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Magnetic Curves in C-manifolds

Şaban Güvenç*

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ABSTRACT

In this paper, we study normal magnetic curves in C-manifolds. We prove that magnetic trajectories with respect to the contact magnetic fields are indeed θ_α-slant curves with certain curvature functions. Then, we give the parametrizations of normal magnetic curves in $\mathbb{R}^{2n+s}$ with its structures as a C-manifold.

Keywords: *C-manifold, magnetic curve θ_α-slant curve.*

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1. Introduction

Let (M, g) be a Riemannian manifold, F a closed 2-form and let us denote the Lorentz force on M by Φ, which is a $(1,1)$-type tensor field. If F is associated by the relation

$$g(\Phi X, Y) = F(X, Y), \ \ \forall X, Y \in \chi(M), \tag{1.1}$$

then it is called a *magnetic field* ([1], [2] and [5]). Let ∇ be the Riemannian connection associated to the Riemannian metric g and $\gamma : I \to M$ a smooth curve. If γ satisfies the Lorentz equation

$$\nabla_{\gamma'(t)}\gamma'(t) = \Phi(\gamma'(t)), \tag{1.2}$$

then it is called a *magnetic curve* or a *trajectory* for the magnetic field F. The Lorentz equation can be considered as a generalization of the equation for geodesics. Magnetic trajectories have constant speed. If the speed of the magnetic curve γ is equal to 1, then it is called a *normal magnetic curve* [6]. For fundamentals of almost contact metric manifolds, we refer to Blair's book [4]. This paper is based on a similar idea of Ozgur and the present author's previous paper [7].

2. Preliminaries

Let $\left(M^{2n+s}, g\right)$ be a differentiable manifold, φ a $(1,1)$-type tensor field, η^α 1-forms, ξ_α vector fields for $\alpha = 1, 2, ..., s$, satisfying

$$\varphi^2 X = -X + \sum_{\alpha=1}^{s} \eta^\alpha (X) \xi_\alpha, \tag{2.1}$$

$$\eta^\alpha (\xi_\beta) = \delta^\alpha_\beta, \ \varphi\xi_\alpha = 0, \ \eta^\alpha (\varphi X) = 0, \ \eta^\alpha (X) = g (X, \xi_\alpha),$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

where $X, Y \in TM$. Then $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is called *framed φ-structure* and $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called *framed φ-manifold.* The *fundamental 2-form* and *Nijenhuis tensor* is given by:

$$\Omega(X, Y) = g (X, \varphi Y),$$

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*** Corresponding author***

$$N_{\varphi}(X,Y) = -2\sum_{\alpha=1}^{s} d\eta^{\alpha}(X,Y)\,\xi_{\alpha}.$$

If $d\Omega = 0$ and $d\eta^{\alpha} = 0$, $M = (M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is called a C-manifold. In a C-manifold, it is known that

$$(\nabla_X \varphi)\,Y = 0$$

and

$$\nabla_X \xi_{\alpha} = 0,$$

(see [3] and [4]).

3. Magnetic Curves in C-manifolds

Let $\gamma : I \to M$ be a unit-speed curve in an n-dimensional Riemannian manifold (M, g). The curve γ is called a *Frenet curve of osculating order* r $(1 \le r \le n)$, if there exists orthonormal vector fields $T, v_2, ..., v_r$ along the curve validating the Frenet equations

$$\begin{aligned} T &= \gamma' = v_1, \\ \nabla_T T &= \kappa_1 v_2, \\ \nabla_T v_2 &= -\kappa_1 v_1 + \kappa_2 v_3, \\ &\ \ \cdots \\ \nabla_T v_r &= -\kappa_{r-1} v_{r-1}, \end{aligned} \tag{3.1}$$

where $\kappa_1, ..., \kappa_{r-1}$ are positive functions called the curvatures of γ. If $\kappa_1 = 0$, then γ is called a *geodesic*. If κ_1 is a non-zero positive constant and $r = 2$, γ is called a *circle*. If $\kappa_1, ..., \kappa_{r-1}$ are non-zero positive constants, then γ is called a *helix of order* r $(r \ge 3)$. If $r = 3$, it is shortly called a *helix*.

