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A Study of Kenmotsu-like Statistical Submersions

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Abstract: In this paper, we first define a Kenmotsu-like statistical manifold ($K.l.s.m$) with examples. Then, we switch to Kenmotsu-like statistical submersions ($K.l.s.s$), where we investigate the fact that, for such submersions, each fiber is a statistical manifold that is similar to $K.l.s.m$, and the base manifold is similar to the Kähler-like statistical manifold. Subsequently, assuming the postulate that the curvature tensor with regard to the affine connections of the total space obeys certain criteria, we analyze such statistical submersions to those developed by Kenmotsu. Lastly, we talk about statistical submersions (SS) with conformal fibers (CFs) that are $K.l.s.m$.

Keywords: statistical manifolds; Kenmotsu-like statistical manifolds; statistical submersions; Kenmotsu-like statistical submersions

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1. Introduction

A Reimannian manifold is a statistical manifold of probability distributions possessing a Riemannian metric and two dual (conjugate) affine connections without torsion [1]. A statistical framework of a Riemannian metric and its extension are a Riemannian connection. The theory of statistical submanifolds and statistical manifolds is a recent geometry that plays a crucial role in several fields of mathematics. Various results have been derived by distinguished geometers in this area.

K. Kenmotsu [2] found interesting results and studied the warped product spaces of the type $\mathbb{R} \times_f \mathbf{B}$, where $\mathbf{B}$ is a Kaehlerian manifold with a maximal dimension that falls under Tanno's categorization of connected nearly contact metric manifolds (called the third class). Then, the author examined the characteristics of $\mathbb{R} \times_f \mathbf{B}$ and described it using tensor relations. A manifold of this type is referred to as a Kenmotsu manifold. A new notion in the statistical manifold, the Kenmotsu statistical manifold, was initiated by Furuhata et al. in [3]. Locally, it is the warped product of a holomorphic statistical manifold and a line. By establishing a natural affine connection to a Kenmotsu manifold, they developed a Kenmotsu statistical manifold in the same publication. Recently, Murathan et al. [4] talked about the term β-$K.l.s.m$.

On the other hand, the concept of submersion in differential geometry was first reported by O'Neill [5] and Gray [6], and Watson [7] later brought the concept of almost Hermitian submersions by using Riemannian submersions (RS) from almost Hermitian manifolds.

Afterwards, there have been several subclasses of almost Hermitian manifolds between which almost Hermitian submersions have been found. Additionally, under the heading of contact RS, Şahin in [8] extended RS to a wide variety of subclasses of virtually contact metric manifolds. In [9] the majority of studies on Riemannian, almost Hermitian, or contact RS are contained.

Barndroof-Nielsen and Jupp [10] discussed RS from the viewpoint of statistics. Abe and Hasegawa introduced and studied the SS between statistical manifolds in [11]. The

SS of the space of the multivariate normal distribution, statistical manifolds with virtually contact structures, and statistical manifolds with almost complex structures were among the topics that K. Takano found intriguing to research (see [12–14]). Remarkable statistical submersions have recently been studied, including para-Kähler-like statistical submersions [15], cosymplectic-like statistical submersions [16], and quaternionic Kähler-like statistical submersions [17]. Most of the research related to the various submersion can be found in [18–24].

Inspired by the affirmative works, we consider $K.l.s.m$ with some examples. Then, we study Kenmotsu-like statistical submersions ($K.l.s.s$) and give many results for such submersions with new examples. This study contributes to developing the SS literature.

2. Kenmotsu-like Statistical Manifolds ($K.l.s.m$)

Let $\bar{\mathbf{M}}$ be a semi-Riemannian manifold and nondegenerate metric $\bar{g}$, and a torsion-free affine connection by $\bar{\nabla}$. Triplet $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$ is a *statistical manifold* with symmetric $\bar{\nabla}\bar{g}$ [12]. For a statistical manifold $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$, we describe a second connection $\bar{\nabla}^*$ as

$$W\bar{g}(U,V) = \bar{g}(\bar{\nabla}_W U, V) + \bar{g}(U, \bar{\nabla}^*_W V), \tag{1}$$

for any $U, V, W \in T_r\bar{\mathbf{M}}$, $r \in \bar{\mathbf{M}}$. Here, affine connection $\bar{\nabla}^*$ is referred to as a *conjugate* (or *dual*) of the connection $\bar{\nabla}$ with respect to $\bar{g}$. Affine connection $\bar{\nabla}^*$ is torsion-free with symmetric $\bar{\nabla}^*\bar{g}$ and obeys

$$(\bar{\nabla}^*)^* = \bar{\nabla}, \qquad 2\bar{\nabla}^0 = \bar{\nabla} + \bar{\nabla}^*,$$

where in the Levi-Civita connection $\bar{\nabla}^0$ on $\bar{\mathbf{M}}$.

A statistical manifold is $(\bar{\mathbf{M}}, \bar{\nabla}^*, \bar{g})$. For example, let $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$ be a semi-Riemannian manifold along its Riemannian connection $\bar{\nabla}$ is a *trivial statistical manifold*. In this case, $\bar{\mathcal{R}}$ ($\bar{\mathcal{R}}^*$) stands for the curvature tensor on $\bar{\mathbf{M}}$ with respect to affine connection $\bar{\nabla}$ (its conjugate $\bar{\nabla}^*$). Now, we produce

$$\bar{g}(\bar{\mathcal{R}}(U,V)W, W') = -\bar{g}(W, \bar{\mathcal{R}}^*(U,V)W'), \tag{2}$$

for any $U, V, W, W' \in T_r\bar{\mathbf{M}}$ [12].

Let $(\bar{\mathbf{M}}, \bar{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold that admits the almost contact structure (φ, ξ, ν) that contains another tensor field, φ^*, of type $(1,1)$ that fulfils

$$\bar{g}(\varphi U, V) = -\bar{g}(U, \varphi^* V), \tag{3}$$

for any $U, V \in T_r\bar{\mathbf{M}}$. Then, $\bar{\mathbf{M}}$ is a metric manifold with almost contact structure (φ, ξ, ν, g) of a specific sort [14]. Then,

$$\varphi^{*2}U = -U + \nu(U)\xi \quad \text{and} \quad \bar{g}(\varphi U, \varphi^* V) = \bar{g}(U,V) - \nu(U)\nu(V). \tag{4}$$

In fact, φ is a nonsymmetric tensor field, which shows that $\varphi + \varphi^* \neq 0$ everywhere. The almost contact manifold also entertain the following equations:

$$\nu(\xi) = 1, \quad \varphi(\xi) = 0 \quad \text{and} \quad \nu \circ \varphi = 0. \tag{5}$$

