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Some Fixed-Disc Results in Double Controlled Quasi-Metric Type Spaces

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Abstract: In this paper, we introduce new types of general contractions for self mapping on double controlled quasi-metric type spaces, where we prove the existence and uniqueness of fixed disc and circle for such mappings.

Keywords: double controlled quasi-metric spaces; fixed disc; fixed circle

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1. Introduction

Lately, proving the existence of a fixed circle or a fixed disk on metric spaces or on a generalized form of a metric space has been the focus of many researchers (see [1–5]). For instance, in [1], they proved the existence of a fixed circle for the Caristi-type contraction on a regular metric spaces. Further, adopting the techniques of the Wardowski contractive mapping, the authors in [5] were able to prove some interesting fixed-circle theorems. In [2,3], the existence of a fixed-circle problem was investigated on the three dimensions of space, the so-called S-metric space. In [6], the authors also proved some new fixed-circle results for mappings that satisfies the modified Khan-type contraction on S-metric spaces. Some generalized fixed-circle results with geometric viewpoint were obtained on S_b-metric spaces and parametric N_b-metric spaces (see [7,8]). Moreover, it was an open question to study the existence of a fixed circle on extended M_b-metric spaces [9]. Further, an application of the obtained fixed-circle results was given to discontinuous activation functions on metric spaces (see [1,4,10]).

One of the most useful tools to show that a fractional differential equation has a solution is a fixed-point theory, see [11,12]; however, the existence of a fixed point leads us to conclude that such types of equations have a solution, but in some cases a map has a fixed point but it is not necessarily unique. So, in a case where we have more than one fixed point, what can we say about the set of a fixed point? In this manuscript, we are interested in the type of mapping that the set of fixed points is a disc or a circle. Of course the shape of a circle or a disc varies from one metric space to another. For our purposes, we study the existence and uniqueness of fixed circle and disc in double controlled quasi-metric type spaces. In the next section, we present some preliminaries.

2. Preliminaries

Definition 1 ([13])**.** *Consider the set $\mathbb{B} \neq \emptyset$. Given non-comparable functions $\mathbb{K}, \mathbb{L} : \mathbb{B} \times \mathbb{B} \to [1, \infty)$. If $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, \infty)$ satisfies*

$(\mathbb{M}1)$ $\mathbb{M}(x, \upsilon) = 0 \Leftrightarrow x = \upsilon$,

$(\mathbb{M}2)$ $\mathbb{M}(x, \upsilon) \leq \mathbb{K}(x, \beta)\mathbb{M}(x, \beta) + \mathbb{L}(\beta, \upsilon)\mathbb{M}(\beta, \upsilon)$,

for all $x, \upsilon, \beta \in \mathbb{B}$. Then $\mathbb{M}$ is called a double controlled quasi-metric type with the functions $\mathbb{K}$, $\mathbb{L}$ and $(\mathbb{B}, \mathbb{M})$ is a double controlled quasi-metric type space.

Throughout the rest of this manuscript we denote double controlled quasi-metric type space by $(DCQMS)$.

Example 1 ([13]). *Let $\mathbb{B} = \{0, 1, 2\}$. Define $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, \infty)$ by*

$$\begin{aligned} \mathbb{M}(0,1) &= 4, \mathbb{M}(0,2) = 1, \\ \mathbb{M}(1,0) &= \mathbb{M}(1,2) = 3, \\ \mathbb{M}(2,0) &= 0, \mathbb{M}(2,1) = 2, \\ \mathbb{M}(0,0) &= \mathbb{M}(1,1) = \mathbb{M}(2,2) = 0. \end{aligned}$$

Then $\mathbb{M}$ is a double controlled quasi-metric type with the functions $\mathbb{K}, \mathbb{L} : \mathbb{B} \times \mathbb{B} \to [1, \infty)$ defined as

$$\begin{aligned} \mathbb{K}(0,1) &= \mathbb{K}(1,0) = \mathbb{K}(1,2) = 1, \\ \mathbb{K}(0,2) &= \frac{5}{4}, \mathbb{K}(2,0) = \frac{10}{9}, \mathbb{K}(2,1) = \frac{20}{19}, \\ \mathbb{K}(0,0) &= \mathbb{K}(1,1) = \mathbb{K}(2,2) = 1 \end{aligned}$$

and

$$\begin{aligned} \mathbb{L}(0,1) &= \mathbb{L}(1,0) = \mathbb{L}(0,2) = \mathbb{L}(1,2) = 1, \\ \mathbb{L}(2,0) &= \frac{3}{2}, \mathbb{L}(2,1) = \frac{11}{8}, \\ \mathbb{L}(0,0) &= \mathbb{L}(1,1) = \mathbb{L}(2,2) = 1. \end{aligned}$$

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers, and $\mathbb{N}$ represents the set of all positive integers.

Example 2. *Let $\mathbb{B} = l_1$ be defined by*

$$l_1 = \left\{ \{\varpi_n\}_{n \geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} |\varpi_n| < \infty \right\}.$$

Consider $\mathbb{M} : \mathbb{B} \times \mathbb{B} \to [0, \infty)$ such that

$$\mathbb{M}(\eta, \varpi) = \sum_{n=1}^{\infty} (\varpi_n - \eta_n)^+$$

where $\alpha^+ := \max\{\alpha, 0\}$ denotes the positive part of a number $\alpha \in \mathbb{R}$., and $\varpi = \{\varpi_n\}$ and $\eta = \{\eta_n\}$ are in $\mathbb{B}$. Further, let $\mathbb{K}(\varpi, \eta) = \max\{\varpi, \eta\} + 2$ and $\mathbb{L}(\varpi, \eta) = \max\{\varpi, \eta\} + 3$.
Note that $(\mathbb{B}, \mathbb{M})$ is a $(DCQMS)$ with control functions $\mathbb{K}$, $\mathbb{L}$.

Now, we remind the reader of the topological properties of $(DCQMS)$.

Definition 2. *Let $(\mathbb{B}, \mathbb{M})$ be a $(DCQMS)$, $\{\varpi_n\}$ be a sequence in $\mathbb{B}$ and $\varpi \in \mathbb{B}$. The sequence $\{\varpi_n\}$ converges to $\varpi \Leftrightarrow$*

$$\lim_{n \to \infty} \mathbb{M}(\varpi_n, \varpi) = \lim_{n \to \infty} \mathbb{M}(\varpi, \varpi_n) = 0. \quad (1)$$

Remark 1. *In a $(DCQMS)$, $(\mathbb{B}, \mathbb{M})$, the limit for a convergent sequence is unique. Further, if $\varpi_n \to \varpi$, we have for all $\eta \in \mathbb{B}$*

$$\lim_{n \to \infty} \mathbb{M}(\varpi_n, \eta) = \mathbb{M}(\varpi, \eta) \text{ and } \lim_{n \to \infty} \mathbb{M}(\eta, \varpi_n) = \mathbb{M}(\eta, \varpi).$$

In fact, $\varpi_n \to \varpi$ and $\eta_n \to \eta \Rightarrow \mathbb{M}(\varpi_n, \eta_n) \to \mathbb{M}(\varpi, \eta)$.