A submanifold of a C-manifold is said to be an *integral submanifold* if $\eta^{\alpha}(X) = 0,\ \alpha \in \{1, 2, ..., s\}$, where X is tangent to the submanifold. A *Legendre curve* is a 1-dimensional integral submanifold of a C-manifold $(M^{2n+s}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$. More precisely, a unit-speed curve $\gamma : I \to M$ is a Legendre curve if T is g-orthogonal to all ξ_{α} $(\alpha = 1, 2, ...s)$, where $T = \gamma'$.

Definition 3.1. Let γ be a unit-speed curve in a C-manifold $(M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$. γ is called a $\theta_{\alpha}-$slant curve if there exist constant contact angles such that $\eta^{\alpha}(T) = \cos\theta_{\alpha}$, $\alpha = 1, 2, ..., s$. If $\theta_{\alpha} = \theta$ for all $\alpha = 1, 2, ..., s$, then γ is shortly called slant. Moreover, if $\theta_{\alpha} = \frac{\pi}{2}$ for all $\alpha = 1, 2, ..., s$, then γ is called a Legendre curve.

For $\theta_{\alpha}-$slant curves, we can give the following inequality for the constant contact angles:

$$\sum_{\alpha=1}^{s} \cos^2\theta_{\alpha} \le 1.$$

The equality case is only valid when γ is a geodesic as an integral curve of $\pm \sum\limits_{\alpha=1}^{s} \cos\theta_{\alpha}\xi_{\alpha}$.

Let γ be a unit-speed Legendre curve in a C-manifold $(M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$. If we differentiate $\eta^{\alpha}(T) = 0$, we obtain $\eta^{\alpha}(v_2) = 0$. We can continue this process until we find $\eta^{\alpha}(v_r) = 0$. Thus, we can state the following proposition:

Proposition 3.1. *If γ is a unit-speed Legendre curve in a C-manifold $(M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$, then ξ_{α} is g-orthogonal to sp $\{T, v_2, ..., v_r\}$, for all $\alpha = 1, 2, ..., s$.*

If we consider equations (1.1), (1.2) and (3.1) together, for a normal magnetic curve of a magnetic field F with charge q, we find

$$\nabla_T T = \Phi T,$$

$$F(X,Y) = g(\Phi X, Y),$$

$$\begin{aligned} F_q(X,Y) &= q\Omega(X,Y) \\ &= qg(X, \varphi Y), \end{aligned}$$

which gives us

$$\Phi_q = -q\varphi.$$

Here, T denotes the tangential vector field of the normal magnetic curve γ for the magnetic field F_q in M. Then, we have the following equations:

$$\nabla_T T = -q\varphi T, \tag{3.2}$$

$$\nabla_T \xi_\alpha = 0,$$

$$\begin{aligned} \nabla_T \varphi T &= (\nabla_T \varphi) T + \varphi \nabla_T T \\ &= \varphi(-q\varphi T) \\ &= -q\varphi^2 T \\ &= -q\left(-T + \sum_{\alpha=1}^{s} \eta^\alpha(T)\xi_\alpha\right) \\ &= qT - q\sum_{\alpha=1}^{s} \eta^\alpha(T)\xi_\alpha. \end{aligned}$$

If we take the inner product of equation (3.2) with ξ_α, we obtain

$$\begin{aligned} 0 &= g(-q\varphi T, \xi_\alpha) = g(\nabla_T T, \xi_\alpha) \\ &= \frac{d}{dt} g(T, \xi_\alpha). \end{aligned}$$

Integrating both sides, we get

$$\eta^\alpha(T) = \cos\theta_\alpha = constant,$$

for all $\alpha = 1, 2, ..., s$. Equations (3.1) and (3.2) give us

$$\nabla_T T = \kappa_1 v_2 = -q\varphi T, \tag{3.3}$$

$$\begin{aligned} g(\varphi T, \varphi T) &= g(T, T) - \sum_{\alpha=1}^{s} (\eta^\alpha(T))^2 \\ &= 1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha \end{aligned}$$

and

$$\|\varphi T\| = \sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}.$$

From equation (3.3), we find

$$\kappa_1 = |q|\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha} = constant, \tag{3.4}$$

$$-q\varphi T = \kappa_1 v_2 = |q|\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha v_2}$$

and

$$\varphi T = -sgn(q)\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha v_2}. \tag{3.5}$$