We also obtained the almost contact metric manifold of a specific sort [14], such that

$$\varphi^*\xi = 0 \quad \text{and} \quad \nu(\varphi^*(E)) = 0. \tag{6}$$

Murathan et al. [4] produced a method of how to construct $K.l.s.m$ relying on the idea of a statistical manifold similar to that of the Kähler-like statistical manifold. They defined

β-$K.l.s.m$ and said that an almost contact metric such as statistical manifold $(\bar{\mathbf{M}}, \bar{\nabla}, \varphi, \xi, \nu, \bar{g})$ is referred to as a β-Kenmotsu-like statistical manifold if

$$(\bar{\nabla}_U \varphi)V = \beta\{\bar{g}(U, \varphi V)\xi + \nu(V)\varphi U\}, \quad (7)$$
$$\bar{\nabla}_U \xi = \beta\varphi^2 U, \quad (8)$$

where β is differentiable function on $\bar{\mathbf{M}}$. They proved the following theorem [4]:

Theorem 1. *Let* $(\mathbf{B}, g_{\mathbf{B}}, \nabla^{\mathbf{B}}, J)$ *be a Kähler-like statistical manifold, and* $(\mathbb{R}, \nabla^{\mathbb{R}}, dt)$ *be trivial statistical manifold.* $\mathbb{R} \times \mathbf{B}$. *Under Proposition 2.2 (see [4]),* $\mathbb{R} \times_f \mathbf{B}$ *is a* $\beta = \frac{f'}{f}$ *K.l.s.m.*

Now, $(\bar{\mathbf{M}}, \bar{\nabla}, \varphi, \xi, \nu, \bar{g})$ is called a $K.l.s.m$ if the following conditions hold:

$$\bar{\nabla}_U \xi = U - \nu(U)\xi, \quad \bar{\nabla}^*_U \xi = U - \nu(U)\xi, \quad (9)$$

$$(\bar{\nabla}_U \varphi)V = \bar{g}(\varphi U, V)\xi - \nu(W)\varphi U \quad \text{and} \quad (\bar{\nabla}^*_U \varphi^*)V = \bar{g}(\varphi^* U, V)\xi - \nu(W)\varphi^* U. \quad (10)$$

Consequently, we have the following lemma:

Lemma 1. $(\bar{\mathbf{M}}, \bar{\nabla}, \varphi, \xi, \nu, \bar{g})$ *is a K.l.s.m if and only if* $(\bar{\mathbf{M}}, \bar{\nabla}^*, \varphi^*, \xi, \nu, \bar{g})$ *is a K.l.s.m.*

In [25], certain bounds for statistical curvatures of submanifolds with any codimension of $K.l.s.m$ were obtained. Now, we give the following examples on β-$K.l.s.m$:

Example 1. *Let us assume a Kähler-like statistical manifold* $(\mathbf{B}^2, \nabla^{\mathbf{B}}, g_{\mathbf{B}}, J)$*, where* $\mathbf{B}^2 = \{(x_1, x_2) \in \mathbb{R}^2\}$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$J^* = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},$$

$$g_{\mathbf{B}} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

and the flat affine connection $\nabla^{\mathbf{B}}$*. Also,* $(\mathbb{R}, \nabla^{\mathbb{R}}, dt^2)$ *is a trivial statistical manifold with constant curvature* 0*. From Theorem 1, warped product manifold* $(\bar{\mathbf{M}} = \mathbb{R} \times_f \mathbf{B}^2, \bar{\nabla}, \bar{g} = dt^2 + f^2 g_{\mathbf{B}})$ *is a* β*-K.l.s.m.*

Example 2. *A Euclidean space* $\mathbb{R}^4$ *with local coordinate system* $\{x_1, x_2, y_1, y_2\}$ *that admits the following almost complex structure J:*

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

the metric $g_{\mathbb{R}^4} = 2dx_1^2 + 2dx_2^2 - dy_1^2 - dy_2^2$ *with a flat affine connection* $\nabla^{\mathbb{R}^4}$ *is referred as a Kähler-like statistical manifold (see [14]). If* $(\mathbb{R}, \nabla^{\mathbb{R}}, dt^2)$ *is a trivial statistical manifold. In view of [4], the product manifold* $\left(\mathbb{R} \times_f \mathbb{R}^4, \bar{\nabla}, \bar{g} = dt^2 + f^2 g_{\mathbb{R}^4}\right)$ *is called a* $\beta-$ *K.l.s.m.*

Let us define φ, ξ and η by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad \xi = dt = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $\eta = (1, 0, -y_1, 0, -y_2)$. We also find

$$\varphi^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

Example 3. *From [26], we know that $\bar{\mathbf{M}}_v^{n+1} = \{(x_1, \ldots, x_n, x_{n+1}) | x_{n+1} > 0\}$ the half upper space with $\left(\bar{g}, \bar{\nabla}^{(1)}, \bar{J}^{(1)}\right)$ that was described as in [26], $\left(\bar{\mathbf{M}}, \bar{g}, \bar{\nabla}^{(1)}, \bar{J}^{(1)}\right)$ is a Kähler-like statistical manifold. So, If $(\mathbb{R}, \nabla^{\mathbb{R}}, dt^2)$ is a trivial statistical manifold, It is recognised by [4] that the product manifold $\left(\mathbb{R} \times_f \bar{\mathbf{M}}, \nabla, g = dt^2 + f^2\bar{g}\right)$ is a $\beta-$ K.l.s.m.*

We examine curvature tensor $\bar{\mathcal{R}}$ on a statistical manifold similar to that of *K.l.s.m* with respect to $\bar{\nabla}$, such that

$$\begin{aligned} \bar{\mathcal{R}}(U,V)W &= \frac{c-3}{4}\{\bar{g}(V,W)U - \bar{g}(U,W)V\} \\ &+\frac{c+1}{4}\{\bar{g}(\varphi V,W)\varphi U - \bar{g}(\varphi U,W)\varphi V \\ &-2\bar{g}(\varphi U,V)\varphi W - \bar{g}(V,\xi)\bar{g}(W,\xi)U \\ &+\bar{g}(U,\xi)\bar{g}(W,\xi)V + \bar{g}(V,\xi)\bar{g}(W,U)\xi \\ &-\bar{g}(U,\xi)\bar{g}(W,V)\xi\}, \end{aligned} \tag{11}$$

where $c \in \mathbb{R}$. Afterwards, shifting φ to φ^* in (11), we produce the expression for the curvature tensor $\bar{\mathcal{R}}^*$ in terms of $\bar{\nabla}^*$.

