(f,\frac{1}{n}\right)_{p(.),W,\theta}\leq\frac{c}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p(.),W,\theta}$$

with a constant $c>0$ independent of n.

To prove main results we need some lemmas and propositions given below.

Lemma 3.5. *Let* $W(x) = |x - x_0|^{\gamma}$*,* $\theta > 0$*,* $p(.) \in \mathrm{P}_0(\mathbb{T})$ *and* $r \in \mathbb{N}$*. If* $f \in \mathcal{W}^r_{0,p(.),W,\theta}$*, then*

$$\Omega_r(f,\delta)_{p(.),W,\theta} \le c\delta^r \left\| f^{(r)} \right\|_{p(.),W,\theta}$$

with a constant $c > 0$ *independent of* n*.*

Proof. Since

$$\triangle_t^r f(.) = \int_0^t \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r)\, dt_1 \dots dt_r,$$

applying (r times) the generalized Minkowski's inequality we get

$$\left\| \frac{1}{h} \int_0^h \triangle_t^r f dt \right\|_{p(.),W,\theta} \le \frac{c_1(p)}{h} \int_0^h \|\triangle_t^r f\|_{p(.),W,\theta}\, dt$$

$$\begin{aligned}
&\le h^r \frac{c_1(p)}{h^{r+1}} \int_0^h \left\| \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r)\, dt_1 \dots dt_r \right\|_{p(.),W,\theta} dt \\
&= h^r \frac{c_1(p)}{h} \int_0^h \left\| \frac{1}{h} \int_0^t \left| \frac{1}{h^{r-1}} \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r)\, dt_1 \dots dt_{r-1} \right| dt_r \right\|_{p(.),W,\theta} dt \\
&\le h^r \frac{c_2(p)}{h} \int_0^h \left\| \frac{1}{h^{r-1}} \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_{r-1})\, dt_1 \dots dt_{r-1} \right\|_{p(.),W,\theta} dt \\
&\le \dots \le h^r \frac{c_3(p,r)}{h} \int_0^h \left\| \left\{ \frac{1}{h} \int_0^h f^{(r)}(. + t_1)\, dt_1 \right\} \right\|_{p(.),W,\theta} dt \\
&\le c_4(p,r) h^r \left\| f^{(r)} \right\|_{p(.),W,\theta} \frac{1}{h} \int_0^h dt = c_4(p,r) h^r \left\| f^{(r)} \right\|_{p(.),W,\theta},
\end{aligned}$$

and taking supremum on $0 < h \le \delta$, we obtain the required inequality

$$\Omega_r(f,\delta)_{p(.),W,\theta} \le c\delta^r \left\| f^{(r)} \right\|_{p(.),W,\theta}.$$

□

Definition 3.6. Let $W(x) = |x - x_0|^{\gamma}$, $\theta > 0$, $p(.) \in P_0(\mathbb{T})$, $r \in \mathbb{N}$ and $f \in L^{p(.),\theta}_{0,W}$. We define Peetre's K-functional as

$$K_r(f,\delta)_{p(.),W,\theta} := \inf \left\{ \|f - g\|_{p(.),W,\theta} + \delta^r \left\| g^{(r)} \right\|_{p(.),W,\theta} : g \in \mathcal{W}^r_{0,p(.),W,\theta},\ \delta > 0 \right\}.$$

Theorem 3.7. *Let* $W(x) = |x - x_0|^{\gamma}$*,* $\theta > 0$*,* $p(.) \in \mathrm{P}_0(\mathbb{T})$*,* $r \in \mathbb{N}$*. If* $f \in L^{p(.),\theta}_{0,W}$*, then there are some constants* c_6*,* $c_7 > 0$ *independent of* δ *such that*

$$c_6 \Omega_r(f,\delta)_{p(.),W,\theta} \le K_r(f,\delta)_{p(.),W,\theta} \le c_7 \Omega_r(f,\delta)_{p(.),W,\theta}.$$

Proof. Let $f \in L^{p(.),\theta}_{0,W}$ and $g \in \mathcal{W}^r_{0,p(.),W,\theta}$. By Lemma 3.5 and Remark 2.17,

$$\begin{aligned}
\Omega_r(f,\delta)_{p(.),W,\theta} &\le \Omega_r(f - g,\delta)_{p(.),W,\theta} + \Omega_r(g,\delta)_{p(.),W,\theta} \\
&\le c\left(\|f - g\|_{p(.),W,\theta} + \delta^r \left\| g^{(r)} \right\|_{p(.),W,\theta} \right),
\end{aligned}$$

and taking infimum with respect to $g\in\mathcal{W}^{r}_{0,p(.),W,\theta}$ in the last inequality we have

$$\Omega_r(f,\delta)_{p(.),W,\theta}\le cK_r(f,\delta)_{p(.),W,\theta}.$$

In order to prove the reverse of the last inequality we define the function

$$f_{r,\delta}(x)=\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}(\frac{1}{h^r}\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\int\limits_0^h\ldots\int\limits_0^h f(x+\frac{r-s}{r}[t_1+\ldots+t_r])dt_1\ldots dt_r)dh \quad (3.1)$$

for $\delta>0$ and $r\ge 1$. Then, differentiating $r-1$ times and setting $t:=\frac{r-s}{r}t_r$ we see that