Definition 3 ([14]). *Let $(\mathbb{B}, \mathbb{M})$ be a $(DCQMS)$, and $\{\varpi_n\}$ be a sequence in $\mathbb{B}$. We say that $\{\varpi_n\}$ is right DCQ-Cauchy $\Leftrightarrow$ for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\mathbb{M}(\varpi_n, \varpi_m) < \varepsilon$ for all $n \geq m > N$.*

Definition 4 ([14]). *Let $(\mathbb{B}, \mathbb{M})$ be a $(DCQMS)$, and $\{\varpi_n\}$ be a sequence in $\mathbb{B}$. We say that $\{\varpi_n\}$ is left DCQ-Cauchy $\Leftrightarrow$ for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\mathbb{M}(\varpi_n, \varpi_m) < \varepsilon$ for all $m \geq n > N$.*

Definition 5. *Let $(\mathbb{B}, \mathbb{M})$ be a $(DCQMS)$, and $\{\varpi_n\}$ be a sequence in $\mathbb{B}$. We say that $\{\varpi_n\}$ is DCQ-Cauchy $\Leftrightarrow$ for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\mathbb{M}(\varpi_n, \varpi_m) < \varepsilon$ for all $m, n > N$.*

Remark 2. *A sequence $\{\varpi_n\}$ in a $(DCQMS)$, is DCQ-Cauchy $\Leftrightarrow$ it is right DCQ-Cauchy and left DCQ-Cauchy.*

Definition 6 ([15]). *Let $(\mathbb{B}, \mathbb{M})$ DCQMS.*
(1) $(\mathbb{B}, \mathbb{M})$ *is said left-complete $\Leftrightarrow$ each left DCQ-Cauchy sequence in $\mathbb{B}$ is convergent.*
(2) $(\mathbb{B}, \mathbb{M})$ *is said right-complete $\Leftrightarrow$ each right DCQ-Cauchy sequence in $\mathbb{B}$ is convergent.*
(3) $(\mathbb{B}, \mathbb{M})$ *is said complete $\Leftrightarrow$ each DCQ-Cauchy sequence in $\mathbb{B}$ is convergent.*

Remark 3. *If $\mathbb{M}$ is a DCQMS on $\mathbb{B}$, then $\overline{\mathbb{M}}(x, v) = \mathbb{M}(v, x)$ for all $x, v \in \mathbb{B}$ is another quasi-metric, called the conjugate of $\mathbb{M}$ and $\mathbb{M}^s(x, v) = \max\{\mathbb{M}(x, v), \overline{\mathbb{M}}(x, v)\}$ for all $x, v \in \mathbb{B}$ is a metric on $\mathbb{B}$. Moreover, we have*

1. $\varpi_n \to_{\mathbb{M}} \varpi \quad \Leftrightarrow \quad \lim_{n\to\infty} \mathbb{M}(\varpi, \varpi_n) = 0;$
2. $\varpi_n \to_{\overline{\mathbb{M}}} \varpi \quad \Leftrightarrow \quad \lim_{n\to\infty} \overline{\mathbb{M}}(\varpi, \varpi_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \mathbb{M}(\varpi_n, \varpi) = 0.$

   Further, note that

$$\begin{aligned} \varpi_n \to_{\mathbb{M}^s} \varpi \Leftrightarrow &\quad \lim_{n\to\infty} \mathbb{M}(\varpi, \varpi_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{M}(\varpi_n, \varpi) = 0 \\ \Leftrightarrow &\quad \varpi_n \to_{\mathbb{M}} \varpi \quad \text{and} \quad \varpi_n \to_{\overline{\mathbb{M}}} \varpi. \end{aligned}$$

   Hence, $\varpi_n \to_{\mathbb{M}} \varpi$ implies $\varpi_n \to_{\mathbb{M}^s} \varpi$.

Lemma 1. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS and $\mathbb{W} : \mathbb{B} \to \mathbb{B}$ be a self-mapping. Suppose that $\mathbb{W}$ is continuous at $\varpi \in \mathbb{B}$. Then for each sequence $\{\varpi_n\}$ in $\mathbb{B}$ such that $\varpi_n \to \varpi$, we have $\mathbb{W}\varpi_n \to \mathbb{W}\varpi$, that is,*

$$\lim_{n\to\infty} \mathbb{M}(\mathbb{W}\varpi_n, \mathbb{W}\varpi) = \lim_{n\to\infty} \mathbb{M}(\mathbb{W}\varpi, \mathbb{W}\varpi_n) = 0.$$

3. Main Results

One way to generalize the fixed-point results is to study the geometric properties of the set of fixed points when we do not have a unique fixed point. Let $(\mathbb{B}, \mathbb{M})$ be a *DCQMS*, $x_0 \in \mathbb{B}$ and $r > 0$. The upper closed ball of radius r centered x_0 and the lower closed ball of radius r centered x_0 are defined by,

$$\overline{B^+}(x_0, r) = \{x \in \mathbb{B} : \mathbb{M}(x, x_0) \leq r\}$$

and

$$\overline{B^-}(x_0, r) = \{x \in \mathbb{B} : \mathbb{M}(x_0, x) \leq r\},$$

Next, we present the definitions of a circle and a disc on a *DCQMS* $(\mathbb{B}, \mathbb{M})$: Let $r \geq 0$ and $x_0 \in \mathbb{B}$. The circle $C^{\mathbb{M}}_{x_0,r}$ and the disc $D^{\mathbb{M}}_{x_0,r}$ are

$$C^{\mathbb{M}}_{x_0,r} = \{x \in \mathbb{B} : \mathbb{M}(x_0, x) = \mathbb{M}(x, x_0) = r\}$$

and

$$D^{\mathbb{M}}_{x_0,r} = \overline{B^+}(x_0,r) \cap \overline{B^-}(x_0,r) = \{x \in \mathbb{B} : \mathbb{M}(x_0,x) \le r \text{ and } \mathbb{M}(x,x_0) \le r\}.$$

Notice that the disc $D^{\mathbb{M}}_{x_0,r}$ form a closed ball with respect to the associated metric $\mathbb{M}^s$. That is,

$$\mathbb{M}(x_0,x) \le r \text{ and } \mathbb{M}(x,x_0) \le r \Leftrightarrow \max\{\mathbb{M}(x_0,x),\mathbb{M}(x,x_0)\} \le r \Leftrightarrow \mathbb{M}^s(x,x_0) \le r.$$

Let $(\mathbb{B},\mathbb{M})$ be a *DCQMS* and $\mathbb{W}$ be a self-mapping on $\mathbb{B}$. Further, let

$$r = \inf\{\mathbb{M}(x,\mathbb{W}x) \mid \ x \in \mathbb{B},\ \mathbb{W}x \neq x\}. \tag{2}$$

3.1. DCQMS-$F_{\mathbb{M}}$-Contractions

In [16], Wardowski defined a new class of functions as follows.