If $\kappa_2 = 0$, then $r = 2$ and γ is a circle. If we apply η^α to equation (3.5), we obtain

$$\eta^\alpha(v_2) = 0,$$

which gives us

$$\begin{aligned}\nabla_T \eta^\alpha(v_2) &= 0\\ &= g(\nabla_T v_2, \xi_\alpha) + g(T, \nabla_T \xi_\alpha)\\ &= -\kappa_1 \cos\theta_\alpha.\end{aligned}$$

As a result, we get $\cos\theta_\alpha = 0$, for all $\alpha = 1, 2, ..., s$. Hence, γ is a Legendre circle, $\|\varphi T\| = 1$ and $\kappa_1 = |q|$. Let $\kappa_2 \neq 0$. Using equations (2.1) and (3.1), we calculate

$$\begin{aligned}\nabla_T \varphi T &= (\nabla_T \varphi) T + \varphi \nabla_T T\\ &= \varphi(-q\varphi T)\\ &= -q\left(-T + \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha\right).\end{aligned} \tag{3.6}$$

Differentiating equation (3.5), we also have

$$\nabla_T \varphi T = -sgn(q)\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}\,(-\kappa_1 T + \kappa_2 v_3) \tag{3.7}$$

In view of (3.4), (3.6) and (3.7), it is easy to see that

$$q\left[\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha - \left(\sum_{\alpha=1}^{s} \cos^2\theta_\alpha\right) T\right] = sgn(q)\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}\,\kappa_2 v_3. \tag{3.8}$$

Note that

$$g(T,T) = 1,\ g\left(T, \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha\right) = \sum_{\alpha=1}^{s} \cos^2\theta_\alpha,$$

$$g\left(\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha, \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha\right) = \sum_{\alpha=1}^{s} \cos^2\theta_\alpha,\ g(v_3, v_3) = 1.$$

So, if we calculate the norm of both sides of equation (3.8), we get

$$\kappa_2 = |q|\sqrt{\sum_{\alpha=1}^{s} \cos^2\theta_\alpha}. \tag{3.9}$$

If we write (3.9) in (3.8), we have

$$\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha = \left(\sum_{\alpha=1}^{s} \cos^2\theta_\alpha\right) T + \sqrt{\sum_{\alpha=1}^{s} \cos^2\theta_\alpha}\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}\, v_3 \tag{3.10}$$

If we differentiate (3.10), we find $\kappa_3 = 0$. From equations (3.5) and (3.10), we can write

$$v_2 = \frac{-sgn(q)}{\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}}\varphi T \tag{3.11}$$

$$v_3 = \frac{1}{\sqrt{\sum_{\alpha=1}^{s} \cos^2\theta_\alpha}\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}}\left(\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha - \left(\sum_{\alpha=1}^{s} \cos^2\theta_\alpha\right) T\right) \tag{3.12}$$

Finally, if $\kappa_1 = 0$, after some calculations, by (2.1) and (3.5), we obtain $T = \pm \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha$, where $\sum_{\alpha=1}^{s} \cos^2\theta_\alpha = 1$. So, we can give the following theorem:

Theorem 3.1. *Let $\gamma : I \to M = (M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a unit-speed curve in a C-manifold M. Then γ is a normal magnetic curve for F_q $(q \neq 0)$ in M if and only if*

i) γ is a geodesic $\theta_\alpha-$slant curve as an integral curve of $\pm \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha$, where $\sum_{\alpha=1}^{s} \cos^2\theta_\alpha = 1$; or

ii) γ is a Legendre circle with $\kappa_1 = |q|$ having the Frenet frame field

$$\{T, -sgn(q)\varphi T\};$$

or

iii) γ is a non-Legendre $\theta_\alpha-$slant helix with

$$\kappa_1 = |q| \sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha},$$

$$\kappa_2 = |q| \sqrt{\sum_{\alpha=1}^{s} \cos^2\theta_\alpha},$$

having the Frenet frame field

$$\{T, v_2, v_3\},$$

where $\sum_{\alpha=1}^{s} \cos^2\theta_\alpha < 1$, v_2 and v_3 are given in equations (3.11) and (3.12), respectively.