$$\begin{aligned}&\left\{\int\limits_0^h\ldots\int\limits_0^h f\left(x+\frac{r-s}{r}[t_1+\ldots+t_r]\right)dt_1\ldots dt_r\right\}^{(r-1)}\\
=&\left\{\int\limits_0^h\left(\frac{r}{r-s}\right)^{r-1}\sum_{m=0}^{r-1}\binom{r-1}{m}(-1)^{r+m}f(x+\frac{r-s}{r}t_r+m\frac{r-s}{r}h)dt_r\right\}\\
=&\int\limits_0^h\left(\frac{r}{r-s}\right)^{r-1}\triangle^{r-1}_{\frac{r-s}{r}h}f(x+t)\,dt,\end{aligned}$$

and then by (3.1)

$$f^{(r-1)}_{r,\delta}(x):=\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\frac{1}{h^r}\left\{\sum_{s=0}^{r-1}\int\limits_x^{x+\frac{r-s}{r}h}(-1)^{r+s+1}\binom{r}{s}\triangle^{r-1}_{\frac{r-s}{r}h}f(t)dt\right\}dh. \quad (3.2)$$

Now we prove $f^{(r)}_{r,\delta}\in L^{p(.),\theta}_{0,W}$. Differentiating the relation (3.2) we obtain

$$f^{(r)}_{r,\delta}(x):=\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\frac{1}{h^r}\left\{\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\triangle^{r}_{\frac{r-s}{r}h}f(x)\right\}dh$$

and denoting $t:=\frac{r-s}{r}h$ we have

$$\begin{aligned}\left|f^{(r)}_{r,\delta}(x)\right|&\le\frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left|\frac{1}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\triangle^{r}_{\frac{r-s}{r}h}f(x)dh\right|\\
&=\frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left|\frac{1}{\frac{r-s}{r}\delta}\int\limits_{\frac{r-s}{r}(\frac{\delta}{2})}^{\frac{r-s}{r}\delta}\triangle^{r}_{t}f(x)dt\right|\end{aligned}$$

$$\le\frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left\{\left|\frac{1}{\frac{r-s}{r}\delta}\int\limits_0^{\frac{r-s}{r}\delta}\triangle^{r}_{t}f(x)dt\right|+\left|\frac{1}{\frac{r-s}{r}\delta}\int\limits_0^{\frac{r-s}{r}(\frac{\delta}{2})}\triangle^{r}_{t}f(x)dt\right|\right\},$$

which implies the inequality

$$\left\|f^{(r)}_{r,\delta}\right\|_{p(.),W,\theta}\le 2c(r)\delta^{-r}\Omega_r(f,\delta)_{p(.),W,\theta}\le c_5(p,r)\|f\|_{p(.),W,\theta}. \quad (3.3)$$

Since $f\in L^{p(.),\theta}_{0,W}$, then $f^{(r)}_{r,\delta}\in L^{p(.),\theta}_{0,W}$.

Let $f\in L^{p(.),\theta}_{0,W}$. For $\delta>0$ and $r=1,2,...$, we have

$$|f_{r,\delta}(x)-f(x)|=\left|\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\left\{\frac{1}{h^r}\int\limits_0^h...\int\limits_0^h \triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\right\}dh\right|$$

and by the generalized Minkowski's inequality

$$\|f_{r,\delta}-f\|_{p(.),W,\theta}\leq c_6(p,r)\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\left\{\frac{1}{h^{r-1}}\int\limits_0^h...\int\limits_0^h\left\|\frac{1}{h}\int\limits_0^h \triangle^r_{\frac{t_1+...+t_r}{r}}fdt_1\right\|_{p(.),W,\theta}dt_2...dt_r\right\}dh$$

$$=c_6(p,r)\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\left\{\frac{1}{h^{r-1}}\int\limits_0^h...\int\limits_0^h\left\|\frac{1}{h}\int\limits_{t_2+...+t_r}^{h+t_2+...+t_r} \triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta}dt_2...dt_r\right\}dh. \quad (3.4)$$

Since

$$\begin{aligned}\left\|\frac{1}{h}\int\limits_{t_2+...+t_r}^{h+t_2+...+t_r}\triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta} &= \left\|\frac{1}{h}\left(\int\limits_0^{h+t_2+...+t_r}\triangle^r_{\frac{t}{r}}fdt-\int\limits_0^{t_2+...+t_r}\triangle^r_{\frac{t}{r}}fdt\right)\right\|_{p(.),W,\theta}\\ &\leq \left\|\frac{1}{(h+t_2+...+t_r)/r}\int\limits_0^{(h+t_2+...+t_r)/r}\triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta}\\ &\quad+\left\|\frac{1}{(t_2+...+t_r)/r}\int\limits_0^{(t_2+...+t_r)/r}\triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta}\end{aligned}$$