Definition 7 ([16])**.** *Let $\mathbb{F}$ be the family of all functions $F : (0,\infty) \to \mathbb{R}$ such that*

(F_1) F is strictly increasing;

(F_2) For each sequence $\{\alpha_n\}$ in $(0,\infty)$, the following holds

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Next, we presnt the following contractive type of mappings.

Definition 8. *Let $(\mathbb{B},\mathbb{M})$ be a DCQMS, $\mathbb{W}$ a self-mapping on $\mathbb{B}$ and $F \in \mathbb{F}$. Then $\mathbb{W}$ is said to be a DCQMS-$F_{\mathbb{M}}$-contraction if there exist $t > 0$ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x,\mathbb{W}x) > 0 \Rightarrow t + F(\mathbb{M}(x,\mathbb{W}x)) \le F(\mathbb{M}(x_0,x)), \tag{3}$$

for each $x \in \mathbb{B}$.

Denote the set of fixed-points of a map $\mathbb{W}$ by $Fix(\mathbb{W})$.

Theorem 1. *Let $(\mathbb{B},\mathbb{M})$ be a DCQMS, $\mathbb{W}$ be a DCQMS-$F_{\mathbb{M}}$-contraction with $x_0 \in \mathbb{B}$ on $\mathbb{B}$ and r defined as in (2). Then we have $\overline{B^-}(x_0,r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. First of all, we show that x_0 is a fixed point of $\mathbb{W}$. Assume that $\mathbb{M}(x_0,\mathbb{W}x_0) > 0$. By the quasi-$F_{\mathbb{M}}$-contractive property of $\mathbb{W}$ we deduce that

$$t + F(\mathbb{M}(x_0,\mathbb{W}x_0)) \le F(\mathbb{M}(x_0,x_0)).$$

Thus, $F(\mathbb{M}(x_0,\mathbb{W}x_0)) < F(0)$, which leads to a contradiction and that is due to the fact that F is strictly increasing. Thus, we arrive at $\mathbb{W}x_0 = x_0$.

If $r = 0$ then we obtain $\overline{B^-}(x_0,r) = D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ and clearly, $\mathbb{W}$ fixes the center of the disc $D^{\mathbb{M}}_{x_0,r}$ and the whole disc $D^{\mathbb{M}}_{x_0,r}$.

Let $r > 0$ and $x \in \overline{B^-}(x_0,r)$ with $\mathbb{W}x \neq x$. By the definition of r, we have $r \le \mathbb{M}(x,\mathbb{W}x)$. Hence, by the *DCQMS-$F_{\mathbb{M}}$*-contractive property, there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in \mathbb{B}$ such that

$$t + F(\mathbb{M}(x,\mathbb{W}x)) \le F(\mathbb{M}(x_0,x)) \le F(r) \le F(\mathbb{M}(x,\mathbb{W}x)),$$

for all $x \in \mathbb{B}$ which leads us to a contradiction; therefore, we deduce that $\mathbb{W}x = x$, hence $\overline{B^-}(x_0,r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

Now, we introduce a new rational type contractive condition.

Definition 9. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a self-mapping on $\mathbb{B}$ and $F \in \mathbb{F}$. Then $\mathbb{W}$ is said to be DCQMS-$F_{\mathbb{M}}$-rational contraction if there exist $t > 0$ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Rightarrow t + F(\mathbb{M}(x, \mathbb{W}x)) \leq F(M_R^{\mathbb{M}}(x_0, x)), \tag{4}$$

for all $x \in \mathbb{B}$, where

$$M_R^{\mathbb{M}}(x, v) = \max \left\{ \begin{array}{c} \mathbb{M}(x, v), \mathbb{M}(x, \mathbb{W}x), \mathbb{M}(y, \mathbb{W}y), \\ \frac{\mathbb{M}(x, \mathbb{W}x)\mathbb{M}(y, \mathbb{W}y)}{1+\mathbb{M}(x,v)}, \frac{\mathbb{M}(x, \mathbb{W}x)\mathbb{M}(y, \mathbb{W}y)}{1+\mathbb{M}(\mathbb{W}x, \mathbb{W}y)} \end{array} \right\}.$$

Theorem 2. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a DCQMS-$F_{\mathbb{M}}$-rational contraction self-mapping with $x_0 \in \mathbb{B}$ on $\mathbb{B}$, $\mathbb{W}x_0 = x_0$ and r defined as in (2). Then we have $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D_{x_0,r}^{\mathbb{M}}$.*

Proof. Suppose that $r = 0$. So we have $\overline{B^-}(x_0, r) = D_{x_0,r}^{\mathbb{M}} = \{x_0\}$. Using the hypothesis $\mathbb{W}x_0 = x_0$, $\mathbb{W}$ fixes the disc $D_{x_0,r}^{\mathbb{M}}$.

Let $r > 0$ and $x \in \overline{B^-}(x_0, r)$ with $\mathbb{W}x \neq x$. By the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. Because of the *DCQMS-$F_{\mathbb{M}}$*-rational contractive property, there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in \mathbb{B}$ such that

$$t + F(\mathbb{M}(x, \mathbb{W}x)) \leq F(M_R^{\mathbb{M}}(x_0, x)),$$

for all $x \in \mathbb{B}$. Then we obtain

$$\begin{aligned} t + F(\mathbb{M}(x, \mathbb{W}x)) &\leq F(M_R^{\mathbb{M}}(x_0, x)) \\ &= F\left(\max \left\{ \begin{array}{c} \mathbb{M}(x_0, x), \mathbb{M}(x_0, \mathbb{W}x_0), \mathbb{M}(x, \mathbb{W}x), \\ \frac{\mathbb{M}(x_0, \mathbb{W}x_0)\mathbb{M}(x, \mathbb{W}x)}{1+\mathbb{M}(x_0,x)}, \frac{\mathbb{M}(x_0, \mathbb{W}x_0)\mathbb{M}(x, \mathbb{W}x)}{1+\mathbb{M}(\mathbb{W}x_0, \mathbb{W}x)} \end{array} \right\} \right) \\ &\leq F(\max\{r, \mathbb{M}(x, \mathbb{W}x)\}) = F(\mathbb{M}(x, \mathbb{W}x)), \end{aligned}$$

a contradiction. Hence it should be $\mathbb{W}x = x$. Consequently, $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D_{x_0,r}^{\mathbb{M}}$. □

3.2. DCQMS-α-x_0-Contractive Type Mappings

First, we present the definition of an x_0-contractive mapping in *DCQMS*.

