APPROXIMATION PROPERTIES OF BERNSTEIN'S SINGULAR INTEGRALS IN VARIABLE EXPONENT LEBESGUE SPACES ON THE REAL AXIS

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ABSTRACT. In generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent $p(\cdot)$ defined on the real axis, we obtain several inequalities of approximation by integral functions of finite degree. Approximation properties of Bernstein singular integrals in these spaces are obtained. Estimates of simultaneous approximation by integral functions of finite degree in $L^{p(\cdot)}$ are proved.

1. INTRODUCTION

In this work we consider approximation properties of Bernstein's singular integrals for functions given in the variable exponent Lebesgue spaces $L^{p(x)}(\mathbb{R})$. This scale of function spaces were studied in detail in books Uribe-Fiorenza [15], Diening, Harjulehto, Hästö, Růžička [17] and Sharapudinov [40]. $L^{p(x)}(\mathbb{R})$ has many applications in several branches of mathematics such as elasticity theory [50], fluid mechanics [38], [37], differential operators [38], [18], nonlinear Dirichlet boundary value problems [32], nonstandard growth [50] and variational calculus. Variable exponent works started with W. Orlicz [35] and developed in many directions. For example, $L^{p(x)}(\mathbb{R})$ is a modular space ([33]) and under the condition $p^+ := esssup_{x \in \mathbb{R}} p(x) < \infty$, $L^{p(x)}(\mathbb{R})$ becomes a particular case of the Musielak-Orlicz spaces [33]. Starting from nineties, studies on $L^{p(x)}(\mathbb{R})$ has reached a positive momentum: See [32], [39], [20], [16] and many others.

In variable exponent Lebesgue spaces on $[0, 2\pi]$ (or $[0, 1]$), some fundamental results corresponding to the approximation of function have been obtained by Sharapudinov [41–45]. Some results on approximation in $L^{p(x)}([0, 2\pi])$ or other function classes can be seen e.g. in $[1, 3–6, 8, 9, 19, 21–25, 27–30, 48]$.

In this work, we aim to obtain simultaneous theorems on approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis $\mathbb{R}$.

Approximation by entire function of finite degree in the real axis started by the works of Bernstein [11, 12], N. Wiener and R. Paley [36], N. I. Ahiezer [2], S. M. Nikolskii [34]. Note that an entire function of finite exponential type is merely an entire function of order 1 and finite type that in approximation theory, these often play an important role similar to trigonometric polynomials in the case of approximation of periodic functions.

Note that, some results on approximation by entire integral functions of finite degree were obtained by Ibragimov [26] and Taberski [46, 47] in the classical Lebesgue spaces $L^p(\mathbb{R})$.

We can give some required definitions. We denote by P the class of exponents $p(x) : \mathbb{R} \to [1, \infty)$ such that $p(x)$ is a measurable function and $p(x)$ satisfy conditions

$$1 \leq p_- := essinf_{x \in \mathbb{R}} p(x), \quad p^+ := esssup_{x \in \mathbb{R}} p(x) < \infty. \tag{1}$$

We define $L^{p(\cdot)} := L^{p(\cdot)}(\mathbb{R})$ as the set of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$I_{p(\cdot)}\left(\frac{f}{\lambda}\right) := \int_{\mathbb{R}} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty \tag{2}$$

for some $\lambda > 0$. The set of functions $L^{p(\cdot)}$, with norm

$$\|f\|_{p(\cdot)} := \inf\left\{\eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) < 1\right\}$$

is a Banach space.

For $p \in P$ we define its conjugate $p'(x) := \frac{p(x)}{p(x)-1}$ for $p(x) > 1$ and $p'(x) := \infty$ for $p(x) = 1$.

For $i \in \mathbb{N}$, all constants $\mathbf{c}_i$ (or $\mathbf{c}$) will be some positive numbers such that $\mathbf{c}_i$ will depend on main parameters of the problem. In some cases we will use temporaryly some generic constans $C, c > 0$ for clarity (for example in statements of some theorems). We will give explicit constants in the proofs but these constants are not best constants.