Corollary 3.1. *If γ is a unit-speed slant curve in a C-manifold M, then it is a normal magnetic curve if and only if*

i) it is a geodesic as an integral curve of $\frac{\pm 1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha$; or

ii) γ is a Legendre circle with $\kappa_1 = |q|$ having the Frenet frame field

$$\{T, -sgn(q)\varphi T\};$$

or

iii) γ is a non-Legendre slant helix with $\kappa_1 = |q| \sqrt{1 - s\cos^2\theta}$, $\kappa_2 = |q| \sqrt{s}\varepsilon\cos\theta$, having the Frenet frame field

$$\left\{T, \frac{-sgn(q)}{\sqrt{1 - s\cos^2\theta}}\varphi T, \frac{\varepsilon}{\sqrt{s}\sqrt{1 - s\cos^2\theta}}\left(\sum_{\alpha=1}^{s} \xi_\alpha - s\cos\theta T\right)\right\},$$

where $\theta \neq \frac{\pi}{2}$ is the contact angle satisfying $|\cos\theta| < \frac{1}{\sqrt{s}}$ and $\varepsilon = sgn(\cos\theta)$.

Proof. Since $\theta_\alpha = \theta$ for all $\alpha = 1, 2, ..., s$, if we use

$$\sum_{\alpha=1}^{s} \cos^2\theta_\alpha = s\cos^2\theta$$

and

$$\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha = \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha$$

in Theorem 3.1, the proof is clear. □

Remark. If we take $s = 1$, we have Proposition 1 in [8].

Let $M = (M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a C-manifold. A Frenet curve of order $r = 2$ is called a φ-curve in M if $sp\{T, v_2, \xi_1, ..., \xi_s\}$ is a $\varphi-$invariant space. A Frenet curve of order $r \geq 3$ is called a φ-curve if $sp\{T, v_2, ..., v_r\}$ is $\varphi-$invariant. A $\varphi-$helix of order r is a $\varphi-$curve with constant curvatures $\kappa_1, ..., \kappa_{r-1}$. A $\varphi-$helix of order 3 is shortly named a $\varphi-$helix.

Proposition 3.2. *If γ is a Legendre $\varphi-$helix in a C-manifold M, then it is a Legendre $\varphi-$circle.*

Proof. Let γ be a Legendre $\varphi-$helix. Then the contact angles $\theta_\alpha = \frac{\pi}{2}$ for all $\alpha = 1, 2, ..., s$ and the Frenet frame field $\{T, v_2, v_3\}$ is $\varphi-$invariant. Since γ is Legendre, we have $g(\varphi T, \varphi T) = 1$. Thus, we can write

$$g(\varphi T, v_2) = \cos\mu, \tag{3.13}$$

$$\varphi T = \cos\mu v_2 \pm \sin\mu v_3, \tag{3.14}$$

for some function $\mu = \mu(t)$. If we differentiate equation (3.13), we find

$$\begin{aligned} -\mu' \sin\mu &= \kappa_2 g(\varphi T, v_3) \\ &= \pm\kappa_2 \sin\mu. \end{aligned} \tag{3.15}$$

Firstly, let us assume that $\mu = 0$, i.e. $\varphi T = v_2$. Since γ is a Legendre curve, applying φ to $\varphi T = v_2$, we obtain $\varphi^2 T = -T = \varphi v_2$. Differentiating both sides of $\varphi T = v_2$, we also have

$$\nabla_T \varphi T = \nabla_T v_2,$$

$$(\nabla_T \varphi) T + \varphi \nabla_T T = -\kappa_1 T + \kappa_2 v_3,$$

$$\kappa_1 \varphi v_2 = -\kappa_1 T + \kappa_2 v_3,$$

$$-\kappa_1 T = -\kappa_1 T + \kappa_2 v_3,$$

which is equivalent to $\kappa_2 = 0$. Likewise, if $\mu = \pi$, we obtain $\kappa_2 = 0$. Finally, let us assume that $\mu \neq 0, \pi$. In this case, since γ is a helix, using (3.15), we have

$$\kappa_1 = constant,$$

$$\kappa_2 = \mp\mu' = constant.$$

If we differentiate (3.14) and use $\kappa_2 = \mp\mu'$, we calculate

$$\kappa_1 \varphi v_2 = -\kappa_1 \cos\mu T.$$

If we apply φ to both sides, we conclude $\varphi T = \pm v_2$, which gives $\kappa_2 = 0$. This completes the proof. □