Definition 10. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a self-mapping on $\mathbb{B}$ and $0 < k < 1$. Then $\mathbb{W}$ is said to be a DCQMS-x_0-contractive mapping if there exist $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) \leq k\mathbb{M}(x_0, x), \tag{5}$$

for every $x \in \mathbb{B}$.

Clearly, x_0 is always a fixed point of $\mathbb{W}$ in Definition 10. Now, we show that if $\mathbb{W}$ is a *DCQMS-x_0*-contractive mapping, then $Fix(\mathbb{W})$ contains a disc.

Theorem 3. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a DCQMS-x_0-contractive self-mapping with $x_0 \in \mathbb{B}$ on $\mathbb{B}$ and r defined as in (2). Then we have $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D_{x_0,r}^{\mathbb{M}}$.*

Proof. In the case $r = 0$, it is clear that $\overline{B^-}(x_0, r) = D_{x_0,r}^{\mathbb{M}} = \{x_0\}$ is a fixed disc of $\mathbb{W}$.

Suppose that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ be such that $\mathbb{W}x \neq x$. By the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. On the other hand, using the *DCQMS*-x_0-contractive property of $\mathbb{W}$, we obtain

$$0 < \mathbb{M}(x, \mathbb{W}x) \leq k\mathbb{M}(x_0, x) \leq kr < r,$$

which leads us to a contradiction. Thus, $\mathbb{W}x = x$ for every $x \in \overline{B^-}(x_0, r)$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$. In particular, $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

Now, we define the concept of *DCQMS*-α-x_0-contractive self-mappings in quasi-metric spaces.

Definition 11. *Let $\mathbb{W}$ be a self mapping on a DCQMS $(\mathbb{B}, \mathbb{M})$. Then $\mathbb{W}$ is said to be a DCQMS-α-x_0-contractive self-mapping if there exist a function $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$, $0 < k < 1$ and $x_0 \in \mathbb{B}$ such that*

$$\alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq k\mathbb{M}(x_0, x), \tag{6}$$

for all $x \in \mathbb{B}$.

We recall α-x_0-admissible maps as follows:

Definition 12. *Let $\mathbb{B}$ be a non-empty set. Given a function $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$ and $x_0 \in \mathbb{B}$. Then $\mathbb{W}$ is said to be an α-x_0-admissible if for every $x \in \mathbb{B}$,*

$$\alpha(x_0, x) \geq 1 \quad \Rightarrow \quad \alpha(x_0, \mathbb{W}x) \geq 1.$$

Theorem 4. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a DCQMS-α-x_0-contractive self-mapping with $x_0 \in \mathbb{B}$ on $\mathbb{B}$ and r defined as in (2). Assume that $\mathbb{W}$ is α-x_0-admissible and $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B^-}(x_0, r)$. Then we have $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. By the definition of a *DCQMS*-α-x_0-contractive self-mapping, it is easy to see that x_0 is always a fixed point of $\mathbb{W}$; therefore, if $r = 0$ then we have $\overline{B^-}(x_0, r) = D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ and the proof follows.

Suppose that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ such that $\mathbb{W}x \neq x$. By the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. On the other hand, we have $\alpha(x_0, x) \geq 1$. Using the α-x_0-admissible property and the *DCQMS*-α-x_0-contractive property of $\mathbb{W}$, we find

$$0 < \mathbb{M}(x, \mathbb{W}x) \leq \alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq k\mathbb{M}(x_0, x) \leq kr < r,$$

which leads us to a contradiction. Thus, $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

The concept of a *DCQMS*-$F^{\alpha}_{\mathbb{M}}$-contractive mapping is defined as follows.

Definition 13. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a self-mapping on $\mathbb{B}$ and $F \in \mathbb{F}$. Then $\mathbb{W}$ is called a DCQMS-$F^{\alpha}_{\mathbb{M}}$-contraction if there exist $t > 0$, a function $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Rightarrow t + \alpha(x_0, \mathbb{W}x)F(\mathbb{M}(x, \mathbb{W}x)) \leq F(\mathbb{M}(x_0, x)), \tag{7}$$

for all $x \in \mathbb{B}$.

Theorem 5. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a DCQMS-$F^{\alpha}_{\mathbb{M}}$-contractive self-mapping with $x_0 \in \mathbb{B}$ and r defined as in (2). Suppose that $\mathbb{W}$ is α-x_0-admissible and $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B^-}(x_0, r)$. Then we have $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. At first, using the *DCQMS*-$F^{\alpha}_{\mathbb{M}}$-contractive property, one can easily deduce that $\mathbb{W}x_0 = x_0$. Hence we have $\overline{B^-}(x_0, r) = D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ if $r = 0$. Clearly, $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.

Assume that $r > 0$. Let $x \in \overline{B^-}(x_0, r)$ where $\mathbb{W}x \neq x$; therefore, by the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. On the other hand, we have $\alpha(x_0, x) \geq 1$ and $\mathbb{W}$ is α-x_0-admissible. So, using the *DCQMS*-$F^{\alpha}_{\mathbb{M}}$-contractive property of $\mathbb{W}$, we deduce

$$F(\mathbb{M}(x, \mathbb{W}x)) < t + \alpha(x_0, \mathbb{W}x)F(\mathbb{M}(x, \mathbb{W}x)) \leq F(\mathbb{M}(x_0, x)) \leq F(r) \leq F(\mathbb{M}(x, \mathbb{W}x)).$$

Thus, by the fact that F is strictly increasing and $t > 0$ we come to a contradiction. Hence, we have $\overline{B^-}(x_0, r) \subseteq Fix(T)$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

Definition 14. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a self-mapping on $\mathbb{B}$ and $F \in \mathbb{F}$. Then $\mathbb{W}$ is called a Ćirić-type DCQMS-$F_{\mathbb{M}}$-contraction if there exist $t > 0$ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Longrightarrow t + \alpha(x_0, \mathbb{W}x)F(\mathbb{M}(x, \mathbb{W}x)) \leq F(M^{\mathbb{M}}_C(x_0, x)), \quad (8)$$

for all $x \in \mathbb{B}$, where

$$M^{\mathbb{M}}_C(x, v) = \max\left\{\mathbb{M}(x, v), \mathbb{M}(x, \mathbb{W}x), \mathbb{M}(y, \mathbb{W}y), \frac{\mathbb{M}(x, \mathbb{W}y) + \mathbb{M}(y, \mathbb{W}x)}{2}\right\}. \quad (9)$$