Throughout this paper symbol $\mathfrak{A} \lesssim \mathcal{B}$ will mean that there exists a constant C depending only on unimportant parameters in question such that inequality $\mathfrak{A} \leq C\mathcal{B}$ holds.

Definition 1. *Let P^{Log} be a subclass ([17]) of P such that there exist constants $\mathbf{c}_1, \mathbf{c}_2 > 0$, $\mathbf{c}_3 \in \mathbb{R}$ with properties*

$$|p(x) - p(y)| \ln\left(e + 1/|x - y|\right) \leq \mathbf{c}_1 < \infty, \quad \forall x, y \in \mathbb{R}, \tag{3}$$

$$|p(x) - \mathbf{c}_3| \ln (e + |x|) \leq \mathbf{c}_2 < \infty, \quad \forall x \in \mathbb{R}. \tag{4}$$

2. Transference Result

Let C_0^∞ be class of infinitely times continuously differentiable functions ϕ with compact support $spt\phi := \overline{\{x \in \mathbb{R} : \phi(x) \neq 0\}}$. Let $C(A)$ be the class of continuous functions defined on A. Define $\|f\|_{C(A)} := \sup \{|f(x)| : x \in A\}$ for $f \in C(A)$.

For given $f \in L^{p(\cdot)}$ we can define an auxiliary function as follows: Define

$$F_f(u) := F_{f,G}(u) := \int_\mathbb{R} f(u+x) |G(x)| \, dx, \quad u \in \mathbb{R}, \tag{5}$$

where $G \in L^{p'(\cdot)} \cap C_0^\infty$ and $\|G\|_{p'(\cdot)} \leq 1$. Also we set $\mathbf{c}_0 := \|G\|_{C(\mathbb{R})}$.

Theorem 1. *Let $p \in P^{Log}$ and $f,g \in L^{p(\cdot)}$. If*

$$\|F_{f,G}\|_{C(\mathbb{R})} \lesssim \|F_{g,G}\|_{C(\mathbb{R})},$$

with an absolute positive constant, then, we have following norm inequality

$$\|f\|_{p(\cdot)} \lesssim \|g\|_{p(\cdot)}$$

with a positive constant depending only on p.

3. Mollifiers and forward Steklov means in $L^{p(\cdot)}$

Definition 2. *Suppose that $0 < \delta < \infty$ and $\tau \in \mathbb{R}$. We define ([44]) family of translated Steklov operators $\{\mathcal{S}_{\delta,\tau}f\}$, by*

$$\mathcal{S}_{\delta,\tau}f(x) := \frac{1}{\delta} \int_{x+\tau-\delta/2}^{x+\tau+\delta/2} f(t) \, dt, \quad x \in \mathbb{R} \tag{6}$$

for locally integrable function f defined on $\mathbb{R}$.

Let f and g be two real-valued measurable functions on $\mathbb{R}$. We define the convolution $f * g$ of f and g by setting $(f * g)(x) = \int_\mathbb{R} f(y)g(x-y)dy$ for $x \in \mathbb{R}$ for which the integral exists in $\mathbb{R}$.

The following result on mollifiers in variable exponent Lebesgue spaces is obtained by D. Cruz-Uribe and A. Fiorenza (see [14]).

Definition 3. *Let $\phi \in L^1(\mathbb{R})$ and $\int_\mathbb{R} \phi(t) \, dt = 1$. For each $t > 0$ we define $\phi_t(x) = \frac{1}{t}\phi\left(\frac{x}{t}\right)$. Such sequence $\{\phi_t\}$ will be called approximate identity. A function*

$$\tilde{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|$$

will be called radial majorant of ϕ. If $\tilde{\phi} \in L^1(\mathbb{R})$, then, sequence $\{\phi_t\}$ will be called potential-type approximate identity.

