Filomat 36:12 (2022), 4087–4097
https://doi.org/10.2298/FIL2212087B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Roots of Fibonacci Polynomials

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Abstract. In this paper, we investigate Fibonacci polynomials as complex hyperbolic functions. We examine the roots of these polynomials. Also, we give some exciting identities about images of the roots of Fibonacci polynomials under another member of the Fibonacci polynomials class. Finally, we obtain some excellent relationships between the roots of Fibonacci polynomials and the modular group, Hecke groups and generalized Hecke groups with geometric interpretations.

1. Introduction

In recent years, many recursive sequences have been extensively studied from many points of view in the literature. The most famous sequences are Fibonacci and Lucas. They are used in various fields of science and art. Interesting large classes of Fibonacci and Lucas polynomials can be defined by Fibonacci-like recurrence relation. On the other hand, the sum of the coefficients of the polynomials is the Fibonacci and Lucas number. Besides, the ratio of two consecutive numbers or polynomials of Fibonacci and Lucas families converges to the golden ratio, which appears in many fields in the literature, such as nature, art, architecture, biology, physics, chemistry, cosmos, theology, finance, and so on (see [13], [20], [25], [28], [32], [34]). There are many interesting studies related to the number sequences, polynomials, and the golden ratio mentioned above (see [12], [25], [35] for more details). The recursive formula of the Fibonacci sequence is

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers can also be calculated by the Binet formula

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} \tag{2}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Catalan defined the recursive formula for Fibonacci polynomials in the following manner [25].

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 11B39; Secondary 65H04, 11F06, 20H10
Keywords. Fibonacci polynomials, roots of polynomial equations, the modular group, Hecke groups.
Received: 14 March 2021; Revised: 06 July 2022; Accepted: 25 July 2022
Communicated by Dragan S. Djordjević
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where $n \geq 2$ and $F_0(x) = 0, F_1(x) = 1$.

Fibonacci polynomials hold the following properties. The Binet formula of Fibonacci polynomials is obtained as

$$F_n(x) = \frac{\varphi^n(x) - \psi^n(x)}{\varphi(x) - \psi(x)} \tag{4}$$

where $\varphi(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\psi(x) = \frac{x-\sqrt{x^2-4}}{2}$ [25]. Also, the Fibonacci polynomials are generated by a matrix $Q(x)$ [22].

$$Q(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},\ Q^n(x) = \begin{bmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{bmatrix}.$$

In [25], the Cassini-like (Simpson) formula for Fibonacci polynomial $F_n(x)$ is given as

$$F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n. \tag{5}$$

Using the above recurrence relation, we can reach the following equation [41].

$$F_{n+2}(x) = (1 + x^2)F_n(x) + xF_{n-1}(x). \tag{6}$$

The relations between Fibonacci polynomials and the diagonal of Pascal's triangle were generalized in [22] by Hoggatt and Bicknell in 1973. In the same year, they expressed Fibonacci polynomials as complex hyperbolic functions. They obtained the root formula for these polynomials in [21]. Then, the roots of the derivative of Fibonacci polynomials were obtained in [39].

There are many studies in the literature that state the conditions under which these groups or semi-groups are free groups (semi-groups) or not free groups (semi-groups). The main results can be given as [2], [4], [8], [10], [11], [14], [29], [30], [33], [40], [41], [42]. In these studies, the provision of different conditions about the freeness of linear groups or semi-groups that have two or more generators is obtained with similar approaches. We briefly summarize some of these studies: In [33], when $a, b, c, d, \alpha, \beta, \gamma, \delta \geq 0$; $d - a \geq 2$; $\delta - \alpha \geq 2$; the matrices

$$A = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix}, B = \begin{bmatrix} -\alpha & -\beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{R})$$

generate a free group. In [2], Bachmuth proved when $x, y, z \in \mathbb{C}$; $|x|, |y|, |z| \geq 4.45$; the following three matrices

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1-z & -z \\ z & 1-z \end{bmatrix}$$

generate a free group.

In [8], [10], [40], [41], the matrix representations of the generators were defined by the following form of a linear group in the same form and complementary studies were carried out for different conditions of this form in these articles. We briefly express these studies as follows.

They examine the generators of the linear group which

$$A_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ and } B_b = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

when a and $b \in \mathbb{C}$.