Remark. For $s = 1$, we obtain Proposition 2 of [8]. Likewise, the following theorem generalizes Theorem 1 of [8] to C-manifolds:

Theorem 3.2. *Let γ be a $\varphi-$helix of order $r \leq 3$ in a C-manifold $M = (M, \varphi, \xi_\alpha, \eta^\alpha, g)$. Then, the following statements are valid:*

i) If $\cos\theta_\alpha$ $(\alpha = 1, 2, ..., s)$ are constants such that $\sum\limits_{\alpha=1}^{s} \cos^2\theta_\alpha = 1$, then γ is an integral curve of $\pm \sum\limits_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha$, hence it is a normal magnetic curve for arbitrary q.

ii) If $\cos\theta_\alpha = 0$ for all $\alpha = 1, 2, ..., s$, i.e. γ is a Legendre $\varphi-$curve, then it is a magnetic circle generated by the magnetic field $F_{\pm\kappa_1}$.

iii) If $\cos\theta_\alpha$ $(\alpha = 1, 2, ..., s)$ are constants such that $\sum\limits_{\alpha=1}^{s} \cos^2\theta_\alpha = \frac{\kappa_2^2}{\kappa_1^2+\kappa_2^2}$, then γ is a magnetic curve for $F_{\pm\sqrt{\kappa_1^2+\kappa_2^2}}$.

iv) Except above cases, γ cannot be a magnetic curve for any magnetic field F_q.

Proof. In view of Theorem 3.1 and Proposition 3.2, it is straightforward to show that $\nabla_T T = -q\varphi T$ for valid q. □

4. Magnetic Curves of $\mathbb{R}^{2n+s}$ with its structures as a C-manifold

In this section, we consider parameterizations of normal magnetic curves in $M = \mathbb{R}^{2n+s}$ as a C-manifold. Let $\{x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_s\}$ be the coordinate functions and define

$$X_i = \frac{\partial}{\partial x_i},\ Y_i = \frac{\partial}{\partial y_i},\ \xi_\alpha = \frac{\partial}{\partial z_\alpha},$$

for $i = 1, ..., n$ and $\alpha = 1, 2, ..., s$. $\{X_i, Y_i, \xi_\alpha\}$ is an orthonormal basis of $\chi(M)$ with respect to the usual metric

$$g = \sum_{i=1}^{n} \left[(dx_i)^2 + (dy_i)^2 \right] + \sum_{\alpha=1}^{s} (dz_\alpha)^2 .$$

Let us define a $(1,1)$-type tensor field φ as

$$\varphi X_i = -Y_i, \ \varphi Y_i = X_i, \ \varphi \xi_\alpha = 0.$$

Finally, let $\eta^\alpha = dz_\alpha$ for $\alpha = 1, 2, ..., s$. It is well-known that $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is a C-manifold, since $d\eta^\alpha = 0$ and $d\Omega = 0$, where $\Omega(X, Y) = g(X, \varphi Y)$ for all $X, Y \in \chi(M)$ (see [3] and [4]).

Let us denote normal magnetic curve by

$$\gamma = (\gamma_1, ..., \gamma_n, \gamma_{n+1}, ..., \gamma_{2n}, \gamma_{2n+1}, ..., \gamma_{2n+s}) .$$

Then

$$T = \gamma' = \left(\gamma_1', ..., \gamma_n', \gamma_{n+1}', ..., \gamma_{2n}', \gamma_{2n+1}', ..., \gamma_{2n+s}'\right) ,$$

which gives us

$$\nabla_T T = \left(\gamma_1'', ..., \gamma_n'', \gamma_{n+1}'', ..., \gamma_{2n}'', \gamma_{2n+1}'', ..., \gamma_{2n+s}''\right) ,$$
$$\varphi T = \left(\gamma_{n+1}', ..., \gamma_{2n}', -\gamma_1', ..., -\gamma_n', 0, ..., 0\right) .$$