Proposition 1. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS. If $\mathbb{W}$ is a Ćirić-type DCQMS-$F_{\mathbb{M}}$-contraction with $x_0 \in \mathbb{B}$ such that $\alpha(x_0, \mathbb{W}x_0) \geq 1$, then we have $\mathbb{W}x_0 = x_0$.*

Proof. Assume that $\mathbb{W}x_0 \neq x_0$. From the definition of a Ćirić-type *DCQMS*-$F_{\mathbb{M}}$-contraction, we obtain

$$\begin{aligned} \mathbb{M}(x_0, \mathbb{W}x_0) &> 0 \Longrightarrow t + \alpha(x_0, \mathbb{W}x_0)F(\mathbb{M}(x_0, \mathbb{W}x_0)) \leq F(M^{\mathbb{M}}_C(x_0, x_0)) \\ &= F\left(\max\left\{\begin{array}{c} \mathbb{M}(x_0, x_0), \mathbb{M}(x_0, \mathbb{W}x_0), \mathbb{M}(x_0, \mathbb{W}x_0), \\ \frac{\mathbb{M}(x_0, \mathbb{W}x_0) + \mathbb{M}(x_0, \mathbb{W}x_0)}{2} \end{array}\right\}\right) \\ &= F(\mathbb{M}(x_0, \mathbb{W}x_0)), \end{aligned}$$

which is a contradiction since $t > 0$. Then we have $\mathbb{W}x_0 = x_0$. □

Theorem 6. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a Ćirić-type DCQMS-$F_{\mathbb{M}}$-contraction with $x_0 \in \mathbb{B}$ and r defined as in (2). Assume that $\mathbb{W}$ is α-x_0-admissible and if for every $x \in D^{\mathbb{M}}_{x_0,r}$, we have $\mathbb{M}(x_0, \mathbb{W}x) \leq r$. Then $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. If $r = 0$, clearly $D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ is a fixed-disc (point) by Proposition 1.

Assume that $r > 0$. Let $x \in D^{\mathbb{M}}_{x_0,r}$. By the definition of r, we have $\mathbb{M}(x, \mathbb{W}x) \geq r$. So using the Ćirić-type *DCQMS*-$F_{\mathbb{M}}$-contractive property and the fact that $\mathbb{W}$ is α-x_0-admissible and F is increasing, we obtain

$$\begin{aligned} F(\mathbb{M}(x, \mathbb{W}x)) &\leq \alpha(x_0, \mathbb{W}x)F(\mathbb{M}(x, \mathbb{W}x)) + t \leq F(M^{\mathbb{M}}_C(x_0, x)) \\ &= F\left(\max\left\{\mathbb{M}(x_0, x), \mathbb{M}(x_0, \mathbb{W}x_0), \mathbb{M}(x, \mathbb{W}x), \frac{\mathbb{M}(x_0, \mathbb{W}x) + \mathbb{M}(x, \mathbb{W}x_0)}{2}\right\}\right) \\ &\leq F(\max\{r, \mathbb{M}(x, \mathbb{W}x), 0, r\}) \leq F(\mathbb{M}(x, \mathbb{W}x)), \end{aligned}$$

which leads to a contradiction. Therefore, $\mathbb{M}(x, \mathbb{W}x) = 0$ and so $\mathbb{W}x = x$. Hence, $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

3.3. DCQMS-α-φ-x_0-Contractive Type Mappings

At first, we recall the notion of (c)-comparison functions [17] (see also [18]).

Definition 15 ([17])**.** *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a (c)-comparison function if*

$(i)_\varphi$ *φ is increasing;*

$(ii)_\varphi$ *There exist* $k_0 \in \mathbb{N}$, $a \in (0,1)$ *and a convergent series of nonnegative terms* $\sum_{k=1}^{\infty} v_k$ *such that*

$$\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k,$$

for $k \geq k_0$ *and any* $t \in \mathbb{R}_+$.

The class of (c)-comparison functions will be denoted by Ψ_c.

Lemma 2 ([17])**.** *If* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a* (c)*-comparison function, then the followings hold*:
(i) φ *is a comparison function;*
(ii) $\varphi(t) < t$ *for any* $t \in \mathbb{R}_+$;
(iii) φ *is continuous at* 0;
(iv) *the series* $\sum_{k=0}^{\infty} \varphi^k(t)$ *converges for any* $t \in \mathbb{R}_+$.

Next, we introduce two new contractions and obtain two new fixed-disc theorems as follows:

Definition 16. *Let* $(\mathbb{B}, \mathbb{M})$ *be a DCQMS and* $\mathbb{W}$ *a self-mapping on* $\mathbb{B}$*. Then* $\mathbb{W}$ *is said to be a DCQMS-*α*-*φ*-*x_0*-contraction if there exist* $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$, $\varphi \in \Psi_c$ *and* $x_0 \in \mathbb{B}$ *such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Longrightarrow \alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq \varphi(\mathbb{M}(x_0, \mathbb{W}x)),$$

for each $x \in \mathbb{B}$.

Theorem 7. *Let* $(\mathbb{B}, \mathbb{M})$ *be a DCQMS,* $\mathbb{W}$ *a DCQMS-*α*-*φ*-*x_0*-contractive self-mapping with* $x_0 \in \mathbb{B}$ *and* r *defined as in* (2)*. Assume that* $\mathbb{W}$ *is* α*-*x_0*-admissible. If* $\alpha(x_0, x) \geq 1$ *for* $x \in \overline{B^-}(x_0, r)$ *and* $0 < \mathbb{M}(x_0, \mathbb{W}x) \leq r$ *for* $x \in \overline{B^-}(x_0, r) - \{x_0\}$*, then we have* $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$*, in particular* $\mathbb{W}$ *fixes the disc* $D^{\mathbb{M}}_{x_0,r}$.

Proof. Using the *DCQMS-*α*-*φ*-*x_0-contractive property, we have $\mathbb{W}x_0 = x_0$. Indeed, we assume $\mathbb{W}x_0 \neq x_0$, that is, $\mathbb{M}(x_0, \mathbb{W}x_0) > 0$. Then using the condition (ii) in Lemma 2 and α-x_0-admissibility, we obtain

$$\alpha(x_0, \mathbb{W}x_0)\mathbb{M}(x_0, \mathbb{W}x_0) \leq \varphi(\mathbb{M}(x_0, \mathbb{W}x_0)) < \mathbb{M}(x_0, \mathbb{W}x_0),$$

a contradiction. Thus, $\mathbb{W}x_0 = x_0$.

Suppose that $r = 0$. In this case, $\overline{B^-}(x_0, r) = D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ and the proof follows.