$$\begin{aligned}&=\sup_{(h+t_2+...+t_r)/r\leq\delta}\left\|\frac{1}{(h+t_2+...+t_r)/r}\int\limits_0^{(h+t_2+...+t_r)/r}\triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta}\\ &\quad+\sup_{(t_2+...+t_r)/r\leq\delta}\left\|\frac{1}{(t_2+...+t_r)/r}\int\limits_0^{(t_2+...+t_r)/r}\triangle^r_{\frac{t}{r}}fdt\right\|_{p(.),W,\theta}\\ &=\Omega_r(f,\delta)_{p(.),W,\theta}+\Omega_r(f,\delta)_{p(.),W,\theta}=2\Omega_r(f,\delta)_{p(.),W,\theta}, \end{aligned}\quad (3.5)$$

then combining (3.4) and (3.5) we have

$$\begin{aligned}\|f_{r,\delta}-f\|_{p(.),W,\theta} &\leq c(p,r)\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}\left\{\frac{1}{h^{r-1}}\int\limits_0^h...\int\limits_0^h\Omega_r(f,\delta)_{p(.),W,\theta}dt_2...dt_r\right\}dh\\ &\leq c(p,r)\Omega_r(f,\delta)_{p(.),W,\theta}\frac{2}{\delta}\int\limits_{\frac{\delta}{2}}^{\delta}dh=c(p,r)\Omega_r(f,\delta)_{p(.),W,\theta}\end{aligned}\quad (3.6)$$

Finally, if we use (3.3) and (3.6), then we get

$$\begin{aligned}K_r(f,\delta)_{p(.),W,\theta} &\leq \|f_{r,\delta}-f\|_{p(.),W,\theta}+\delta^r\left\|f^{(r)}_{r,\delta}\right\|_{p(.),W,\theta}\\ &\leq c_7\Omega_r(f,\delta)_{p(.),W,\theta}.\end{aligned}$$

This completes the proof. □

The following lemma is a Bernstein inequality for $L^{p(.),\theta}_W$.

Lemma 3.8. *Let $W(x)=|x-x_0|^{\gamma}$, $\theta>0$, $p(.)\in \mathrm{P}_0(\mathbb{T})$, $r\in\mathbb{N}$. If T_n is a trigonometric polynomial of degree at most n, then*

$$\|T_n'\|_{p(.),W,\theta}\leq cn\|T_n\|_{p(.),W,\theta}.$$

Proof. It is well-known that

$$\sup_n|\sigma_n(x,f)|\leq cMf(x)$$

with a constant $c>0$ independent of f and $x\in\mathbb{T}$, where $\sigma_n(x,f)$ is the Cesàro means for a function $f\in L_W^{p(.),\theta}$ [27]. Using Theorem 2.14 we have

$$\left\|\sup_n|\sigma_n(.,f)|\right\|_{p(.),W,\theta}\leq c\|f\|_{p(.),W,\theta}. \tag{3.7}$$

Since

$$T_n(x)=\frac{1}{\pi}\int_T T_n(t)D_n(t-x)dt, \text{ with } D_n(t)=\frac{1}{2}+\sum_{j=1}^{n}\cos jt,$$

it is well-known that

$$T_n'(x)=2n\sigma_{n-1}(x,T_n)$$

and, hence,

$$\|T_n'\|_{p(.),W,\theta}\leq 2n\|\sigma_{n-1}(.,|T_n|)\|_{p(.),W,\theta}\leq 2cn\|T_n\|_{p(.),W,\theta}.$$

This completes the proof. □

Lemma 3.8 can be generalized for r-th derivative of T_n. For this we need a Minkowski's inequality for integrals. The following results were proved, when $W\equiv 1$, by Danelia and Kokilashvili [12, Proposition 2.4]. The same proof also suits our case below.

Lemma 3.9. *Let $W(x)=|x-x_0|^{\gamma}$, $\theta>0$, $p(.)\in\mathrm{P}_0(\mathbb{T})$, and $f\in L_{0,W}^{p(.),\theta}$. If $f(x,y)$ a measurable function on $\mathbb{T}\times\mathbb{T}$, then, the following integral inequality holds*

$$\left\|\int_{\mathbb{T}}f(.,y)dy\right\|_{p(.),W,\theta}\leq C\int_{\mathbb{T}}\|f(.,y)\|_{p(.),W,\theta}dy.$$

As a corollary of the last two lemmas we get:

Corollary 3.10. *Let $W(x)=|x-x_0|^{\gamma}$, $\theta>0$, $p(.)\in\mathrm{P}_0(\mathbb{T})$ and $r\in\mathbb{N}$. If T_n is a trigonometric polynomial of degree at most n, then*

$$\left\|T_n^{(r)}\right\|_{p(.),W,\theta}\leq cn^r\|T_n\|_{p(.),W,\theta}.$$

4. Proof of main results

Let $n\in\mathbb{N}$ and

$$D_nf(x):=\frac{1}{\pi}\int_{\mathbb{T}}f(x-t)J_{2,\lfloor\frac{n}{2}\rfloor+1}(t)dt \tag{4.1}$$

be the Jackson operator (polynomial), where $\lfloor\frac{n}{2}\rfloor$ denotes the integer part of a real number $\frac{n}{2}$, and $J_{2,n}$ is the Jackson kernel

$$J_{2,n}(x):=\frac{1}{\varkappa_{2,n}}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^4,\quad \varkappa_{2,n}:=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4dt.$$