Theorem 2. *([14]) Suppose $p \in P^{Log}$, $f \in L^{p(\cdot)}$, ϕ is a potential-type approximate identity. Then, for any $t > 0$,*

$$\|f * \phi_t\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

and

$$\lim_{t \to 0} \|f * \phi_t - f\|_{p(\cdot)} = 0$$

hold with a positive constant depend on p.

As a corollary of Theorem 1 we have

Theorem 3. *Suppose that $p \in P^{Log}$, $0 < \delta < \infty$ and $\tau \in \mathbb{R}$. Then, the family of operators $\{\mathcal{S}_{\delta,\tau}f\}$, defined by (6), is uniformly bounded (in δ and τ) in $L^{p(\cdot)}$, namely, for any $0 < \delta < \infty$ and $\tau \in \mathbb{R}$ norm inequality*

$$\|\mathcal{S}_{\delta,\tau}f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p.

As a corollary of Theorem 3 we get

Corollary 1. *Let $p \in P^{Log}$, $0 < \delta < \infty$, $f \in L^{p(\cdot)}$. If $\tau = \delta/2$ then,*

$$T_\delta f(x) := \mathcal{S}_{\delta,\delta/2}f(x) = \frac{1}{\delta} \int_0^\delta f(x+t)\, dt$$

and

$$\|T_\delta f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p.

4. Modulus of Smoothness and K-functional

If $f \in L^{p(\cdot)}$ and $0 \le \delta < \infty$, $r \in \mathbb{N}$, then

$$\Omega_r(f,\delta)_{p(\cdot)} := \|(I - T_\delta)^r f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \qquad (7)$$

Here I is the identity operator. In what follows $W_r^{p(\cdot)}$, $r \in \mathbb{N}$, will be the class of functions $f \in L^{p(\cdot)}$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^{p(\cdot)}$.

Remark 1. *For $p \in P^{Log}$, f, $g \in L^{p(\cdot)}$ and $0 \le \delta < \infty$, the modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot)}$, has the following usual properties:*

(i) $\Omega_r(f,\delta)_{p(\cdot)}$ is non-negative; non-decreasing function of $\delta \ge 0$;

(ii) $\Omega_r(f+g,\cdot)_{p(\cdot)} \le \Omega_r(f,\cdot)_{p(\cdot)} + \Omega_r(g,\cdot)_{p(\cdot)}$;

(iii) $\lim_{\delta \to 0^+} \Omega_r(f,\delta)_{p(\cdot)} = 0$;

(iv) $\Omega_r(f,\delta)_{p(\cdot)} \lesssim \delta^r \|f^{(r)}\|_{p(\cdot)}$ for $r \in \mathbb{N}$, $f \in W_r^{p(\cdot)}$ and $\delta > 0$.

Indeed: (ii) follows from definition. (iii) is follow from (7), (3.4) and Theorem 3.1 of [7]. (iv) follows from Lemma 3.2 of [7]. (i) follows from Lemma 1 given below.

Definition 4. *Define, for $f \in L^{p(\cdot)}$, $p \in P^{Log}$, and $\delta > 0$,*

$$(\mathfrak{R}_\delta f)(\cdot) := \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\frac{1}{h} \int_0^h f(\cdot + t)\, dt \right) dh.$$

Remark 2. *Note that, for $0 < \delta < \infty$, $p \in P^{Log}$ we know from Corollary 1 that*

$$\|\mathfrak{R}_\delta f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

and, hence, $f - \mathfrak{R}_\delta f \in L^{p(\cdot)}$ for $f \in L^{p(\cdot)}$.

We set $\mathfrak{R}_\delta^r f := (\mathfrak{R}_\delta f)^r$.

Lemma 1. *Let $0 < h \le \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then*

$$\|(I - T_h)f\|_{p(\cdot)} \lesssim \|(I - T_\delta)f\|_{p(\cdot)} \tag{8}$$

holds with a positive constant depend on p.

Lemma 2. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then*

$$\|(I - \mathfrak{R}_\delta)f\|_{p(\cdot)} \lesssim \|(I - T_\delta)f\|_{p(\cdot)}$$

holds with a positive constant depend on p.