Sanov, Brenner, and Chang proved the group's freeness while these generators A_a and B_b have the following conditions

$a = b = 2$; $a = b$ and $|a| \geq 2$; $|ab| \geq 2, |ab - 2| \geq 2$ and $|ab + 2| \geq 2$, respectively.

Also, in [41], Słanina proved the group is not free which generators mentioned above A_a and B_a for a that root of Fibonacci polynomials.

In [41], some facts about Fibonacci polynomials are used to solve this problem.

Słanina showed some properties about the linear group as follows.

(i) a is a root of Fibonacci polynomial $F_{2n}(x)$ if and only if $(A_aB_a)^n = I$.
(ii) a is a root of Fibonacci polynomial $F_{2n+1}(x)$ if and only if $(A_aB_a)^n(B_aA_a)^n$ is a lower triangular matrix which commutes with B_a .

Hence, we get if a is a root of $F_{2n}(x)$, $F_{2n+1}(a) = F_{2n-1}(a) = \pm 1$ from (i).

In [19], Hecke introduced the groups $H(\lambda)$ generated by two Möbius transformations

$$T(z) = -\frac{1}{z} \text{ and } U(z) = z + \lambda$$

where λ is a fixed positive real number. Let $S = TU$ i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

The transformation $\frac{az+b}{cz+d}$ is represented by the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $-A$. Notice that, T and S have the matrix representations are

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & -1 \\ 1 & \lambda \end{bmatrix}.$$

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$ or $\lambda \geq 2$. These groups have come to be known as the Hecke groups and denoted by $H(\lambda_q)$, $H(\lambda)$ for $q \geq 3, \lambda \geq 2$, respectively. Hecke group $H(\lambda_q)$ is the Fuchsian group of the first kind when $\lambda = \lambda_q$ or $\lambda = 2$ and $H(\lambda)$ is the Fuchsian group of the second kind when $\lambda > 2$. In this study, we focus the case $\lambda = \lambda_q, q \geq 3$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q and it has a presentation

$$H(\lambda_q) = H_q = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q.$$

Some important Hecke groups H_q are $H_3 = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$, and $H_6 = H(\sqrt{3})$.

Lehner studied in [27], a more general class $H_{p,q}$ of Hecke groups H_q, by taking

$$X(z) = \frac{-1}{z-\lambda_p} \text{ and } V(z) = z + \lambda_p + \lambda_q$$

where $2 \leq p \leq q$, $p + q > 4$. Here, if we take $Y = XV = \frac{-1}{z+\lambda_q}$ then the group presentation is

$$H_{p,q} = \langle X, Y \mid X^p = Y^q = I \rangle \cong C_p * C_q$$

These groups are named as generalized Hecke groups $H_{p,q}$. Also, from [27] $H_{2,q} = H_q$. Furthermore, all Hecke groups H_q are included in generalized Hecke groups $H_{p,q}$. The modular group, Hecke groups, and generalized Hecke groups have been studied extensively. (See for more details [5], [6], [9], [19], [23], [24], [36], [37].) Moreover, there are many remarkable studies on $2\cos\frac{\pi}{q}$ and $\cos\frac{2\pi}{q}$ in the literature. Finding the minimal polynomial of $\cos\frac{2\pi}{q}$ is an old problem due to its connection to the cyclotomic polynomials. The algebraic numbers are investigated in many papers related to Chebyshev polynomials, Gaussian periods, Dickson polynomials, Ramanujan sums, and Möbius inversion (see for more details [1], [3], [17], [26]).

In this study, we focus on the roots of Fibonacci polynomials. In Section 2, we give some background knowledge about Fibonacci polynomials. Then, we study Fibonacci polynomials in terms of complex hyperbolic functions in Section 3. We examine the roots of Fibonacci polynomials. Also, we investigate the image of a root of a polynomial under another member of the family. Finally, we obtain strong relationships between the roots of Fibonacci polynomials and the modular group, Hecke groups, generalized Hecke groups in Section 4.

2. Motivation and background

In [21], V. E. Hoggatt and M. Bicknell have been obtained the roots of large classes of polynomials Fibonacci and Lucas using hyperbolic trigonometric functions. Hence, the general root formulas for the polynomials have been obtained. This contribution is quite remarkable for the fundamental theorem of algebra and the Abel-Ruffini theorem. There are numerous papers on this topic from different aspects (see [7], [16], [18], [31], [38], [39]).