It is known that ([15, p.147])

$$\frac{3}{2\sqrt{2}}n^3\leq\varkappa_{2,n}\leq\frac{5}{2\sqrt{2}}n^3.$$

Jackson kernel $J_{2,n}$ satisfies the relations

$$\left.\begin{array}{l}\frac{1}{\pi}\int_{\mathbb{T}}J_{2,n}(u)du=1,\\ \frac{1}{\pi}\int_0^{\pi}uJ_{2,n}(u)du\leq\frac{1}{2n},\end{array}\right\} \tag{4.2}$$

Lemma 4.1. *Let* $W(x) = |x - x_0|^{\gamma}$, $\theta > 0$, $p(.) \in \mathrm{P}_0(\mathbb{T})$, *and* $f \in L_{0,W}^{p(.),\theta}$. *If* $f \in \mathcal{W}^1_{0,p(.),W,\theta}$, *then*

$$E_n(f)_{p(.),W,\theta} \leq \|f - D_n f\|_{p(.),W,\theta} \leq \frac{c}{n} \|f'\|_{p(.),W,\theta} \tag{4.3}$$

holds for $n \in \mathbb{N}$.

Proof of Lemma 4.1. From (4.1), Theorem 2.14, and (4.2), we have

$$\begin{aligned}
\|f - D_n f\|_{p(.),W,\theta} &= \left\| \frac{1}{\pi} \int_{\mathbb{T}} (f(x) - f(x-t))(1/t) t J_{2,\lfloor \frac{n}{2} \rfloor + 1}(t) dt \right\|_{p(.),W,\theta} \\
&= \left\| \frac{1}{\pi} \int_{\mathbb{T}} t J_{2,\lfloor \frac{n}{2} \rfloor + 1}(t) \frac{1}{t} \int_{x-t}^{x} f'(\tau) d\tau dt \right\|_{p(.),W,\theta} \\
&\leq \frac{1}{\pi} \int_{\mathbb{T}} t J_{2,\lfloor \frac{n}{2} \rfloor + 1}(t) \left\| \frac{1}{t} \int_{x-t}^{x} f'(\tau) d\tau \right\|_{p(.),W,\theta} dt \\
&\leq \|M f'\|_{p(.),W,\theta} \frac{1}{\pi} \int_0^{\pi} t J_{2,\lfloor \frac{n}{2} \rfloor + 1}(t) dt \\
&\leq \frac{C}{2\left(\lfloor \frac{n}{2} \rfloor + 1\right)} \|f'\|_{p(.),W,\theta} \leq \frac{c}{n} \|f'\|_{p(.),W,\theta}.
\end{aligned}$$

Hence (4.3) holds. □

Proof of Theorem 3.1. Let $f \in \mathcal{W}^1_{0,p(.),W,\theta}$, $n \in \mathbb{N}$, $\Theta_n \in \mathcal{T}_n$, $E_n(f')_{p(.),W,\theta} = \|f' - \Theta_n\|_{p(.),W,\theta}$ and $\beta/2$ be the constant term of Θ_n, namely,

$$\beta = \frac{1}{\pi} \int_{\mathbb{T}} \Theta_n(t)\, dt = \frac{1}{\pi} \int_{\mathbb{T}} (\Theta_n(t) - f'(t))\, dt.$$

Then

$$\begin{aligned}
|\beta/2| &\leq \frac{1}{2\pi} \|f' - \Theta_n\|_{L_1} \\
&\leq \frac{c}{2\pi} \|f' - \Theta_n\|_{p(.),W,\theta} = \frac{c}{2\pi} E_n(f')_{p(.),W,\theta}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|f' - (\Theta_n - \beta/2)\|_{p(.),W,\theta} &\leq E_n(f')_{p(.),W,\theta} + \|\beta/2\|_{p(.),W,\theta} \\
&\leq E_n(f')_{p(.),W,\theta} + \frac{c}{2\pi} \|W\|_{L_1} E_n(f')_{p(.),W,\theta} \\
&= \left(1 + \frac{c}{2\pi} \|W\|_{L_1}\right) E_n(f')_{p(.),W,\theta}.
\end{aligned}$$

Set $u_n \in \mathcal{T}_n$ so that $u_n' = \Theta_n - \beta/2$. Then

$$\begin{aligned}
E_n(f)_{p(.),W,\theta} &= E_n(f - u_n)_{p(.),W,\theta} \\
&\leq \frac{c}{n} \|f' - (\Theta_n - \beta/2)\|_{p(.),W,\theta} \\
&\leq \left(c + \frac{C}{2\pi} \|W\|_{L_1}\right) \frac{1}{n} E_n(f')_{p(.),W,\theta}
\end{aligned}$$

for all $f \in \mathcal{W}^1_{0,p(.),W,\theta}$. If $f \in \mathcal{W}^r_{0,p(.),W,\theta}$ for some r, the last inequality gives

$$\begin{aligned}
E_n(f)_{p(.),W,\theta} &\leq C\left(1 + \frac{c}{2\pi} \|W\|_{L_1}\right)^r \frac{1}{n^r} E_n\left(f^{(r)}\right)_{p(.),W,\theta} \\
&= \frac{c}{n^r} E_n\left(f^{(r)}\right)_{p(.),W,\theta}.
\end{aligned}$$