Since

$$\nabla_T T = -q\varphi T,$$

we have

$$\eta^\alpha (T) = \gamma_{2n+\alpha}' = \cos\theta_\alpha = constant$$

and

$$\gamma_{2n+\alpha} = \cos\theta_\alpha t + h_\alpha.$$

We also get

$$\gamma_i'' = -q\gamma_{n+i}', \tag{4.1}$$
$$\gamma_{n+i}'' = q\gamma_i' \tag{4.2}$$

for $i = 1, ..., n$. As a result, we obtain

$$\gamma_i'\gamma_i'' + \gamma_{n+i}'\gamma_{n+i}'' = 0,$$

i.e.

$$(\gamma_i')^2 + \left(\gamma_{n+i}'\right)^2 = c_i^2.$$

Since γ is unit-speed, that is $g(T, T) = 1$, we have

$$\sum_{i=1}^{n} c_i^2 + \sum_{\alpha=1}^{s} \cos^2\theta_\alpha = 1.$$

If we consider differentiable functions $f_i : I \to \mathbb{R}$, we can write

$$\gamma_i' = c_i \cos f_i, \tag{4.3}$$

$$\gamma_{n+i}' = c_i \sin f_i. \tag{4.4}$$

Then, we have

$$\gamma_i'' = -c_i f_i' \sin f_i, \tag{4.5}$$

$$\gamma_{n+i}'' = c_i f_i' \cos f_i. \tag{4.6}$$

If we write (4.4) and (4.5) in (4.1), or likewise (4.3) and (4.6) in (4.2), we find

$$-c_i f_i' \sin f_i = -qc_i \sin f_i \tag{4.7}$$

$$c_i f_i' \cos f_i = qc_i \cos f_i. \tag{4.8}$$

Let us analyze equations (4.7) and (4.8):

i) If $c_i \neq 0$, $\sin f_i \neq 0$ and $\cos f_i \neq 0$, $\forall i$, then we have $f_i' = q$, that is,

$$f_i(t) = qt + d_i.$$

Hence, we find

$$\gamma_i = \frac{c_i}{q} \sin(qt + d_i) + b_i,$$

$$\gamma_{n+i} = \frac{-c_i}{q} \cos(qt + d_i) + b_{n+i}.$$

ii) If $c_i = 0$, $\exists i$, then (4.3) and (4.4) give us $\gamma_i' = c_i = 0$ and $\gamma_{n+i}' = c_i = 0$, respectively. So we have $\gamma_i = b_i$ and $\gamma_{n+i} = b_{n+i}$, which can also be obtained from above parameterization by writing $c_i = 0$.

iii) If $\sin f_i = 0, \exists i$, then $f_i = k\pi, (k \in \mathbb{Z})$, which is a constant, so $\cos f_i = \pm 1$. Thus (4.8) gives $c_i = 0$, since $q \neq 0$ and $f_i' = 0$. So, this is the same as Case ii).

iv). If $\cos f_i = 0, \exists i$, then $f_i = \frac{\pi}{2} + k\pi, (k \in \mathbb{Z})$, which is a constant, so $\sin f_i = \pm 1$. Therefore (4.7) gives $c_i = 0$, since $q \neq 0$ and $f_i' = 0$. This is again the same as Case ii).

As a result, we can give all four cases in one parameterization and state the following theorem:

Theorem 4.1. *The normal magnetic curves on $\mathbb{R}^{2n+s}$ satisfying the Lorentz equation $\nabla_T T = -q\varphi T$ have the parametric equations*

$$\gamma_i = \frac{c_i}{q} \sin(qt + d_i) + b_i,$$

$$\gamma_{n+i} = \frac{-c_i}{q} \cos(qt + d_i) + b_{n+i},$$

$$\gamma_{2n+\alpha} = \cos\theta_\alpha t + h_\alpha,$$

where $i = 1, ..., n$, $\alpha = 1, 2, ..., s$, b_i, b_{n+i}, d_i, h_α are arbitrary constants, θ_α are the constant contact angles and c_i are arbitrary constants satisfying

$$\sum_{i=1}^{n} c_i^2 = 1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha \geq 0.$$

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Affiliations

ŞABAN GÜVENÇ

ADDRESS: Balikesir University, Dept. of Mathematics, 10145, Balikesir-Turkey.

E-MAIL: sguvenc@balikesir.edu.tr

ORCID ID: orcid.org/0000-0001-6254-4693