Hoggatt and Bicknell studied on hyperbolic function represent of Fibonacci polynomials as follows.

Theorem 2.1. *[21] Let $x = 2\sinh z$ then,*

$$F_{2n}(x) = \frac{e^{2nz} - (-1)^{2n} e^{-2nz}}{e^{z} + e^{-z}} = \frac{\sinh 2nz}{\cosh z} \tag{7}$$

$$F_{2n+1}(x) = \frac{e^{(2n+1)z} - (-1)^{2n+1} e^{-(2n+1)z}}{e^{z} + e^{-z}} = \frac{\cosh (2n+1)z}{\cosh z} \tag{8}$$

Theorem 2.2. *[21] Let $x = 2i\cosh z$ then,*

$$F_{n}(x) = \frac{i^{n} e^{nz} - i^{n} e^{-nz}}{ie^{z} - ie^{-z}} = i^{n-1} \frac{\sinh nz}{\sinh z} \tag{9}$$

It is known from W. N. H. Abel that an algebraic equation of degree five or more has no solution. Also, considering the Abel-Ruffini theorem, we can interpret that the general root formulas for Fibonacci polynomials are very valuable. At that point, the existence of the formulas is outstanding and significant for this study.

3. Main results

In this section, we prove the root of the Fibonacci polynomial by clear expression compared with [21]. We give some results about roots of Fibonacci polynomials. We consider Fibonacci polynomials as complex hyperbolic functions. Then, we get interesting identities about images of a root of a Fibonacci polynomial under another member of the family.

Theorem 3.1. *The roots of Fibonacci polynomials are*

$$F_{2n}(x) = 0 : x = \pm 2i \sin \frac{k\pi}{2n} \tag{10}$$

$$F_{2n+1}(x) = 0 : x = \pm 2i \sin \frac{(2k+1)\pi}{(2n+1)2} \tag{11}$$

where $k = 0, 1, \ldots, n-1$.

Proof. Firstly, we deal the roots of the even subscripted Fibonacci polynomials. Consider the Theorem 2.1. If $F_{2n}(x) = 0$ then, $\frac{\sinh 2nz}{\cosh z} = 0$ which yields $\sinh 2nz = 0$ and $\cosh z \neq 0$. Therefore,

$$\sinh 2nz = \sinh(2na + i2nb) = \sinh 2na \cos 2nb + i \cosh 2na \sin 2nb = 0$$

$$\cosh z = \cosh(a + ib) = \cosh a \cos b + i \sinh a \sin b \neq 0$$

for $z = a + ib$, where $a, b \in \mathbb{R}$. Since $\cosh 2na \geq 1$ for $n \in \mathbb{N}$, $\sin 2nb$ must be zero. Hence, $b = \frac{k\pi}{2n}$ for $0 \leq k \leq 2n-1$ and $k \in \mathbb{N}$. We use this in the real part of the preceding equation in the above line.

$$\sinh 2na \cos 2n \frac{k\pi}{2n} = \sinh 2na \cos k\pi = 0$$

Here, $a = 0$ because $\sinh 2na$ must be zero. The error now we have is the Fibonacci polynomial $F_{2n}(x)$ has degree $2n-1$. Hence, we must collect at most $2n-1$ zeros. Unlikely, we have one value of b which should not be a member. That one is obtained when $k = n$ is impossible because it leads the denominator $F_{2n}(x) = \frac{\sinh 2nz}{\cosh z}$ to be zero. Therefore, we omit it. It can be easily seen that $\cosh a \cos b + i \sinh a \sin b \neq 0$ for other possible values of k. Since $F_{2n}(x)$ is an odd function, we can restrict k as $0 \leq k \leq n-1$ and give roots as $x = \pm 2i \sin \frac{k\pi}{2n}$.
Roots of odd subscripted Fibonacci polynomials can be calculated similarly. □

Theorem 3.2. *The roots of the n^{th} Fibonacci polynomial $F_n(x)$ are $x = 2i \cos \frac{k\pi}{n}$ for $k = 1, 2, \ldots, n-1$.*

Proof. Let $F_n(x) = i^{n-1} \frac{\sinh nz}{\sinh z} = 0$ for $x = 2i \cosh z$ from Theorem 2.2. Then, the nominator $\sinh nz = 0$ and the denominator $\sinh z \neq 0$. For $z = a + ib$, where a and b are real numbers.