□

Proof of Theorem 3.3. Let $f \in L_{0,W}^{p(.),\theta}$. Using Theorem 3.1 and Corollary 3.2 we have

$$\begin{aligned} E_n(f)_{p(.),W,\theta} &\leq E_n(f-g)_{p(.),W,\theta} + E_n(g)_{p(.),W,\theta} \\ &\leq c\left\{\|f-g\|_{p(.),W,\theta} + \delta^r \left\|g^{(r)}\right\|_{p(.),W,\theta}\right\}. \end{aligned}$$

for $g \in \mathcal{W}_{0,p(.),W,\theta}^r$ and $\delta = \frac{1}{n}$. Using Theorem 3.7 and taking infimum on $g \in \mathcal{W}_{0,p(.),W,\theta}^r$, we obtain

$$E_n(f)_{p(.),W,\theta} \leq c\Omega_r\left(f, \frac{1}{n}\right)_{p(.),W,\theta}, \quad n \in \mathbb{N}.$$

□

Proof of Theorem 3.4. Let T_n be a best approximation trigonmetric polynomial for $f \in L_{0,W}^{p(.),\theta}$. For any $n \in \mathbb{N}$ we choose $n \in \mathbb{N}$ such that $2^m \leq n < 2^{m+1}$. If we use the subadditivity property of $\Omega_r(f,\delta)_{p(.),W,\theta}$, then we have

$$\Omega_r(f,\delta)_{p(.),W,\theta} \leq \Omega_r(f - T_{2^{m+1}}, \delta)_{p(.),W,\theta} + \Omega_r(T_{2^{m+1}}, \delta)_{p(.),W,\theta}. \tag{4.4}$$

On the other hand, it is well-known that

$$2^{(i+1)r} E_{2^i}(f)_{p(.),W,\theta} \leq 2^{2r} \sum_{j=2^{i-1}+1}^{2^i} j^{r-1} E_j(f)_{p(.),W,\theta} \tag{4.5}$$

by Theorem 3.1 in [26]. If we take $\delta = \frac{1}{n}$, then we get

$$\begin{aligned} \Omega_r(f - T_{2^{m+1}}, \delta)_{p(.),W,\theta} &\leq c\|f - T_{2^{m+1}}\|_{p(.),W,\theta} \\ &= cE_{2^{m+1}}(f)_{p(.),W,\theta} \\ &\leq \frac{c}{n^r} 2^{2(m+1)r} E_{2^m}(f)_{p(.),W,\theta} \\ &\leq c\delta^r 2^{2r} \sum_{k=2^{m-1}+1}^{2^m} k^{r-1} E_k(f)_{p(.),W,\theta}. \end{aligned} \tag{4.6}$$

Using Lemma 3.5, Lemma 3.8 and (4.5) one can find that

$$\begin{aligned} &\Omega_r(T_{2^{m+1}}, \delta)_{p(.),W,\theta} \\ \leq\ & c\delta^r \left\|T_{2^{m+1}}^{(r)}\right\|_{p(.),W,\theta} \\ \leq\ & c\delta^r \left\{\left\|T_1^{(r)} + \sum_{i=0}^{m}\left(T_{2^{i+1}}^{(r)} - T_{2^i}^{(r)}\right)\right\|_{p(.),W,\theta}\right\} \\ \leq\ & c\delta^r \left\{\|T_1\|_{p(.),W,\theta} + \sum_{i=0}^{m} 2^{(i+1)r}\left\|T_{2^{i+1}}^{(r)} - T_{2^i}^{(r)}\right\|_{p(.),W,\theta}\right\} \\ \leq\ & c\delta^r \left\{E_0(f)_{p(.),W,\theta} + \sum_{i=0}^{m} 2^{(i+1)r} E_{2^i}(f)_{p(.),W,\theta}\right\} \\ =\ & c\delta^r \left\{E_0(f)_{p(.),W,\theta} + 2^r E_1(f)_{p(.),W,\theta} + 2^{2r}\sum_{i=1}^{m}\sum_{k=2^{i-1}+1}^{2^i} k^{r-1}E_k(f)_{p(.),W,\theta}\right\} \\ \leq\ & c\delta^r \left\{E_0(f)_{p(.),W,\theta} + \sum_{k=1}^{2^m} k^{r-1}E_k(f)_{p(.),W,\theta}\right\}. \end{aligned} \tag{4.7}$$

If we combine (4.4), (4.6) and (4.7), then we find

$$\Omega_r\left(f,\frac{1}{n}\right)_{p(.),W,\theta} \leq \frac{c}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p(.),W,\theta}\,,\ n\in\mathbb{N}.$$

□

The notation $\mathcal{O}$ indicates that $A=\mathcal{O}(B)$ if and only if there exists a positive constant c, independent of essential parameters, such that $A\leq cB$.