Let $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$ be a statistical manifold and $\mathbf{M}$ be a submanifold of $\bar{\mathbf{M}}$. Then $(\mathbf{M}, \nabla, g)$ is also a statistical manifold with the induced statistical structure (∇, g) on $\mathbf{M}$ from $(\bar{\nabla}, \bar{g})$ and we call $(\mathbf{M}, \nabla, g)$ as a statistical submanifold in $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$.

In the statistical setting, Gauss and Weingarten equations are respectively specified by [27]

$$\left.\begin{aligned} \bar{\nabla}_U V = \nabla_U V + h(U,V), \quad & \bar{\nabla}^*_U V = \nabla^*_U V + h^*(U,V), \\ \bar{\nabla}_U \eta = -A_\eta(U) + \nabla^\perp_U \eta, \quad & \bar{\nabla}^*_U \eta = -A^*_\eta(U) + \nabla^{\perp *}_U \eta, \end{aligned}\right\} \tag{12}$$

for any $U, V \in T_r\mathbf{M}$ and $\eta \in T_r^\perp\mathbf{M}$, where $\bar{\nabla}$ and $\bar{\nabla}^*$ are the dual connections on $\bar{\mathbf{M}}$. Similarly, on $\mathbf{M}$, we denote them with ∇ and ∇^*. For $\bar{\nabla}$ and $\bar{\nabla}^*$, the symmetric and bilinear imbedding curvature tensor of $\mathbf{M}$ in $\bar{\mathbf{M}}$ are indicated by h and h^*, respectively. The finest relation between h (h^*) and A (A^*) is [27]:

$$\bar{g}(h(U,V),\eta) = g(A^*_\eta U, V) \quad \text{and} \quad \bar{g}(h^*(U,V),\eta) = g(A_\eta U, V). \tag{13}$$

We indicate the curvature tensor fields of $\bar{\nabla}$ and ∇ as $\bar{\mathcal{R}}$ and $\mathcal{R}$, respectively. Then, for any $U, V, W, W' \in T_r\mathbf{M}$, the corresponding Gaussian equations are [27]

$$\begin{aligned} \bar{g}(\bar{\mathcal{R}}(U,V)W,W') &= g(\mathcal{R}(U,V)W,W') + \bar{g}(h(U,W),h^*(V,W')) \\ &\quad -\bar{g}(h^*(U,W'),h(V,W)) \end{aligned} \tag{14}$$

and

$$\begin{aligned} \bar{g}(\bar{\mathcal{R}}^*(U,V)W,W') &= g(\mathcal{R}^*(U,V)W,W') + \bar{g}(h^*(U,W),h(V,W')) \\ &\quad -\bar{g}(h^*(U,W'),h(V,W)). \end{aligned} \tag{15}$$

Thus, the statistical curvature tensor fields of $\bar{\mathbf{M}}$ and $\mathbf{M}$ are, respectively, specified by

$$\bar{\mathcal{S}} = \frac{1}{2}(\bar{\mathcal{R}} + \bar{\mathcal{R}}^*) \quad \text{and} \quad \mathcal{S} = \frac{1}{2}(\mathcal{R} + \mathcal{R}^*).$$

For $U \in T_r\mathbf{M}$, we put

$$\begin{aligned} \varphi U &= tan(\varphi U) + nor(\varphi U) \\ &= \mathbf{P}U + \mathbf{F}U, \end{aligned}$$

where $\mathbf{P}U$ ($\mathbf{P}^*U$) and $\mathbf{F}U$ ($\mathbf{F}^*U$) indicate the tangential and normal components of φU ($\varphi^* U$), respectively. Likewise, we can write

$$\begin{aligned} \varphi^* U &= tan(\varphi^* U) + nor(\varphi^* U) \\ &= \mathbf{P}^*U + \mathbf{F}^*U. \end{aligned}$$

3. Background of Statistical Submersions

This segment provides the prior knowledge required for SS.

Let us consider two semi-Riemannian manifolds, $\bar{\mathbf{M}}$ and $\mathcal{N}$, and let a semi-Riemannian submersion $\omega : \bar{\mathbf{M}} \longrightarrow \mathcal{N}$ such that $\omega*$ maintains the lengths of horizontal vectors, and all the fibers are semi-Riemannian submanifolds of $\bar{\mathbf{M}}$ (for more details, see [9,21]). Abe and Hasegawa [11] investigated affine submersions with horizontal distribution from a statistical manifold. Furthermore, SS was discussed by Takano in [12,13].

Let a semi-Riemannian submersion $\omega : (\bar{\mathbf{M}}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{g})$ between the semi-Riemannian manifolds $(\bar{\mathbf{M}}, \bar{g})$ and $(\mathcal{N}, \hat{g})$. The semi-Riemannian submanifold $\omega^{-1}(x)$ has $n-2$ dimensions and an induced metric g' known as a *fiber* and denoted by $\mathbf{M}'$ for any point $x \in \mathcal{N}$. The vertical and horizontal distributions in the tangent bundle $T\bar{\mathbf{M}}$ of $\bar{\mathbf{M}}$ are indicated by $\mathcal{V}(\bar{\mathbf{M}})$ and $\mathcal{H}(\bar{\mathbf{M}})$, respectively. Thus, we have

$$T(\bar{\mathbf{M}}) = \mathcal{V}(\bar{\mathbf{M}}) \oplus \mathcal{H}(\bar{\mathbf{M}}).$$

If there is a vector field X on $\bar{\mathbf{M}}$, we refer to it as projectable. Vector field $\hat{X}$ on $\mathcal{N}$, such that $\omega_*(X_r) = \hat{X}_\omega(r)$, for each $r \in \bar{\mathbf{M}}$. In this instance, X and $\hat{X}$ are referred to as ω-related. A vector field X on $\mathcal{H}(\bar{\mathbf{M}})$ if it is projectable, it is referred to as basic [5]. We have the following information if X and Y are the fundamental vector fields, ω-related to $\hat{X}$, $\hat{Y}$:

1. $\hat{g}(\hat{X}, \hat{Y}) \circ \omega = \bar{g}(X, Y)$,
2. $\mathcal{H}[X, Y]$ is a fundamental vector field is $\mathcal{H}[X, Y]$, and $\omega * \mathcal{H}[X, Y] = [\hat{X}, \hat{Y}] \circ \omega$. vector field and $\omega_* \mathcal{H}[X, Y] = [\hat{X}, \hat{Y}] \circ \omega$,
3. For any vertical vector field U, $[X, U]$ is vertical.