$$\sinh nz = \sinh(na + inb) = \sinh na \cos nb + i \cosh na \sin nb = 0$$

$$\sinh z = \sinh(a + ib) = \sinh a \cos b + i \cosh a \sin b \neq 0$$

Hence, the real numbers a and b must satisfy both of the following equalities.

$$\sinh na \cos nb = 0 \tag{12}$$

and

$$\cosh na \sin nb = 0 \tag{13}$$

Furthermore, the real numbers a and b which have the equations above must also satisfy at least one of the followings.

$$\sinh a \cos b \neq 0 \tag{14}$$

or

$$\cosh a \sin b \neq 0 \tag{15}$$

From Equation 13, we have $\sin nb = 0$, since $\cosh nb = \frac{e^{nb}+e^{-nb}}{2} \geq 1$ for all $b \in \mathbb{R}$. Therefore, $b = \frac{k\pi}{n}$ for some $k \in \mathbb{Z}$ excluding multiples of n because of Equation 15. In addition, one can solve $a = 0$ from Equation 12. As a result, we have $F_n(x) = 0$ if and only if $x = 2i \cos \frac{k\pi}{n}$. Observe that, $F_n(x) = F_n(-x) = 0$ where x is a zero of $F_n(x)$. Owing to the fact that Fibonacci polynomial $F_n(x)$ has degree $n-1$, we restrict $k = 1, 2, \ldots, n-1$. □

Now, we are ready to calculate images of the roots of a Fibonacci polynomial under other members of the family.

Theorem 3.3. *If a is a root of the Fibonacci polynomial $F_{2n-1}(x)$, i.e. $F_{2n-1}(a) = 0$, then $F_{2n}(a) = \pm i$ and $F_{2n+1}(a) = \pm ai$.*

Proof. Let $x = a$ be a root of $F_{2n-1}(x)$. Using the Cassini-like formula for Fibonacci polynomials, we get

$$F_{2n}^2(a) = -1$$

Therefore,

$$F_{2n}(a) = \pm i.$$

Now, we need the image of a under the polynomial $F_{2n+1}(x)$. For this purpose, we use the recurrence formula

$$F_{2n+1}(x) = xF_{2n}(x) + F_{2n-1}(x)$$

If we put $x = a$, we obtain $F_{2n+1}(a) = aF_{2n}(a) = \pm ai$. Thus, we get the desired result. □

Corollary 3.4. *$F_{2n-1}(a) = 0$, implies $F_{2n}(a).F_{2n+1}(a) = -a$.*

Theorem 3.5. *If a is zero of the Fibonacci polynomial $F_{2n-1}(x)$ i.e. $F_{2n-1}(a) = 0$, we have $F_{2n}(a)F_{2n+1}(a) \neq 0$.*

Proof. We know from Corollary 3.4 that if $F_{2n-1}(a) = 0$ then $F_{2n}(a).F_{2n+1}(a) = -a$. Hence, if $F_{2n}(a).F_{2n+1}(a) = 0$ then a must be zero. The roots of $F_{2n-1}(x)$ can be calculated by Theorem 3.2 as

$$a = 2i \cos \frac{k\pi}{2n-1}$$

for $k = 1, 2, \ldots, 2n-2$. By considering all possible values of k, we have $\frac{k\pi}{2n-1} = \frac{\pi}{2}$. By elementary calculations, one has $n - k = \frac{1}{2}$ which is a contradiction. □

It is able to verify the following theorems by using the same techniques. So, we leave the proofs to the readers.

Theorem 3.6. *If a is a root of the Fibonacci polynomial $F_{2n+1}(x)$, i.e. $F_{2n+1}(a) = 0$ then, $F_{2n}(a) = \pm i$ and $F_{2n-1}(a) = \mp ai$.*

Proof. It can be proved using by recurrence relation and the Cassini formula for Fibonacci polynomials. □

Corollary 3.7. *$F_{2n+1}(a) = 0$, implies $F_{2n}(a).F_{2n-1}(a) = a$.*

Theorem 3.8. *If a is root of the Fibonacci polynomial $F_{2n+1}(x)$ i.e. $F_{2n+1}(a) = 0$, we have $F_{2n}(a)F_{2n-1}(a) \neq 0$.*

Proof. It can be proved by using the same technique in Theorem 3.5. □

Now, we need some identities proved in [15] and [25].