Corollary 4.2. *If $E_n(f)_{p(.),W,\theta}=\mathcal{O}(n^{-\alpha})$, $\alpha>0$, then under the conditions of Theorem 3.4 we have*

$$\Omega_r(f,\delta)_{p(.),W,\theta}=\begin{cases}\mathcal{O}(\delta^\alpha) & ,\ r>\alpha,\\ \mathcal{O}\left(\delta^\alpha\log\left(\frac{1}{\delta}\right)\right) & ,\ r=\alpha,\\ \mathcal{O}(\delta^r) & ,\ r<\alpha.\end{cases}$$

Definition 4.3. Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in P_0(\mathbb{T})$, $f\in L_{0,W}^{p(.),\theta}$, $\alpha>0$ and $r:=[\alpha]+1$ ($[\alpha]$ is the integer part of α). We define the generalized Lipschitz class as

$$Lip_{p(.),W,\theta}^{\alpha,r}=\left\{f\in L_W^{p(.),\theta}:\Omega_r(f,\delta)_{p(.),W,\theta}=\mathcal{O}(\delta^\alpha)\right\}.$$

Corollary 4.4. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in \mathrm{P}_0(\mathbb{T})$, $f\in L_{0,W}^{p(.),\theta}$ and $\alpha>0$. Then the following statements are equivalent:*

(i) $f\in Lip_{p(.),W,\theta}^{\alpha,r}$

(ii) $E_n(f)_{p(.),W,\theta}=O(n^{-\alpha})$, $n\in\mathbb{N}$.

Theorem 4.5. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in \mathrm{P}_0(\mathbb{T})$, $f\in L_{0,W}^{p(.),\theta}$ and $r\in\mathbb{N}$. If*

$$\sum_{k=1}^{\infty}k^{r-1}E_k(f)_{p(.),W,\theta}<\infty,$$

then, $f\in\mathcal{W}_{p(.),0,W,\theta}^r$ and

$$E_n\left(f^{(r)}\right)_{p(.),W,\theta}\leq c\left(n^rE_n(f)_{p(.),W,\theta}+\sum_{k=n+1}^{\infty}k^{r-1}E_k(f)_{p(.),W,\theta}\right)$$

with a positive constant c independent of f and n.

Proof of Theorem 4.5. For the polynomial T_n of the best approximation to f we have by Lemma 3.8 that

$$\begin{aligned}\left\|T_{2^{i+1}}^{(r)}-T_{2^i}^{(r)}\right\|_{p(.),W,\theta} &\leq C(r)\,2^{(i+1)r}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),W,\theta}\\ &\leq 2C(r)\,2^{(i+1)r}E_{2^i}(f)_{p(.),W,\theta}\,.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),W,\theta}^r &= \sum_{i=1}^{\infty}\left\|T_{2^{i+1}}^{(r)}-T_{2^i}^{(r)}\right\|_{p(.),W,\theta}+\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),W,\theta}\\ &\leq c\sum_{m=2}^{\infty}m^{r-1}E_m(f)_{p(.),W,\theta}<\infty.\end{aligned}$$

Therefore

$$\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),W,\theta}^r\to 0 \text{ as } i\to\infty.$$

This means that $\{T_{2^i}\}$ is a Cauchy sequence in $L_W^{p(.),\theta}$. Since $T_{2^i}\to f$ in $L_W^{p(.),\theta}$ and $\mathcal{W}_{p(.),W,\theta}^r$ is a Banach space we obtain $f\in\mathcal{W}_{p(.),W,\theta}^r$.

On the other hand since

$$\left\|f^{(r)}-T_n^{(r)}\right\|_{p(.),W,\theta}\leq\left\|T_{2^{m+2}}^{(r)}-T_n^{(r)}\right\|_{p(.),W,\theta}+\sum_{k=m+2}^{\infty}\left\|T_{2^{k+1}}^{(r)}-T_{2^k}^{(r)}\right\|_{p(.),W,\theta}$$

for $2^m\leq n<2^{m+1}$, we have

$$\left\|T_{2^{m+2}}^{(r)}-T_n^{(r)}\right\|_{p(.),W,\theta}\leq c2^{(m+2)r}E_n\left(f\right)_{p(.),W,\theta}\leq c\left(n+1\right)^rE_n\left(f\right)_{p(.),W,\theta}.$$

Also we find

$$\begin{aligned}\sum_{k=m+2}^{\infty}\left\|T_{2^{k+1}}^{(r)}-T_{2^k}^{(r)}\right\|_{p(.),W,\theta} &\leq c\sum_{k=m+2}^{\infty}2^{(k+1)r}E_{2^k}\left(f\right)_{p(.),W,\theta}\\ &\leq c\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^k}\mu^{r-1}E_\mu\left(f\right)_{p(.),W,\theta}\\ &= c\sum_{\nu=2^{m+1}+1}^{\infty}\nu^{r-1}E_\nu\left(f\right)_{p(.),W,\theta}\\ &\leq c\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu\left(f\right)_{p(.),W,\theta}.\end{aligned}$$

This completes the proof. □

A polynomial $T\in\Pi_n$ is said to be a *near best approximant* of $f\in L_{0,W}^{p(.),\theta}$ for $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in P_0\left(\mathbb{T}\right)$, if

$$\left\|f-T\right\|_{p(.),W,\theta}\leq cE_n\left(f\right)_{p(.),W,\theta},\quad n=1,2,\ldots.$$