O'Neill's law describes the geometry of semi-Riemannian submersions. Tensors $\mathbf{T}$ and $\mathbf{A}$ are defined as follows using [5]:

$$\mathbf{T}_E F = \mathcal{H}\bar{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\bar{\nabla}_{\mathcal{V}E}\mathcal{H}F, \tag{16}$$

$$\mathbf{A}_E F = \mathcal{H}\bar{\nabla}_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\bar{\nabla}_{\mathcal{H}E}\mathcal{H}F \tag{17}$$

with respect to any vector fields E and F on $\bar{\mathbf{M}}$. It is clear that skew-symmetric operators $\mathbf{T}_E$ and $\mathbf{A}_E$ on the tangent bundle of $\bar{\mathbf{M}}$ reverse the vertical and horizontal distributions. We provide a summary of the characteristics of tensor fields $\mathbf{T}$ and $\mathbf{A}$. If E, F are vertical vector fields on $\bar{\mathbf{M}}$, and X, Y are horizontal vector fields, we possess

$$\mathbf{T}_E F = \mathbf{T}_F E, \tag{18}$$

$$\mathbf{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y] = -\mathbf{A}_Y X. \tag{19}$$

Let $\omega : \bar{\mathbf{M}} \longrightarrow \mathcal{N}$ be a semi-Riemannian submersion from a statistical manifold $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$. Let us use symbols $\nabla^{'}$ and $\nabla^{'*}$ to represent the affine connections on $\mathbf{M}^{'}$. It is obvious that

$$\nabla^{'}_E F = \mathcal{V}\bar{\nabla}_E F \quad \text{and} \quad \nabla^{'*}_E F = \mathcal{V}\bar{\nabla}^*_E F$$

for vertical vector fields E and F on $\bar{\mathbf{M}}$. It is simple to observe that $\nabla^{'}$ and $\nabla^{'*}$ are conjugate to each other and torsion-free with respect to $g^{'}$.

Let submersion $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ between two statistical manifolds be a *statistical submersion* if ω obeys $\omega_*(\bar{\nabla}_X Y)_r = (\hat{\nabla}_{\hat{X}}\hat{Y})_{\omega(r)}$ for basic vector field X, Y and $r \in \bar{\mathbf{M}}$. Shifting ∇ for ∇^* in the aforementioned expressions, we derive $\mathbf{T}^*$ and $\mathbf{A}^*$ [12]. $\mathbf{A}$ and $\mathbf{A}^*$ vanish if and only if $\mathcal{H}(\bar{\mathbf{M}})$ is integrable with respect to ∇ and ∇^*, respectively. For $E, F \in \mathcal{V}(\bar{\mathbf{M}})$ and $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$, we produce

$$\bar{g}(\mathbf{T}_E F, X) = -\bar{g}(F, \mathbf{T}^*_E X) \quad \text{and} \quad \bar{g}(\mathbf{A}_X Y, E) = -\bar{g}(Y, \mathbf{A}^*_X E). \tag{20}$$

4. Properties of Statistical Submersions

In this section, we discuss some useful properties of statistical submersion proposed by Takano [12]. First, we have the following lemmas for this study. Therefore, for a statistical submersion $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$, we have [5,12]

Lemma 2 ([12])**.** *If X and Y are horizontal vector fields, then $\mathbf{A}_X Y = -\mathbf{A}^*_Y X$.*

Lemma 3 ([12])**.** *For $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$ and $E, F \in \mathcal{V}(\bar{\mathbf{M}})$. Then we have*

$$\bar{\nabla}_E F = \mathbf{T}_E F + \nabla^{'}_E F, \qquad \bar{\nabla}^*_E F = \mathbf{T}^*_E F + \nabla^{'*}_E F, \tag{21}$$

$$\bar{\nabla}_E X = \mathbf{T}_E X + \mathcal{H}\bar{\nabla}_E X, \qquad \bar{\nabla}^*_E X = \mathbf{T}^*_E X + \mathcal{H}\bar{\nabla}^*_E X, \tag{22}$$

$$\bar{\nabla}_X E = \mathbf{A}_X E + \mathcal{V}\bar{\nabla}_X E, \qquad \bar{\nabla}^*_X E = \mathbf{A}^*_X E + \mathcal{V}\bar{\nabla}^*_X E, \tag{23}$$

$$\bar{\nabla}_X Y = \mathcal{H}\bar{\nabla}_X Y + \mathbf{A}_X Y, \qquad \bar{\nabla}^*_X Y = \mathcal{H}\bar{\nabla}^*_X Y + \mathbf{A}^*{}_X Y. \tag{24}$$

Furthermore, if X is basic, then $\mathcal{H}\bar{\nabla}_E X = \mathbf{A}_X E$ and $\mathcal{H}\bar{\nabla}^*_E X = \mathbf{A}^*_X E$.

Moreover, let $\hat{\mathcal{R}}(X, Y)Z$ (resp. $\hat{\mathcal{R}}^*(X, Y)Z$) is a horizontal vector field like that

$$\omega_*(\hat{\mathcal{R}}(X, Y)Z) = \hat{\mathcal{R}}(\omega_* X, \omega_* Y)\omega_* Z$$

at each point $r \in \bar{\mathbf{M}}$, where $\hat{\mathcal{R}}$ (resp. $\hat{\mathcal{R}}^*$) be the curvature tensor with respect to the induced affine connection $\hat{\nabla}$(resp. $\hat{\nabla}^*$). Thus we have the following theorem [12].

Theorem 2 ([12,14])**.** *If $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ is a statistical submersion then for $E, F, G, H \in \mathcal{V}(\bar{\mathbf{M}})$ and $X, Y, Z, W \in \mathcal{H}(\bar{\mathbf{M}})$*

$$\bar{g}(\mathcal{R}(E, F)G, H) = \bar{g}(\overline{\mathcal{R}}(E, F)G, H) + \bar{g}(\mathbf{T}_E G, \mathbf{T}^*_F H) - \bar{g}(\mathbf{T}_F G, \mathbf{T}^*_E H), \tag{25}$$

$$\bar{g}(\mathcal{R}(X, Y)Z, W) = \bar{g}(\hat{\mathcal{R}}(X, Y)Z, W) + \bar{g}((\mathbf{A}_X + \mathbf{A}^*_X)Y, \mathbf{A}^*_Z W) \tag{26}$$

$$-\bar{g}(\mathbf{A}_Y Z, \mathbf{A}^*_X W) + \bar{g}(\mathbf{A}_X Z, \mathbf{A}^*_Y W).$$

$$\bar{g}(\mathcal{R}(X,E)F,Y) = \bar{g}((\bar{\nabla}_X \mathbf{T})_E F, Y) - \bar{g}((\bar{\nabla}_E \mathbf{A})_X, F) + \bar{g}(\mathbf{A}_X E, \mathbf{A}^*_Y F) - \bar{g}(\mathbf{T}_E X, \mathbf{T}^*_F Y), \tag{27}$$

$$\bar{g}(\mathcal{R}(X,E)Y,F) = \bar{g}(\bar{\nabla}_X \mathbf{T})_E Y, F) - \bar{g}((\bar{\nabla}_E \mathbf{A})_X Y, F) - \bar{g}(\mathbf{A}_X E, \mathbf{A}_Y F) - \bar{g}(\mathbf{T}_E X, \mathbf{T}_F Y). \tag{28}$$