Theorem 3.9.

$$\sum_{r=1}^{n} F_r(x) = \frac{F_{n+1}(x) + F_n(x) - 1}{x} \tag{16}$$

$$F_k(x) | F_{nk}(x) \tag{17}$$

$$F_{m+n}(x) = F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) \tag{18}$$

$$\frac{dF_n(x)}{dx} = \frac{nF_{n+1}(x) - xF_n(x) + nF_{n-1}(x)}{x^2+4} = \frac{2nF_{n-1}(x) + (n-1)xF_n(x)}{x^2+4} \tag{19}$$

$$\int_0^x F_n(x)\, dx = \frac{1}{n}\Big(F_{n+1}(x) + F_{n-1}(x) - F_{n+1}(0) - F_{n-1}(0)\Big) \tag{20}$$

Proof. This identities were proved in [15] and [25]. □

We obtain some interesting properties about the roots of Fibonacci polynomials via the above equalities.

Theorem 3.10.

(i) $F_{2n+1}(a) = 0 \leftrightarrow F_1(a) + F_2(a) + \ldots + F_{2n}(a) = \frac{\pm i - 1}{a} = \frac{\pm i - 1}{\pm 2i \sin \frac{(2k+1)\pi}{4n+2}}$ *for* $k = 0, 1, \ldots, n-1$

(ii) $F_k(a) = 0 \leftrightarrow F_{nk}(a) = 0$ *for* $k, n \in \mathbb{N}$

(iii) $F_{2n}(a) = 0 \leftrightarrow F_{m+2n+1}(x) = \pm F_{m+1}(x)$

(iv) $F_{2n+1}(a) = 0 \leftrightarrow F_{m+2n+1}(x) = \pm i F_m(x)$

(v) $F_{2n}(a) = 0 \leftrightarrow F_{m+2n}(x) = \pm F_m(x)$

(vi) $F_{2n-1}(a) = 0 \leftrightarrow F_{m+2n}(x) = \pm i F_{m+1}(x)$

Proof. The proof can be seen by Theorem 3.2, Theorem 3.3, Theorem 3.6, and Theorem 3.9 as follows.

(i) It is proved by using Equation 16 and Equation 11 together with Theorem 3.6.

(ii) By using Equation 17 it is obtained.

(iii) This is followed by changing in Equation 18 n by odd subscript $2n+1$ and using $F_{2n+1}(a) = \pm 1$ when $F_{2n}(a) = 0$.

(iv) This is followed by changing in Equation 18 n by odd subscript $2n+1$ and using Theorem 3.6.

(v) This is followed by changing in Equation 18 n by even subscript $2n$ and using $F_{2n+1}(a) = \pm 1$ when $F_{2n}(a) = 0$.

(vi) This is followed by changing in Equation 18 n by even subscript $2n$ and using Theorem 3.3.

□

Theorem 3.11.

(i) $F_{2n}(a) = 0 \leftrightarrow \left.\frac{dF_{2n+1}(x)}{dx}\right|_{x=a} = in \tan \frac{k\pi}{2n} . \sec \frac{k\pi}{2n}$
for $k = 0, 1, \ldots, n-1$

(ii) $F_{2n+1}(a) = 0 \leftrightarrow \left.\frac{dF_{2n+1}(x)}{dx}\right|_{x=a} = \pm \frac{(4n+2)i}{4} . \sec^2 \frac{(2k+1)\pi}{4n+2}$
for $k = 0, 1, \ldots, n-1$

(iii) $F_{2n}(a) = 0 \leftrightarrow \left.\frac{dF_{2n}(x)}{dx}\right|_{x=a} = \pm n \sec^2 \frac{k\pi}{2n}$
for $k = 0, 1, \ldots, n-1$

(iv) $F_{2n-1}(a) = 0 \leftrightarrow \left.\frac{dF_{2n}(x)}{dx}\right|_{x=a} = \mp \frac{(4n-2)}{4} . \cot \frac{k\pi}{2n-1} . \csc \frac{k\pi}{2n-1}$
for $k = 1, 2, \ldots, 2n-2$

(v) $F_{2n+1}(a) = 0 \leftrightarrow \int_0^a F_{2n}(x)\, dx = \frac{\sin \frac{(2k+1)\pi}{4n+2} - 1}{n}$
for $k = 0, 1, \ldots, n-1$

(vi) $F_{2n}(a) = 0 \leftrightarrow \int_0^a F_{2n}(x)\, dx = \frac{\pm 1 - 1}{n}$

(vii) $F_{2n-1}(a) = 0 \leftrightarrow \int_0^a F_{2n}(x)\, dx = \frac{\mp \cos \frac{k\pi}{2n-1} - 1}{n}$
for $k = 1, 2, \ldots, 2n-2$

Proof. The proof can be seen by Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.6, and Theorem 3.9 as follows.