Theorem 4.6. *Let $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in \mathrm{P}_0\left(\mathbb{T}\right)$, $r,n\in\mathbb{N}$. If $T_n\in\Pi_n$ is a near best approximant of $f\in\mathcal{W}_{p(.),W,\theta}^r$, then there exists a constant $c>0$ dependent only on r,W and $p(.)$, such that*

$$\left\|f^{(r)}-T_n^{(r)}\right\|_{p(.),W,\theta}\leq cE_n\left(f^{(r)}\right)_{p(.),W,\theta}.$$

Corollary 4.7. *Suppose that $W(x)=|x-x_0|^\gamma$, $\theta>0$, $p(.)\in \mathrm{P}_0\left(\mathbb{T}\right)$, r,$n\in\mathbb{N}$, $f\in\mathcal{W}_{p(.),W,\theta}^\alpha$, and*

$$\sum_{\nu=1}^{\infty}\nu^{\alpha-1}E_\nu\left(f\right)_{p(.),W,\theta}<\infty$$

for some $\alpha>0$. Hence there exists a constant $c>0$ dependent only on α, r, W and $p(.)$ such that

$$\Omega_r(f^{(\alpha)},\frac{\pi}{n})_{p(.),W,\theta}\leq c\left\{\frac{1}{n^r}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+r-1}E_\nu(f)_{p(.),W,\theta}+\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu)f)_{p(.),W,\theta}\right\}.$$

Proof of Theorem 4.6. We set $W_n(f):=W_n(x,f):=\frac{1}{n+1}\sum\limits_{\nu=n}^{2n}S_\nu(x,f),\quad n=0,1,2,\ldots.$ Since

$$W_n(.,f^{(\alpha)})=W_n^{(\alpha)}(.,f),$$

then we have

$$\begin{aligned}&\left\|f^{(\alpha)}(.)-T_n^{(\alpha)}(.,f)\right\|_{p(.),W,\theta}\leq\left\|f^{(\alpha)}(.)-W_n(.,f^{(\alpha)})\right\|_{p(.),W,\theta}\\ &+\left\|T_n^{(\alpha)}(.,W_n(f))-T_n^{(\alpha)}(.,f)\right\|_{p(.),W,\theta}+\left\|W_n^{(\alpha)}(.,f)-T_n^{(\alpha)}(.,W_n(f))\right\|_{p(.),W,\theta}\\ =\;&I_1+I_2+I_3.\end{aligned}$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_W^{p(.),\theta}$. In this case, from the boundedness of W_n in $L_W^{p(.),\theta}$, we have

$$\begin{aligned} I_1 &\leq \left\| f^{(\alpha)}(.) - T_n^*(.,f^{(\alpha)}) \right\|_{p(.),W,\theta} + \left\| T_n^*(.,f^{(\alpha)}) - W_n(.,f^{(\alpha)}) \right\|_{p(.),W,\theta} \\ &\leq c\,(p,W,\theta)\, E_n\left(f^{(\alpha)}\right)_{p(.),W,\theta} + \left\| W_n(.,T_n^*(f^{(\alpha)}) - f^{(\alpha)}) \right\|_{p(.),W,\theta} \\ &\leq c\,(p,W,\theta)\, E_n\left(f^{(\alpha)}\right)_{p(.),W,\theta}. \end{aligned}$$

From Lemma 3.8 we get

$$I_2 \leq c\,(p,W,\theta)\, n^{\alpha} \left\| T_n(.,W_n(f)) - T_n(.,f) \right\|_{p(.),W,\theta}$$

and

$$\begin{aligned} I_3 &\leq c\,(p,W,\theta)\,(2n)^{\alpha} \left\| W_n(.,f) - T_n(.,W_n(f)) \right\|_{p(.),W,\theta} \\ &\leq c\,(p,W,\theta)\,(2n)^{\alpha}\, E_n\left(W_n(f)\right)_{p(.),W,\theta}. \end{aligned}$$

Now we have

$$\left\| T_n(.,W_n(f)) - T_n(.,f) \right\|_{p(.),W,\theta} \leq \left\| T_n(.,W_n(f)) - W_n(.,f) \right\|_{p(.),W,\theta}$$

$$\begin{aligned} &+ \left\| W_n(.,f) - f(.) \right\|_{p(.),W,\theta} + \left\| f(.) - T_n(.,f) \right\|_{p(.),W,\theta} \\ \leq\ & c\,(p,W,\theta)\, E_n\left(W_n(f)\right)_{p(.),W,\theta} + c\,(p,W,\theta)\, E_n\left(f\right)_{p(.),W,\theta} \\ &+ c\,(p,W,\theta)\, E_n\left(f\right)_{p(.),W,\theta}. \end{aligned}$$

Since

$$E_n\left(W_n(f)\right)_{p(.),W,\theta} \leq c\,(p,W,\theta)\, E_n\left(f\right)_{p(.),W,\theta},$$

then we get

$$\begin{aligned} \left\| f^{(\alpha)}(.) - T_n^{(\alpha)}(.,f) \right\|_{p(.),W,\theta} &\leq c\,(p,W,\theta)\, E_n(f^{(\alpha)})_{p(.),W,\theta} \\ &\quad + c\,(p,W,\theta)\, n^{\alpha} E_n\left(W_n(f)\right)_{p(.),W,\theta} \end{aligned}$$

$$\begin{aligned} &+ c\,(p,W,\theta)\, n^{\alpha} E_n\left(f\right)_{p(.),W,\theta} + c\,(p,W,\theta)\,(2n)^{\alpha}\, E_n\left(W_n(f)\right)_{p(.),W,\theta} \\ \leq\ & c\,(p,W,\theta)\, E_n\left(f^{(\alpha)}\right)_{p(.),W,\theta} + c\,(p,W,\theta)\, n^{\alpha} E_n\left(f\right)_{p(.),W,\theta}. \end{aligned}$$