Now, we describe with $\{K_1, K_2, \dots, K_m\}$, $\{X_1, \dots, X_n\}$ and $\{E_1, E_2 \dots, E_t\}$ the orthonormal frame of $T(\bar{\mathbf{M}})$, $\mathcal{H}(\bar{\mathbf{M}})$ and $\mathcal{V}(\bar{\mathbf{M}})$, respectively, such that $K_i = X_i$, $1 \le i \le n$ and $K_{n+t} = E_t$, $1 \le t \le s$. With ε^a_b and ε^{*b}_a, we jointly define the connection forms in terms of local coordinates $\{K_1, \dots, K_m\}$ with respect to the affine connection $\bar{\nabla}$ and its conjugate $\bar{\nabla}^*$. Adopting (1), we produce

$$\varepsilon^{*b}_a = -\varepsilon^a_b, \quad 1 \le a, b \le m \tag{29}$$

and

$$\bar{g}(\mathbf{T}X, \mathbf{T}Y) = \sum_{t=1}^{s} \bar{g}(\mathbf{T}_{E_t} X, \mathbf{T}_{E_t} Y), \tag{30}$$

for any $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$. The horizontal vector fields accordingly determine the fiber's mean curvature vector field with regard to the affine connection $\bar{\nabla}$ and its conjugate connection $\bar{\nabla}^*$,

$$N = \sum_{t=1}^{s} \mathbf{T}_{E_t} E_t \quad \text{and} \quad N^* = \sum_{t=1}^{s} \mathbf{T}^*_{E_t} E_t.$$

5. Kenmotsu-like Statistical Submersion ($K.l.s.s$)

Assume that $(\bar{\mathbf{M}}, \varphi, \xi, \nu, \bar{g})$ is an almost contact metric manifold. If $\omega : \bar{\mathbf{M}} \longrightarrow \mathcal{N}$ is a semi-Riemannian submersion, each fiber is a φ-invariant semi-Riemannian submersion of $\bar{\mathbf{M}}$ and vector field ξ is tangent to $\bar{\mathbf{M}}$; therefore, ω is an *almost contact metric submersion*. If U is basic on $\bar{\mathbf{M}}$, which is ω-related to $\hat{U}$ on $\mathcal{N}$, then φU (resp. $\varphi^* U$) is basic and φ-related to $\varphi\hat{U}$ (resp. $\varphi^*\hat{U}$) [14].

Analogous to the Sasaki-like statistical submersion [14], we describe $K.l.s.s$ as follows:

Definition 1. *A SS $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ is $K.l.s.s$ if $(\bar{\mathbf{M}}, \bar{\nabla}, \varphi, \xi, \nu, \bar{g})$ is a $K.l.s.m$, if each fiber is a φ-invariant semi-Riemannian submanifold of $\bar{\mathbf{M}}$ and tangent to vector field ξ.*

Therefore, we produced the following results:

Lemma 4. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a $K.l.s.s$ then for $X \in \mathcal{H}(\bar{\mathbf{M}})$ and $E \in \mathcal{V}(\bar{\mathbf{M}})$, we have*

$$\mathbf{A}_X \xi = X, \tag{31}$$

$$\mathbf{T}_E \xi = E, \tag{32}$$

$$\nabla'_E \xi = 0, \tag{33}$$

$$\mathcal{V}\bar{\nabla}_X \xi = 0. \tag{34}$$

Proof. In light of Lemma 3, one produces the above relations. □

Lemma 5. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a $K.l.s.s$; then, we have for $E, F \in \mathcal{V}(\bar{\mathbf{M}})$ and $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$.*

$$(\mathcal{H}\bar{\nabla}_X \varphi)Y = 0, \tag{35}$$

$$\mathbf{A}_X \varphi Y - \overline{\varphi} \mathbf{A}_X Y = \bar{g}(\varphi X, Y)\xi, \tag{36}$$

$$\mathbf{A}_X\overline{\varphi}E - \varphi\mathbf{A}_X E = -\nu(U)\varphi X, \quad \textit{if } X \textit{ is basic}, \tag{37}$$

$$\mathbf{T}_E\varphi X = \overline{\varphi}\mathbf{T}_E X, \tag{38}$$

$$\mathbf{A}_{\varphi X}E = \varphi(\mathbf{A}_X E), \tag{39}$$

$$(\mathcal{V}\bar{\nabla}_X\overline{\varphi})E = 0, \tag{40}$$

$$\nabla'_E\overline{\varphi})F = \bar{g}(\varphi E, F)\xi - \nu(F)\varphi E. \tag{41}$$

Proof. Since vertical and horizontal distributions are φ-invariant for $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$, in view of Lemma 3 and (10), we obtain (35). Now, (36)–(38) follows for $E \in \mathcal{V}(\bar{\mathbf{M}})$ and $X \in \mathcal{H}(\bar{\mathbf{M}})$ with using Lemma 3 and (10). Similarly, we produce (39) and (40) for $E, F \in \mathcal{V}(\bar{\mathbf{M}})$. Immediately, this also gives us (41). □

Adopting Lemmas 4 and 5, the following results entail:

Theorem 3. *Let* $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ *be a K.l.s.s. Then,* $(\mathcal{N}, \hat{\nabla}, \hat{g})$ *is a Kähler-like statistical manifold and* $\mathbf{M}', \nabla', \overline{\varphi}, \xi, \nu, g')$ *a K.l.s.m.*

Proof. The above lemmas show that each fiber is *K.l.s.m*. Now, we prove that $\mathbf{M}', \nabla', \overline{\varphi}, \xi, \nu, g')$ is a Kähler-like statistical manifold. Let X, Y, Z be a basic vector field and ω related to $\hat{X}, \hat{Y}, \hat{Z}$. Now, we have

$$\hat{g}((\hat{\nabla}_{\hat{X}}J)\hat{Y}, \hat{Z}) = \hat{g}(\hat{\nabla}_{\hat{X}}J\hat{Y} - J\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{Z})$$

$$= \hat{g}(\omega_*(\bar{\nabla}_X\varphi Y) - \omega_*(\varphi\bar{\nabla}_X), \omega_* Z)$$

$$= \bar{g}(\bar{\nabla}_X\varphi Y - \varphi\bar{\nabla}_X Y, Z) = \bar{g}((\bar{g}(\varphi X, Y)\xi - \nu(Y)\varphi X, Z)).$$

Since $(\bar{\mathbf{M}}, \bar{\nabla}, \bar{g})$ is a *K.l.s.m*. From the above expression, we produce

$$(\hat{\nabla}_{\hat{X}}J)\hat{Y} = \bar{g}(\varphi X, Y)\xi - \nu(Y)\varphi X, \tag{42}$$

which shows that the base manifold is a Kähler-like statistical manifold. □

Lemma 6. *Let a K.l.s.s* $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$*. Then*

$$\mathbf{A}_X Y = -\bar{g}(X, Y)\xi + \nu(X)\nu(Y)\xi,$$

if $dim(\mathbf{M}') = 1$.