(i) This is followed by changing in Equation 19 n by odd subscript $2n+1$ and using $F_{2n+1}(a) = \pm 1$ when $F_{2n}(a) = 0$ together with Equation 10. Finally, the proof is completed via the below equalities using by some trigonometric function properties and algebraic calculations.
$$\left.\frac{dF_{2n+1}(x)}{dx}\right|_{x=a} = \pm \frac{2na}{a^2+4} = \pm \frac{\pm 4ni \sin \frac{k\pi}{2n}}{4\left(1-\sin^2 \frac{k\pi}{2n}\right)} = in \tan \frac{k\pi}{2n} . \sec \frac{k\pi}{2n}$$
for $k = 0, 1, \ldots, n-1$ when $F_{2n}(a) = 0$.

(ii) This is followed by changing in Equation 19 n by odd subscript $2n+1$ and using Theorem 3.6 together with Equation 11. Finally, the proof is completed via the below equalities using some trigonometric function properties and algebraic calculations.
$$\left.\frac{dF_{2n+1}(x)}{dx}\right|_{x=a} = \frac{\pm(4n+2)i}{a^2+4} = \frac{\pm(4n+2)i}{4\left(1-\sin^2 \frac{(2k+1)\pi}{4n+2}\right)} = \pm \frac{(4n+2)i}{4} . \sec^2 \frac{(2k+1)\pi}{4n+2}$$
for $k = 0, 1, \ldots, n-1$ when $F_{2n+1}(a) = 0$.

(iii) This is followed by changing in Equation 19 n by even subscript $2n$ and using $F_{2n+1}(a) = \pm 1$ when $F_{2n}(a) = 0$ together with Equation 10. From where, after some algebra the desired result is obtained via below equalities.
$$\left.\frac{dF_{2n}(x)}{dx}\right|_{x=a} = \pm \frac{4n}{a^2+4} = \frac{\pm n}{1-\sin^2 \frac{k\pi}{2n}} = \pm n \sec^2 \frac{k\pi}{2n}$$
for $k = 0, 1, \ldots, n-1$ when $F_{2n}(a) = 0$.

(iv) This is followed by changing in Equation 19 n by even subscript $2n$ and using Theorem 3.3 together with Theorem 3.2 as by changing in the theorem n by odd subscript $2n-1$. Finally, the proof is completed via the below equalities using some trigonometric function properties and algebraic calculations.
$\frac{dF_{2n}(x)}{dx}\Big|_{x=a} = \pm\frac{(2n-1)ai}{a^2+4} = \mp\frac{(4n-2)\cos\frac{k\pi}{4n-2}}{4\left(1-\cos^2\frac{k\pi}{2n-1}\right)} = \mp\frac{(4n-2)}{4}.\cot\frac{k\pi}{2n-1}.\csc\frac{k\pi}{2n-1}$
for $k = 1, 2, \ldots, 2n-2$ when $F_{2n-1}(a) = 0$.

(v) This is followed by changing in Equation 20 n by even subscript $2n$ and using Theorem 3.6 together with Equation 11. From where, after some algebra the desired result is obtained via below equalities.
$\int_0^a F_{2n}(x)\,dx = \frac{\mp ai-2}{2n} = \frac{\sin\frac{(2k+1)\pi}{4n+2}-1}{n}$
for $k = 0, 1, \ldots, n-1$ when $F_{2n+1}(a) = 0$.

(vi) This is followed by changing in Equation 20 n by even subscript $2n$ and using $F_{2n+1}(a) = F_{2n-1}(a) = \pm 1$ when $F_{2n}(a) = 0$.