Now we suppose that $r > 0$ and $x \in \overline{B^-}(x_0, r) - \{x_0\}$ such that $x \neq \mathbb{W}x$. Using the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. By the hypothesis, we known $\alpha(x_0, x) \geq 1$. From the *DCQMS-*α*-*φ*-*x_0-contractive property and α-x_0-admissibility, we obtain

$$\alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq \varphi(\mathbb{M}(x_0, \mathbb{W}x)) < \mathbb{M}(x_0, \mathbb{W}x) \leq r,$$

a contradiction. Therefore, $\mathbb{W}x = x$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

Next, we define the following new contraction.

Definition 17. *Let* $(\mathbb{B}, \mathbb{M})$ *be a DCQMS and* $\mathbb{W}$ *a self-mapping on* $\mathbb{B}$*. Then* $\mathbb{W}$ *is said to be a Ćirić-type DCQMS-*α*-*φ*-*x_0*-contraction if there exist* $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$, $\varphi \in \Psi_c$ *and* $x_0 \in \mathbb{B}$ *such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Longrightarrow \alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq \varphi\left(M^{\mathbb{M}}_C(x_0, x)\right),$$

for each $x \in \mathbb{B}$.

Theorem 8. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a Ćirić-type DCQMS-α-φ-x_0-contractive self-mapping with $x_0 \in \mathbb{B}$ and r defined as in (2). Assume that $\mathbb{W}$ is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ and $\mathbb{M}(x_0, \mathbb{W}x) \leq r$ for $x \in D^{\mathbb{M}}_{x_0,r}$, then $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. Using the hypothesis, we have $\mathbb{W}x_0 = x_0$. Indeed, we assume $\mathbb{W}x_0 \neq x_0$, that is, $\mathbb{M}(x_0, \mathbb{W}x_0) > 0$. Then using the condition (ii) in Lemma 2 and α-x_0-admissibility, we obtain

$$M^{\mathbb{M}}_C(x_0, x_0) = \max\left\{ \begin{array}{c} \mathbb{M}(x_0, x_0), \mathbb{M}(x_0, \mathbb{W}x_0), \mathbb{M}(x_0, \mathbb{W}x_0), \\ \frac{\mathbb{M}(x_0, \mathbb{W}x_0) + \mathbb{M}(x_0, \mathbb{W}x_0)}{2} \end{array} \right\} = \mathbb{M}(x_0, \mathbb{W}x_0)$$

and

$$\alpha(x_0, \mathbb{W}x_0)\mathbb{M}(x_0, \mathbb{W}x_0) \leq \varphi\left(M^{\mathbb{M}}_C(x_0, x_0)\right) < \mathbb{M}(x_0, \mathbb{W}x_0),$$

a contradiction. It should be $\mathbb{W}x_0 = x_0$.

Let $r = 0$. In this case, we have $D^{\mathbb{M}}_{x_0,r} = \{x_0\}$.

Now we suppose that $r > 0$ and $x \in D^{\mathbb{M}}_{x_0,r} - \{x_0\}$ such that $x \neq \mathbb{W}x$. Using the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. By the hypothesis, we known $\alpha(x_0, x) \geq 1$. By the Ćirić-type *DCQMS*-α-φ-x_0-contractive property and α-x_0-admissibility, we obtain

$$M^{\mathbb{M}}_C(x_0, x) = \max\left\{ \begin{array}{c} \mathbb{M}(x_0, x), \mathbb{M}(x_0, \mathbb{W}x_0), \mathbb{M}(x, \mathbb{W}x), \\ \frac{\mathbb{M}(x_0, \mathbb{W}x) + \mathbb{M}(x, \mathbb{W}x_0)}{2} \end{array} \right\} \leq \mathbb{M}(x, \mathbb{W}x)$$

and

$$\alpha(x_0, \mathbb{W}x)\mathbb{M}(x, \mathbb{W}x) \leq \varphi\left(M^{\mathbb{M}}_C(x, x_0)\right) < \mathbb{M}(x, \mathbb{W}x),$$

a contradiction. Therefore, $\mathbb{W}x = x$, that is, $D^{\mathbb{M}}_{x_0,r}$ is a fixed disc of $\mathbb{W}$. □

3.4. DCQMS-α-ψ-φ-x_0-Contractive Type Mappings

We recall the notion of an altering distance function.

Definition 18 ([19])**.** *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the followings hold*:

(i) *ψ is continuous and nondecreasing;*
(ii) $\psi(t) = 0 \Leftrightarrow t = 0$.

Using this definition, we present two new contractive conditions and two new fixed-disc results.

Definition 19. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS and $\mathbb{W}$ a self-mapping on $\mathbb{B}$. Then $\mathbb{W}$ is said to be a DCQMS-α-ψ-φ-x_0-contraction if there exist $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$, two altering distance functions ψ, φ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Longrightarrow \alpha(x_0, \mathbb{W}x)\psi(\mathbb{M}(x, \mathbb{W}x)) \leq \psi(\mathbb{M}(x_0, x)) - \varphi(\mathbb{M}(x_0, x)),$$

for each $x \in \mathbb{B}$.

Theorem 9. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a DCQMS-α-ψ-φ-x_0-contractive self-mapping with $x_0 \in \mathbb{B}$ and r defined as in (2). Assume that $\mathbb{W}$ is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ for $x \in \overline{B^-}(x_0, r)$, then we have $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$, in particular $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. Using the *DCQMS*-α-ψ-φ-x_0-contractive property, we have $\mathbb{W}x_0 = x_0$. Indeed, we assume $\mathbb{W}x_0 \neq x_0$, that is, $\mathbb{M}(x_0, \mathbb{W}x_0) > 0$. Then using the condition (ii) in Definition 18 and α-x_0-admissibility, we obtain

$$\begin{aligned} \alpha(x_0, \mathbb{W}x_0)\psi(\mathbb{M}(x_0, \mathbb{W}x_0)) &\leq \psi(\mathbb{M}(x_0, x_0)) - \varphi(\mathbb{M}(x_0, x_0)) \\ &= \psi(0) - \varphi(0) = 0, \end{aligned}$$

a contradiction. It should be $\mathbb{W}x_0 = x_0$.

Suppose that $r = 0$. In this case, we obtain $\overline{B^-}(x_0, r) = D^{\mathbb{M}}_{x_0,r} = \{x_0\}$.