Proof. Consider $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ is a *K.l.s.s* . Thus,

$$(\bar{\nabla}_X\varphi)Y = \bar{\nabla}_X\varphi Y - \varphi\bar{\nabla}_X Y.$$

Setting $Y = \varphi Y$ in the above expression, we find

$$= -\bar{\nabla}_X Y + \bar{g}(\bar{\nabla}_X Y, \xi)\xi + \bar{g}(Y, \bar{\nabla}^*_X\xi)\xi + \nu(Y)\bar{\nabla}_X\xi - \varphi\bar{\nabla}_X\varphi Y.$$

Adopting Lemma 3, we produce

$$= -\mathbf{A}_X Y - \mathcal{H}\bar{\nabla}_X Y + \bar{g}(\mathbf{A}_X Y, \xi)\xi - \varphi\mathbf{A}_X\varphi Y - \varphi\mathcal{H}\bar{\nabla}_X\varphi Y. \tag{43}$$

Hence, the vertical parts from (43) hold

$$\bar{g}(\varphi X, \varphi Y)\xi = -\mathbf{A}_X Y + \bar{g}(\mathbf{A}_X Y, \xi)\xi - \varphi\mathbf{A}_X\varphi Y.$$

Because $\bar{g}(\mathbf{A}_X Y, \xi) = 0$, $\bar{g}(\varphi X, \varphi Y)\xi = -\mathbf{A}_X Y$. Because $dim(\mathbf{M}') = 1$, we obtain the required results. □

By virtue of Lemma 4, we obtain $(\overline{\varphi} + \overline{\varphi}^*)\mathbf{A}_X Y = 0$. This entails the following.

Theorem 4. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s. Then, for $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$, we have*

$$\mathbf{A}_X Y = -\bar{g}(X,Y)\xi + \nu(X)\nu(Y)\xi,$$

if rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}^{'}) - 1$.

Again, in view of Lemma 1 and using $(\overline{\varphi} + \overline{\varphi}^*)\mathbf{A}_X Y = 0$, we obtain the following corollary:

Corollary 1. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s. Then, for $X, Y \in \mathcal{H}(\bar{\mathbf{M}})$, we have*

$$\mathbf{A}_X Y = -\bar{g}(X,Y)\xi + \nu(X)\nu(Y)\xi,$$

if $\overline{\varphi} = \overline{\varphi}^*$.

6. Curvature-Based Characteristics of Kenmotsu-like Statistical Submersion

Statistical manifolds on almost Hermite-like manifolds were proposed by Takano in [12]. if J is parallel with respect to the $\bar{\nabla}$, then $(\bar{\mathbf{M}}, \bar{\nabla}, J, \bar{g})$ is called a *Kähler-like statistical manifold* [12]. Moreover, curvature tensor $\mathcal{R}$ on a Kähler-like manifold $(\bar{\mathbf{M}}, \bar{\nabla}, J, \bar{g})$ with respect to $\bar{\nabla}$ is given by

$$\mathcal{R}(X,Y)Z = \frac{c}{4}[\bar{g}(Y,Z)X - \bar{g}(X,Z)Y - \bar{g}(Y,JZ)JX + \bar{g}(X,JZ)JY + (\bar{g}(X,JY) - \bar{g}(Y,JX))JZ]. \tag{44}$$

Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a *K.l.s.s*. Then, the expression for the curvature tensor of $(\bar{\mathbf{M}}, \bar{\nabla}, \varphi, \xi, \nu, \bar{g})$ is given by (11). Adopting Theorem 2, we produce

$$\begin{aligned} &\bar{g}(\overline{\mathcal{R}}(E,F)G,H) + \bar{g}(\mathbf{T}_E G, \mathbf{T}^*_F H) - \bar{g}(\mathbf{T}_F G, \mathbf{T}^*_E H) \\ = &\frac{c-3}{4}\{g(E,G)\bar{g}(E,H) - g(E,G)\bar{g}(F,H)\} \\ &+\frac{c+1}{4}\{g(\varphi F,G)\varphi E - g(\varphi E,G)\varphi F \\ &-2g(\varphi E,F)\bar{g}(\varphi G,H) - \nu(F)\nu(G)g(E,H) \\ &+\nu(E)\nu(G)g(F,H) + \nu(F)\nu(G)\nu(H)g(G,E) \\ &-\nu(E)\nu(H)g(G,F)\}, \end{aligned}$$

$$\bar{g}((\bar{\nabla}_E\mathbf{T})_F G, X) - \bar{g}((\bar{\nabla}_F\mathbf{T}_E G, X) = 0, \tag{45}$$

$$\bar{g}((\bar{\nabla}_E\mathbf{T})_F X, G) - \bar{g}((\bar{\nabla}_F\mathbf{T}_E X, G) = 0, \tag{46}$$

$$\begin{aligned} &\bar{g}((\bar{\nabla}_E\mathbf{A})_X F, Y) - \bar{g}((\bar{\nabla}_F\mathbf{A}_X E, T) + \bar{g}(\mathbf{T}_E X, \mathbf{T}^*_V Y) \\ &- \bar{g}(\mathbf{A}_X E, \mathbf{A}^*_Y F) + \bar{g}(\mathbf{A}_X F, \mathbf{A}^*_Y E) \\ &= \frac{c+1}{4}[\bar{g}(E,\varphi F) - \bar{g}(\varphi E, V)]\bar{g}(\varphi, X, Y), \end{aligned} \tag{47}$$

$$\bar{g}([\mathcal{V}\bar{\nabla}_X, \nabla^{'}_E]E, F) - \bar{g}(\bar{\nabla}_{[X,E]}F, G) - \bar{g}(\mathbf{T}_E F, \mathbf{A}^*_X G) + \bar{g}(\mathbf{T}^*_E G, \mathbf{A}_X F) = 0, \tag{48}$$