(vii) This is followed by changing in Equation 20 n by even subscript $2n$ and using Theorem 3.3 together with Theorem 3.2 as by changing in the theorem n by odd subscript $2n-1$. From where, after some algebra the desired result is obtained via below equalities.
$\int_0^a F_{2n}(x)\,dx = \frac{\pm ai-2}{2n} = \frac{\mp\cos\frac{k\pi}{2n-1}-1}{n}$
for $k = 1, 2, \ldots, 2n-2$ when $F_{2n-1}(a) = 0$.

□

4. Relationships between the roots of Fibonacci polynomials and the modular group& Hecke groups& generalized Hecke groups

In this section, we consider the complex numbers as vectors in the complex plane. All the roots of Fibonacci polynomials are pure imaginary complex numbers. Also, each norm of the roots of a Fibonacci polynomial is smaller than two. We interpret the roots in the complex plane as related to the parameter of the modular group, Hecke groups, generalized Hecke groups.

Corollary 4.1. *We examine the relationship between parameter of the modular group and roots of Fibonacci polynomial in the complex plane geometrically. The parameter of the modular group* $\lambda_3 = 2\cos\frac{\pi}{3}$*. All of the roots of Fibonacci polynomial* $F_3(x)$ *are known as* $2i\cos\frac{\pi}{3}$ *and* $2i\cos\frac{2\pi}{3}$ *from Theorem 3.2 for* $k = 1, 2$*. If the first root* $2i\cos\frac{\pi}{3}$ *of the Fibonacci polynomial* $F_3(x)$ *is rotated* 270 *degrees counterclockwise around the origin in the complex plane, the parameter of the modular group is obtained. Therefore, we can state that the Fibonacci polynomial* $F_3(x)$ *generates a parameter for the modular group as a geometric interpretation.*

Corollary 4.2. *The parameter of Hecke group as Fuchsian group of first kind is* $\lambda_q = 2\cos\frac{\pi}{q}$ *for* $q \geq 3$ *and all of the roots of Fibonacci polynomial* $F_q(x)$ *are known as* $2i\cos\frac{\pi}{q}$, $2i\cos\frac{2\pi}{q}$,$\ldots$, $2i\cos\frac{(q-1)\pi}{q}$ *from Theorem 3.2 for* $k = 1, 2, \ldots, q-1$*. If the first root* $2i\cos\frac{\pi}{q}$ *of the Fibonacci polynomial* $F_q(x)$ *is rotated* 270 *degrees counterclockwise around the origin in the complex plane, the parameter of the Hecke group is obtained. Therefore, we can state geometrically that the Fibonacci polynomial* $F_q(x)$ *generates a parameter for the Hecke group as Fuchsian group of first kind.*

Corollary 4.3. *The parameters of generalized Hecke groups are* $\lambda_p = 2\cos\frac{\pi}{p}$ *and* $\lambda_q = 2\cos\frac{\pi}{q}$*. Also, all the roots of Fibonacci polynomial* $F_p(x)$ *are known as* $2i\cos\frac{\pi}{p}$, $2i\cos\frac{2\pi}{p}$,$\ldots$, $2i\cos\frac{(p-1)\pi}{p}$ *from Theorem 3.2 for* $k = 1, 2, \ldots, p-1$*. All the roots of Fibonacci polynomial* $F_q(x)$ *are known as* $2i\cos\frac{\pi}{q}$, $2i\cos\frac{2\pi}{q}$,$\ldots$, $2i\cos\frac{(q-1)\pi}{q}$ *from Theorem 3.2 for* $k = 1, 2, \ldots, q-1$*. If the first roots* $2i\cos\frac{\pi}{p}$ *of the Fibonacci polynomial* $F_q(x)$ *and* $2i\cos\frac{\pi}{q}$ *of the Fibonacci polynomial* $F_q(x)$ *are rotated* 270 *degrees counterclockwise around the origin in the complex plane, the parameters of the generalized Hecke groups are obtained. Therefore, we can state geometrically that the Fibonacci polynomial* $F_p(x)$ *and* $F_q(x)$ *generate parameters for the generalized Hecke groups.*