Now, we suppose that $r > 0$ and $x \in \overline{B^-}(x_0, r) - \{x_0\}$ such that $x \neq \mathbb{W}x$. Using the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. By the hypothesis, we know $\alpha(x_0, x) \geq 1$. By the *DCQMS*-α-ψ-φ-x_0-contractive property and α-x_0-admissibility, we obtain

$$\begin{aligned} \alpha(x_0, \mathbb{W}x)\psi(\mathbb{M}(x, \mathbb{W}x)) &\leq \psi(\mathbb{M}(x_0, x)) - \varphi(\mathbb{M}(x_0, x)) \\ &= \psi(r) - \varphi(r) < \psi(r), \end{aligned}$$

a contradiction. Therefore $\mathbb{W}x = x$, that is, $\overline{B^-}(x_0, r) \subseteq Fix(\mathbb{W})$. In particular, $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$. □

Definition 20. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS and $\mathbb{W}$ a self-mapping on $\mathbb{B}$. Then $\mathbb{W}$ is said to be a Ćirić-type DCQMS-α-ψ-φ-x_0-contraction if there exist $\alpha : \mathbb{B} \times \mathbb{B} \to (0, \infty)$, two altering distance functions ψ, φ and $x_0 \in \mathbb{B}$ such that*

$$\mathbb{M}(x, \mathbb{W}x) > 0 \Longrightarrow \alpha(x_0, \mathbb{W}x)\psi(\mathbb{M}(x, \mathbb{W}x)) \leq \psi\Big(M^{\mathbb{M}}_C(x_0, x)\Big) - \varphi\Big(M^{\mathbb{M}}_C(x_0, x)\Big),$$

for each $x \in \mathbb{B}$.

Theorem 10. *Let $(\mathbb{B}, \mathbb{M})$ be a DCQMS, $\mathbb{W}$ a Ćirić-type DCQMS-α-ψ-φ-x_0-contractive self-mapping with $x_0 \in \mathbb{B}$ and r defined as in (2). Assume that $\mathbb{W}$ is α-x_0-admissible. If $\alpha(x_0, x) \geq 1$ and $\mathbb{M}(x_0, \mathbb{W}x) \leq r$ for $x \in D^{\mathbb{M}}_{x_0,r}$, then $\mathbb{W}$ fixes the disc $D^{\mathbb{M}}_{x_0,r}$.*

Proof. Using the hypothesis, we have $\mathbb{W}x_0 = x_0$. Indeed, we assume $\mathbb{W}x_0 \neq x_0$, that is, $\mathbb{M}(x_0, \mathbb{W}x_0) > 0$ and we obtain

$$\begin{aligned} \alpha(x_0, \mathbb{W}x_0)\psi(\mathbb{M}(x_0, \mathbb{W}x_0)) &\leq \psi\Big(M^{\mathbb{M}}_C(x_0, x_0)\Big) - \varphi\Big(M^{\mathbb{M}}_C(x_0, x_0)\Big) \\ &= \psi(\mathbb{M}(x_0, \mathbb{W}x_0)) - \varphi(\mathbb{M}(x_0, \mathbb{W}x_0)) \\ &< \psi(\mathbb{M}(x_0, \mathbb{W}x_0)), \end{aligned}$$

a contradiction. It should be $\mathbb{W}x_0 = x_0$.

Let $r = 0$. In this case, we have $D^{\mathbb{M}}_{x_0,r} = \{x_0\}$ and the proof follows.

Now, we suppose that $r > 0$ and $x \in D^{\mathbb{M}}_{x_0,r} - \{x_0\}$ such that $x \neq \mathbb{W}x$. Using the definition of r, we have $r \leq \mathbb{M}(x, \mathbb{W}x)$. By the hypothesis, we know that $\alpha(x_0, x) \geq 1$. From the Ćirić-type *DCQMS*-α-ψ-φ-x_0-contractive property and α-x_0-admissibility, we obtain

$$\begin{aligned} \alpha(x_0, \mathbb{W}x)\psi(\mathbb{M}(x, \mathbb{W}x)) &\leq \psi\Big(M^{\mathbb{M}}_C(x_0, x)\Big) - \varphi\Big(M^{\mathbb{M}}_C(x_0, x)\Big) \\ &< \psi(\mathbb{M}(x, \mathbb{W}x)), \end{aligned}$$

a contradiction; therefore, $\mathbb{W}x = x$, that is, $D^{\mathbb{M}}_{x_0,r}$ is a fixed disc of $\mathbb{W}$. □

4. An Application: A Common Fixed-Disc Theorem

Let $(\mathbb{B}, \mathbb{M})$ be a *DCQMS*, $\mathbb{W}, S : \mathbb{B} \to \mathbb{B}$ be two self-mappings and $D^{\mathbb{M}}_{x_0,r}$ be a disc on $\mathbb{B}$. If $\mathbb{W}x = Sx = x$ for all $x \in D^{\mathbb{M}}_{x_0,r}$, then the disc $D^{\mathbb{M}}_{x_0,r}$ is called the common fixed disc of the pair $(\mathbb{W}, S)$.

Following [20,21], we present the following.

$$M_{\mathbb{W},S}^{\mathbb{M}}(x,v)=\max\left\{\mathbb{M}(\mathbb{W}x,Sy),\mathbb{M}(\mathbb{W}x,Sx),\mathbb{M}(\mathbb{W}y,Sy),\frac{\mathbb{M}(\mathbb{W}x,Sy)+\mathbb{M}(\mathbb{W}y,Sx)}{2}\right\}.$$

To obtain a common fixed-disc theorem, we define the following number:

$$\mu^{\mathbb{M}}=\inf\{\mathbb{M}(\mathbb{W}x,Sx):x\in\mathbb{B},\mathbb{W}x\neq Sx\}.$$

In the following theorem, we use the numbers $M_{\mathbb{W},S}^{\mathbb{M}}(x,v)$, r which is defined in (2), $\mu^{\mathbb{M}}$ and ρ defined by

$$\rho=\min\{r,\mu^{\mathbb{M}}\}.$$

Theorem 11. *Let $(\mathbb{B},\mathbb{M})$ be a DCQMS, $\mathbb{W},S:\mathbb{B}\to\mathbb{B}$ two self-mappings and $\mathbb{W}$ an α-x_0-admissible map. Assume that there exist $F\in\mathbb{F}$, $t>0$ and $x_0\in\mathbb{B}$ such that*

$$\mathbb{M}(\mathbb{W}x,Sx)>0\Longrightarrow t+\alpha(x_0,\mathbb{W}x)F(\mathbb{M}(\mathbb{W}x,Sx))\leq F(M_{\mathbb{W},S}^{\mathbb{M}}(x,x_0)),$$

for each $x\in\mathbb{B}$ and

$$\alpha(x_0,x)\geq 1,\ \mathbb{M}(\mathbb{W}x,x_0)\leq\rho,\ \mathbb{M}(x_0,Sx)\leq\rho,$$

for each $x\in D_{x_0,\rho}^{\mathbb{M}}$. If $\mathbb{W}$ is a DCQMS-$F_{\mathbb{M}}$-contraction with $x_0\in\mathbb{B}$ and r and S is an $\mathbb{M}-F_{\mathbb{M}}$-contraction with $x_0\in\mathbb{B}$ and r), then $D_{x_0,\rho}^{\mathbb{M}}$ is a common fixed disc of the pair $(\mathbb{W},S)$ in $\mathbb{B}$.