$$\begin{aligned} &\bar{g}((\bar{\nabla}_X\mathbf{T})_E F, Y) - \bar{g}((\bar{\nabla}_E\mathbf{A}_X F, Y) + \bar{g}(\mathbf{A}_X E, \mathbf{A}^*_Y F) - \bar{g}(\mathbf{T}_E X, \mathbf{T}^*_F Y) \\ &= \frac{c-3}{4}\bar{g}(E,F)\bar{g}(X,Y) - \frac{c+1}{4}[\nu(E)\nu(F)\bar{g}(X,Y) + \bar{g}(E,\overline{\varphi}F)\bar{g}(\varphi X, Y)], \end{aligned} \tag{49}$$

$$\bar{g}((\bar{\nabla}_X\mathbf{T})_E Y, F) - \bar{g}((\bar{\nabla}_E\mathbf{A})_X Y, F) + \bar{g}(\mathbf{T}_E X, \mathbf{T}_F Y) - \bar{g}(\mathbf{A}_X E, \mathbf{A}_Y F) \tag{50}$$

$$= -\frac{c-3}{4}\bar{g}(E,F)\bar{g}(X,Y) + \frac{c+1}{4}[\nu(E)\nu(F)\bar{g}(X,Y) + \bar{g}(\overline{\varphi}E,F)\bar{g}(X,\varphi Y)],$$

$$\bar{g}((\bar{\nabla}_X \mathbf{A})_Y E, Z) - \bar{g}(\mathbf{T}_E X, \mathbf{A}^*_Y Z) - \bar{g}(\mathbf{T}_E Y, \mathbf{A}^*_X Z) + \bar{g}(\mathbf{A}_X Y, \mathbf{T}^*_E Z) = 0, \tag{51}$$

$$\bar{g}(((\bar{\nabla}_X \mathbf{T})_E Y, Z) - \bar{g}((\bar{\nabla}_Y \mathbf{T})_E X, F) - \bar{g}(\bar{\nabla}_E \mathbf{T})_X Y, F) + \bar{g}(\mathbf{T}_E X, \mathbf{T}_F) \tag{52}$$
$$= \frac{c+1}{4}[\bar{g}(X,\varphi Y) - \bar{g}(\varphi X, Y)]\bar{g}(\overline{\varphi}E,F),$$

where $\Lambda_X = \mathbf{A}_X + \mathbf{A}^*_X$. We also produce

$$\bar{g}((\bar{\nabla}_X \mathbf{A})_Y E, Z) - \bar{g}(\bar{\nabla}_Y \mathbf{A})_X E, Z) + \bar{g}(\mathbf{T}^*_E Z, \Lambda_X Y) = 0, \tag{53}$$

similarly

$$\bar{g}((\bar{\nabla}_X \mathbf{A})_Y Z, E) - \bar{g}(\bar{\nabla}_Y \mathbf{A})_X Z, E) + \bar{g}(\mathbf{T}_E Z, \Lambda_X Y) = 0, \tag{54}$$

Now, from Theorem 2, we produce

$$\bar{g}(\hat{R}(X,Y)Z,W) - \bar{g}(\mathbf{A}_Y Z, \mathbf{A}^*_X W) + \bar{g}(\mathbf{A}_X Z, \mathbf{A}^*_X W) + \bar{g}(\Lambda_X Y, \mathbf{A}^*_Z W) \tag{55}$$
$$= \frac{c-3}{4}[\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)] + \frac{c-1}{4}[-\bar{g}(Y,\varphi Z)\bar{g}(\varphi X,W)$$
$$+ \bar{g}(X,\varphi Z)\bar{g}(\varphi Y,W) + (\bar{g}(X,\varphi Y) - \bar{g}(\varphi X,Y))]\bar{g}(\varphi Z,W),$$

for $E,F,G,H \in \mathcal{V}(\bar{\mathbf{M}})$ and $X,Y,Z,W \in \mathcal{H}(\bar{\mathbf{M}})$.

Using Lemma 5, and Theorems 4 and (55) together, we produce the following results:

Theorem 5. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s. Let the total manifold and base manifold be holds of the curvature tensor of the (11) kind with $c \in \mathbb{R}$ and (44) with $c-3$, respectively; then, rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$.*

Corollary 2. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s. If rank $dim(\mathbf{M}') = 1$ and the total manifold holds the curvature tensor of the (11) kind with $c \in \mathbb{R}$, the base manifold obeys the curvature tensor of the (44) kind with $c-3$.*

Once again, using Lemma 1 and Theorem 4, Equation (49) can be reconstructed as below:

$$\bar{g}((\bar{\nabla}_X)\mathbf{T})_E F, Y) - \bar{g}(\mathbf{T}_E X, \mathbf{T}^*_F Y)$$
$$= \frac{(c-3)}{4}[\bar{g}(X,Y)(\bar{g}(E,F) - \nu(E)\nu(F)) - \bar{g}(\varphi X, Y)\bar{g}(E,\overline{\varphi}F)];$$

thus, in light of Lemma 1, we obtain

$$\bar{g}(\bar{\nabla}_X N, Y) - \bar{g}(\mathbf{T}^* X, \mathbf{T}^* Y) = \frac{(c-3)}{4}[(s-1)\bar{g}(X,Y) - (trace(\overline{\varphi}))\bar{g}(\varphi X, Y)].$$

If $\mathcal{H}\bar{\nabla}_X N = 0$, then we obtain $c-3=0$ or $trace(\overline{\varphi}) = 0$. Thus one obtain

Theorem 6. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s and the total manifold holds the curvature tensor of kind (11) with c. Let the rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$ and $\mathcal{H}\bar{\nabla}_X N = 0$ for $X \in \mathcal{H}(\bar{\mathbf{M}})$. Then,*

1. *each fiber is totally geodesic submanifold of $\bar{\mathbf{M}}$ and the base manifold is flat if $c = 3$, such that the curvature holds the kind (11) with 3.*
2. *here $trace(\overline{\varphi}) = 0$ and $s > 1$,*
   - *(i) if g is positive definite, then $c - 3 \leq 0$,*
   - *(ii) $c-3 < 0$ and X is spacelike (timelike) or $c-3>0$ and X is timelike (spacelike) if and only if $\mathbf{T}^* X$ is spacelike (timelike),*

(iii) *horizontal vector X is null if and only if $\mathbf{T}^*X$ is null.*

Corollary 3. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s and the total manifold hold the curvature tensor of kind (11) with c. If rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$ and N is a constant, the result is identical to that of Theorem 7.*