Since, according to Theorem 3.1,

$$E_n\left(f\right)_{p(.),W,\theta} \leq \frac{c\,(p,W,\theta)}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(.),W,\theta}, \tag{4.8}$$

we obtain

$$\left\| f^{(\alpha)}(.) - T_n^{(\alpha)}(.,f) \right\|_{p(.),W,\theta} \leq c\,(p,W,\theta)\, E_n\left(f^{(\alpha)}\right)_{p(.),W,\theta}$$

and the proof is completed. □

Acknowledgment. The authors would like to thank the referee for the helpful comments and suggestions.

References

[1] G. Anatriello, *Iterated grand and small Lebesgue spaces*, Collect. Math. **65**, 273–284, 2014.

[2] R. Akgün, *Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent*, Ukrainian Math. J. **63** (1), 1–26, 2011.

[3] R. Akgün, *Approximating polynomials for functions of weighted Smirnov-Orlicz spaces*, J. Funct. Spaces Appl. **2012**, Art. ID 982360, 2012.

[4] R. Akgün and D.M. Israfilov, *Approximation and moduli of smoothness of fractional order in Smirnov-Orlicz spaces*, Glas. Mat. Ser. III, **42** (2), 121–136, 2008.

[5] R. Akgün and D.M. Israfilov, *Polynomial approximation in weighted Smirnov Orlicz space*, Proc. A. Razmadze Math. Inst. **139**, 89–92, 2005.

[6] R. Akgün and D.M. Israfilov, *Approximation by interpolating polynomials in Smirnov-Orlicz class*, J. Korean Math. Soc. **43** (2), 413–424, 2006.

[7] R. Akgün and D.M. Israfilov, *Simultaneous and converse approximation theorems in weighted Orlicz spaces*, Bull. Belg. Math. Soc. **17** (2), 13–28, 2010.

[8] R. Akgün and V. Kokilashvili, *On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces*, Banach J. Math. Anal. **5** (1), 70-82, 2011.

[9] I. Aydin, *Weighted variable Sobolev spaces and capacity*, J. Funct. Spaces Appl. **2012**, Art. ID 132690, 2012.

[10] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis (Applied and Numerical Harmonic Analysis)*, Birkhäuser/Springer, Heidelberg, 2013.

[11] D. Cruz-Uribe, L. Diening and P. Hästö, *The maximal operator on weighted variable Lebesgue spaces*, Fract. Calc. Appl. Anal. **14** (3), 361–374, 2011.

[12] N. Danelia and V. Kokilashvili, *Approximation by trigonometric polynomials in the framework of variable exponent Lebesgue spaces*, Georgian Math. J. **23** (1), 43–53, 2016.

[13] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7**, 245–253, 2004.

[14] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, **2017**, Springer, Heidelberg, 2011.

[15] V.K. Dzyadyk and I.A. Shevchuk, *Theory of Uniform Approximation of Functions by Polynomials*, Walter de Gruyter GmbH & Co. KG, 10785 Berlin, Germany, 2008.

[16] A. Fiorenza, B. Gupta and P. Jain, *The maximal theorem in weighted grand Lebesgue spaces*, Studia Math. **188** (2), 123–133, 2008.

[17] A. Fiorenza, V. Kokilashvili and A. Meskhi, *Hardy-Littlewood maximal operator in weighted grand variable exponent Lebesgue space*, Mediterr. J. Math. **14** (118), 2017.

[18] L. Greco, T. Iwaniec and C. Sbordone, *Inverting the p-harmonic operator*, Manuscripta Math. **92**, 249–258, 1997.

[19] D.M. Israfilov and R. Akgün, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46** (4), 755–770, 2006.

[20] D.M. Israfilov and A. Testici, *Approximation in weighted generalized grand Lebesgue spaces*, Colloq. Math. **143**, 113–126, 2016.

[21] D.M. Israfilov and A. Testici, *Approximation in weighted generalized grand Smirnov spaces*, Studia Sci. Math. Hungar. **54** (4), 471–488, 2017.

[22] T. Iwaniec and C. Sbordone, *On integrability of the Jacobien under minimal hypotheses*, Arch. Ration. Mech. Anal. **119**, 129–143, 1992.

[23] V. Kokilashvili and A. Meskhi, *Maximal and Calderon-Zygmund operators in grand variable exponent Lebesgue spaces*, Georgian Math. J. **21**, 447–461, 2014.

[24] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (116), 592–618, 1991.
[25] W. Orlicz. *Über konjugierte exponentenfolgen*, Stud. Math. **3**, 200–211, 1931.
[26] R.A. De Vore and G.G. Lorentz, *Constructive Approximation*, Springer, 1993.
[27] A. Zygmund, *Trigonometric Series, Volume I-II*, Cambridge University Press, Cambridge, 1968.