Proof. Let $x=x_0$. If $\mathbb{M}(\mathbb{W}x_0,Sx_0)>0$ then we have

$$M_{\mathbb{W},S}^{\mathbb{M}}(x_0,x_0)=\max\left\{\begin{array}{c}\mathbb{M}(\mathbb{W}x_0,Sx_0),\mathbb{M}(\mathbb{W}x_0,Sx_0),\mathbb{M}(\mathbb{W}x_0,Sx_0),\\ \frac{\mathbb{M}(\mathbb{W}x_0,Sx_0)+\mathbb{M}(\mathbb{W}x_0,Sx_0)}{2}\end{array}\right\}=\mathbb{M}(\mathbb{W}x_0,Sx_0)$$

and

$$\begin{aligned}t+\alpha(x_0,\mathbb{W}x_0)F(\mathbb{M}(\mathbb{W}x_0,Sx_0)) &\leq F(M_{\mathbb{W},S}^{\mathbb{M}}(x_0,x_0))=F(\mathbb{M}(\mathbb{W}x_0,Sx_0))\\ &\Longrightarrow t\leq(1-\alpha(x_0,\mathbb{W}x_0))F(\mathbb{M}(\mathbb{W}x_0,Sx_0)),\end{aligned}$$

a contradiction with $t>0$; therefore $\mathbb{W}x_0=Sx_0$, that is, x_0 is a coincidence point of the pair $(\mathbb{W},S)$. If $\mathbb{W}$ is a $\mathbb{M}-F_{\mathbb{M}}$-contraction (or S is a $\mathbb{M}-F_{\mathbb{M}}$-contraction) then using Theorem 1, we have $\mathbb{W}x_0=x_0$ (or $Sx_0=x_0$) and hence $\mathbb{W}x_0=Sx_0=x_0$.

Now if $\rho=0$, then clearly $D_{x_0,\rho}^{\mathbb{M}}=\{x_0\}$ and this disc is a common fixed disc of the pair $(\mathbb{W},S)$.

Let $\rho>0$ and $x\in D_{x_0,\rho}^{\mathbb{M}}$. Assume that $\mathbb{W}x\neq Sx$, that is, $\mathbb{M}(\mathbb{W}x,Sx)>0$. Using the hypothesis, α-x_0-admissibility of $\mathbb{W}$ and the definition of ρ, we obtain

$$\begin{aligned}M_{\mathbb{W},S}^{\mathbb{M}}(x,x_0) &= \max\left\{\mathbb{M}(\mathbb{W}x,Sx_0),\mathbb{M}(\mathbb{W}x,Sx),\mathbb{M}(\mathbb{W}x_0,Sx_0),\frac{\mathbb{M}(\mathbb{W}x,Sx_0)+\mathbb{M}(\mathbb{W}x_0,Sx)}{2}\right\}\\ &= \max\left\{\mathbb{M}(\mathbb{W}x,x_0),\mathbb{M}(\mathbb{W}x,Sx),\frac{\mathbb{M}(\mathbb{W}x,x_0)+\mathbb{M}(x_0,Sx)}{2}\right\}\\ &\leq \mathbb{M}(\mathbb{W}x,Sx)\end{aligned}$$

and so

$$\begin{aligned}t+\alpha(x_0,\mathbb{W}x)F(\mathbb{M}(\mathbb{W}x,Sx)) &\leq F(M_{\mathbb{W},S}^{\mathbb{M}}(x,x_0))\leq F(\mathbb{M}(\mathbb{W}x,Sx))\\ &\Longrightarrow t\leq(1-\alpha(x_0,\mathbb{W}x))F(\mathbb{M}(\mathbb{W}x,Sx)),\end{aligned}$$

a contradiction with $t>0$. We have found that x is a coincidence point of the pair $(\mathbb{W},S)$, that is, $\mathbb{W}x=Sx$. If $\mathbb{W}$ (or S) is a $\mathbb{M}-F_{\mathbb{M}}$-contraction, then by Theorem 1, we have $\mathbb{W}x=x$ (or $Sx=x$) andhence $\mathbb{W}x=Sx=x$. Consequently, $D_{x_0,\rho}^{\mathbb{M}}$ is a common fixed disc of the pair $(\mathbb{W},S)$. □

We give an illustrative example.

Example 3. *Let $\mathbb{B} = (-\infty, \infty)$ and for all $x, y \in \mathbb{B}$ let $\mathbb{M}(x,y) = |x-y|$ if $x \in (0,1)$ and $\mathbb{M}(0,1) = 1$, $\mathbb{M}(1,0) = 2$. Next, let $\mathbb{K}(x,y) = \max\{x,y\} + 2$ and $\mathbb{L}(x,y) = \max\{x,y\} + 3$. It is not difficult to see that $(\mathbb{B}, \mathbb{M})$ is a DCQMS. Now, define the self-mappings $\mathbb{W} : \mathbb{B} \to \mathbb{B}$ and $S : \mathbb{B} \to \mathbb{B}$ as*

$$\mathbb{W}x = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \in \{-1,1\} \\ x & \text{if} \quad x \in (-1,1) \\ x+2 & \text{otherwise} \end{cases},$$

and

$$Sx = \begin{cases} \frac{1}{|x|} & \text{if} \quad x \in \{-1,1\} \\ x & \text{if} \quad x \in (-1,1) \\ x+1 & \text{otherwise} \end{cases},$$

for all $x \in \mathbb{B}$. Define $\alpha(x,y) = e^{|x-y|}$. First of all, note that both mappings $\mathbb{W}, S$ are α-0-admissible. Moreover, the pair of the self-mappings $(\mathbb{W}, S)$ satisfy the following condition

$$\mathbb{M}(\mathbb{W}x, Sx) > 0 \Longrightarrow t + \alpha(0, \mathbb{W}x) F(\mathbb{M}(\mathbb{W}x, Sx)) \leq F(M^{\mathbb{M}}_{\mathbb{W},S}(x,0)),$$

with $F = \ln x$, $t = \ln \frac{3}{2}$ and $x_0 = 0$. Now, it is not difficult to see that all the hypothesis of Theorem 11, are satisfied. Hence, $D^{\mathbb{M}}_{0,1}$ is a common fixed disk of the pair of mappings $(\mathbb{W}, S)$ as required.

5. Conclusions

We have proved the existence of a fixed disk for self mappings in *DCQMS* that satisfy different types of contractions. We provided an application of our result on common fixed disk for two self mapping on *DCQMS*. In closing, we would like to bring to the readers attention the following question:

Question 1. *Under what conditions these types of mappings in DCQMS have a unique fixed disk?*

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