In addition, (50) clearly shows that

$$\bar{g}((\bar{\nabla}^*_X)\mathbf{T}^*)_E F, Y) - \bar{g}(\mathbf{T}^*_E X, \mathbf{T}_F Y)$$
$$= \frac{(c-3)}{4}[\bar{g}(X,Y)(\bar{g}(E,F) - \nu(E)\nu(F)) - \bar{g}(\varphi X, Y)\bar{g}(E, \overline{\varphi} F)],$$

which implies that, from Lemma 1, we obtain

$$\bar{g}(\bar{\nabla}^*_X N^*, Y) - \bar{g}(\mathbf{T}X, \mathbf{T}Y) = \frac{(c-3)}{4}[(s-1)\bar{g}(X,Y) - (trace(\overline{\varphi}))\bar{g}(\varphi X, Y)].$$

If $\mathcal{H}\bar{\nabla}^*_X N^* = 0$, we obtain $c - 3 = 0$ or $trace(\overline{\varphi}) = 0$. Thus, we produce

Theorem 7. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s and the total manifold hold the curvature tensor of kind (11) with c. Let rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$ and $\mathcal{H}\bar{\nabla}^*_X N^* = 0$ for $X \in \mathcal{H}(\bar{\mathbf{M}})$. Then*

1. *each fiber is totally geodesic submanifold of $\bar{\mathbf{M}}$ and the base manifold is flat if $c - 3 = 0$, such that the curvature hold the (11) kind with 3.*
2. *in the case of $trace(\overline{\varphi}) = 0$ and $s > 1$,*
   - (i) *if g is positive definite, then $c - 3 \leq 0$,*
   - (ii) *$c - 3 < 0$ and X is spacelike (timelike) or $c - 3 > 0$ and X is timelike (spacelike) if and only if $\mathbf{T}X$ is spacelike (timelike),*
   - (iii) *horizontal vector X is null if and only if $\mathbf{T}X$ is null.*

Corollary 4. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s and the total manifold hold the curvature tensor of (11) kind with c. Let the rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$ and N^* is constant, the result is identical to that of Theorem 7.*

7. Kenmotsu-like Statistical Submersion with Conformal Fibers

This section is devoted to the *K.l.s.s* with conformal fibers (*CFs*).

Let us assume that ω, like a *SS*, admits *CF*. For $E, F \in \mathcal{V}(\bar{\mathbf{M}})$ if $\mathbf{T}_E F = 0$ ($\mathbf{T}_E F = \frac{1}{s}\bar{g}(E,F)N$) satisfies, then ω is *SS* with isometric fibers (*CF*). Then, from Lemma 1, we can obtain $\mathbf{T}_E\xi = 0$.

Lemma 7. *If $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s with CFs; then, ω has isometric fibers.*

Theorem 8. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s with CFs. Let the total manifold and each fiber that is a totally geodesic submanifold of $\bar{\mathbf{M}}$ hold the curvature tensor of the (11) kind with c.*

Theorem 9. *Let $\omega : (\bar{\mathbf{M}}, \bar{\nabla}, \bar{g}) \longrightarrow (\mathcal{N}, \hat{\nabla}, \hat{g})$ be a K.l.s.s with CFs and the total manifold hold the curvature tensor of the (11) kind with c. Let the rank $(\overline{\varphi} + \overline{\varphi}^*) = dim(\mathbf{M}') - 1$; then,*

1. *if $c = 3$ the total manifold satisfies the (11) kind;*
2. *the base manifold is flat;*
3. *if $c = 3$, each fiber holds the (11) kind.*

Example 4. *Let* $(\bar{\mathbf{M}} = \mathbb{R} \times_f \mathbf{B}^2, \bar{\nabla}, \bar{g} = dt^2 + f^2 g_{\mathbf{B}})$ *be a K.l.s.m obtained in Example 1. Then, the K.l.s.s*

$$\omega : \bar{\mathbf{M}} \longrightarrow (\mathbb{R}^2, \bar{\nabla}^{\mathbb{R}^2}, g_{\mathbb{R}^2})$$

as the projection mapping is defined by

$$\omega(t, x_1, x_2) = (x_1, x_2).$$

From this, $\mathcal{V}(\bar{\mathbf{M}}) =< \frac{\partial}{\partial t} >$ *and* $\mathcal{H}(\bar{\mathbf{M}}) =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} >$. *It is easy to verify that* $dim(\mathbf{M}') = 1$ *and* $\mathbf{A} = 0$. *Hence,* $\mathcal{H}(\bar{\mathbf{M}})$ *is integrable with respect to* $\bar{\nabla}$.

Example 5. *Let* $\left(\mathbb{R} \times_f \mathbb{R}^4, \widehat{\nabla}, \widehat{g} = dt^2 + f^2 g_{\mathbb{R}^4}\right)$ *be the* $\beta-$*K.l.s.m given in Example 2. Next, we describe the* $\beta-$ *K.l.s.s* $F : \left(\mathbb{R} \times_f \mathbb{R}^4, \widehat{\nabla}, \widehat{g}\right) \to \left(\mathbb{R}^4, \nabla^{\mathbb{R}^4}, g_{\mathbb{R}^4}\right)$ *as the projective mapping*

$$F(t, x_1, x_2, y_1, y_2) = (x_1, x_2, y_1, y_2).$$

Then, we produce $\mathcal{V}(M) =< \frac{\partial}{\partial t} >$ *and* $\mathcal{H}(M) =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} >$. *It is trivial that* $\dim \overline{M} = 1$. *Since* $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \in \mathcal{H}(M)$, *we obtain* $A = 0$.

Example 6. *Let* $\left(\mathbb{R} \times_f \bar{\mathbf{M}}, \nabla, g = dt^2 + f^2 \bar{g}\right)$ *be the* $\beta-$*K.l.s.m given in Example 3. Next we describe the* $\beta-$*K.l.s.s* $F : \left(\mathbb{R} \times_f \bar{\mathbf{M}}, \nabla, g\right) \to \left(\bar{\mathbf{M}}, \bar{g}, \bar{\nabla}^{(1)}\right)$ *as the projective mapping.*

$$F(x, y) = y.$$

8. Discussion

This subject is from differential geometry, which is a traditional yet very active branch of pure mathematics with notable applications in a number of areas of physics. Until recently, applications in the theory of statistics were fairly limited, but within the last few years, there has been intensive interest in the subject. So, the geometric study of *SS* is new and has many research problems.

In this discourse, we defined *K.l.s.s* and exhibited that, for a *K.l.s.s*, the base manifold is a Kähler-like statistical manifold, and the fibers are *K.l.s.m*. Moreover, we characterized the total space and the base space of such submersions. We presented a *K.l.s.s* along conformal fibers having isometric fibers. Using these results, different spaces can be studied for these issues, and many new relationships between intrinsic and extrinsic curvatures can be discussed.

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Abbreviations

Kenmotsu-like statistical manifold: $K.l.s.m$, Kenmotsu-like statistical submersion: $K.l.s.s.$, Riemannian submersion; RS, statistical submersion: SS, conformal fibers: CFs.

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