Remark 4.4. *Using the root of the Fibonacci polynomial, it is not the only way to obtain a modular group parameter. For instance, another way finding parameter of the modular group is using the Fibonacci polynomial $F_6(x)$. Then the second root of the polynomial $F_6(x)$ is obtained as $2i\cos\frac{\pi}{3}$ from Theorem 3.2 for $k = 2$. Also, the parameter is obtained from another Fibonacci polynomial $F_9(x)$ using Theorem 3.2 for $k = 3$. Therefore, we can state that the parameter of the modular group related to the Fibonacci polynomials $F_{3t}(x)$ when $t \in \mathbb{N}$. More generally, the parameter of the Hecke group H_m can be derived from the Fibonacci polynomial $F_{km}(x)$ when k is a whole number and m is an integer greater than two.*

Remark 4.5. *Each Fibonacci polynomial $F_n(x)$ for $n \geq 3$ generates at least one parameter for the Hecke group. For example, the Fibonacci polynomial $F_3(x)$ generates one parameter as $2\cos\frac{\pi}{3}$ via the root $2i\cos\frac{\pi}{3}$ rotated 270 degrees counterclockwise around the origin in the complex plane. The Fibonacci polynomial $F_4(x)$ generates one parameter as $2\cos\frac{\pi}{4}$ via the root $2i\cos\frac{\pi}{4}$ rotated 270 degrees counterclockwise around the origin in the complex plane. The Fibonacci polynomial $F_6(x)$ generates two parameters as $2\cos\frac{\pi}{6}$ and $2\cos\frac{\pi}{3}$ via the roots $2i\cos\frac{\pi}{6}$ and $2i\cos\frac{2\pi}{6}$ rotated 270 degrees counterclockwise around the origin in the complex plane.*

Corollary 4.6. *We set a general way to get the relationship between the parameter of Hecke group as Fuchsian group of the first kind and Fibonacci polynomial $F_n(x)$. All the roots of Fibonacci polynomial $F_n(x)$ are known as $2i\cos\frac{k\pi}{n}$ for $k = 1, 2, \ldots, n-1$ from Theorem 3.2. $F_n(x)$ generates the parameter for Hecke group every provided condition that k divides n except for $k = \frac{n}{2}$ and $k = n$. For instance, $F_{10}(x) = x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$ generates exactly two parameters for Hecke groups denoted by H_{10} and $H_5 = H(\frac{1+\sqrt{5}}{2})$.*

Theorem 4.7. *(Birol-Hecke-Fibonacci Theorem)*
The number of the parameters for Hecke groups generated by $F_n(x)$ is calculated by the formula

$$F(n) = \begin{cases} \prod_{i=1}^{t}(a_i + 1) - 2 & \text{if } n \text{ even} \\ \prod_{i=1}^{t}(a_i + 1) - 1 & \text{if } n \text{ odd} \end{cases}$$

where $n = \prod_{i=1}^{t} p_i^{a_i}$ for p_i distinct prime numbers and a_i positive integers.

Proof. It can be proved using the fundamental theorem of arithmetic, the formula for the total number of divisors of a number considering the root formula of Fibonacci polynomials and the parameter of the Hecke group as $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$. □

Corollary 4.8. *Considering the polynomial space, the $\{F_n(x) : n \geq 3\}$ set of Fibonacci polynomials is a relation with the ability to generate parameter for Hecke groups. This relation has reflection and symmetry properties.*

Remark 4.9. *We call the above relation as ϱ. Notice that, ϱ is not reflexive relation. We give a counterexample to prove that. $\big(F_3(x), F_{21}(x)\big) \in \varrho$ via H_3 and $\big(F_{21}(x), F_7(x)\big) \in \varrho$ via H_7 but $\big(F_3(x), F_7(x)\big) \notin \varrho$. Although $F_3(x)$ and $F_7(x)$ generate one parameter for the Hecke groups, these polynomials do not generate a common parameter for any Hecke group. $F_3(x)$ and $F_7(x)$ generate a parameter for the Hecke groups H_3 and H_7, respectively.*

Definition 4.10. *(Birol-Hecke-Fibonacci Number Sequence)*
We define a new number sequence derived from the Theorem 4.7. This sequence shows the relationship between the root of the Fibonacci polynomial and the Hecke groups interestingly. The first, the fourth and the thirteenth terms of the number sequence are obtained as 1, 2 and 3 from Fibonacci polynomials $F_3(x)$, $F_6(x)$ and $F_{15}(x)$, respectively. This sequence is as follows.

1, 1, 1, 2, 1, 2, 2, 2, 1, 4, 1, 2, 3, 3, 1, 4, 1, 4, 3,...

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