Remark 3. *The function $\mathfrak{R}_\delta f$ is absolutely continuous and differentiable a.e. (almost everywhere) on $\mathbb{R}$ (see [43, (5.2) of Theorem 4]).*

The following lemma is obvious from definitions.

Lemma 3. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in W_1^{p(\cdot)}$. Then*

$$\frac{d}{dx}\mathfrak{R}_\delta f = \mathfrak{R}_\delta \frac{d}{dx} f \quad and \quad \frac{d}{dx}T_\delta f = T_\delta \frac{d}{dx} f \tag{9}$$

a.e. on $\mathbb{R}$.

Lemma 4. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$ be given. Then*

$$\delta \left\| \frac{d}{dx}\mathfrak{R}_\delta f \right\|_{p(\cdot)} \lesssim \|(I - T_\delta)f\|_{p(\cdot)} \tag{10}$$

holds with a positive constant depend on p.

The following lemma can be proved using induction on r.

Lemma 5. *Let $0 < \delta < \infty$, $r - 1 \in \mathbb{N}$, $p \in P^{Log}$, and $f \in L^{p(\cdot)}$ be given. Then*

$$\frac{d^r}{dx^r}\mathfrak{R}_\delta^r f = \frac{d}{dx}\mathfrak{R}_\delta \frac{d^{r-1}}{dx^{r-1}}\mathfrak{R}_\delta^{r-1} f.$$

Modulus of smoothness $\|(I - T_\delta)^r f\|_{p(\cdot)}$ and the K-functional

$$K_r\left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)}\right)_{p(\cdot)} := \inf_{g \in W_r^{p(\cdot)}} \left\{ \|f - g\|_{p(\cdot)} + \delta^r \left\|g^{(r)}\right\|_{p(\cdot)} \right\}$$

are equivalent:

Theorem 4. *If $r \in \mathbb{N}$, $p \in P^{Log}$, $f \in L^{p(\cdot)}$, and $\delta > 0$, then*

$$\frac{\left\| (I - T_\delta)^r f \right\|_{p(\cdot)}}{K_r \left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)} \right)_{p(\cdot)}} \approx 1 \tag{11}$$

holds for some positive constants depend on p, r.

5. Results on Simultaneous Approximation

Let $\mathcal{G}_\sigma (X)$ be the subclass of entire integral functions $f(z)$ of exponential type $\leq \sigma$ that belonging to X and

$$A_\sigma(f)_X := \inf_g \{ \| f - g \|_X : g \in \mathcal{G}_\sigma (X) \}.$$

Let $\mathcal{C}$ be the class of bounded uniformly continuous functions defined on $\mathbb{R}$. We set $\mathcal{G}_{\sigma,\infty} := \mathcal{G}_\sigma (\mathcal{C})$ and $\mathcal{G}_{\sigma,p(\cdot)} := \mathcal{G}_\sigma \left(L^{p(\cdot)} \right)$.

Remark 4. *([10, definition given in (5.3)]) Let $\sigma > 0$, $1 \leq p \leq \infty$, $f \in L^p (\mathbb{R})$,*

$$\vartheta (x) := \frac{2}{\pi} \frac{\sin (x/2) \sin(3x/2)}{x^2}$$

and

$$J (f, \sigma) = \sigma \int_\mathbb{R} f (x - u) \vartheta (\sigma u) \, du$$

be the delà Valèe Poussin operator ([10, definition given in (5.3)]). It is known (see (5.4)-(5.5) of [10]) that, if $f \in L^p (\mathbb{R})$, $1 \leq p \leq \infty$, then,
   (i) $J (f, \sigma) \in \mathcal{G}_{2\sigma} \left(L^p (\mathbb{R}) \right)$,
   (ii) $J (g_\sigma, \sigma) = g_\sigma$ for any $g_\sigma \in \mathcal{G}_\sigma (L^p (\mathbb{R}))$,
   (iii) $\| J (f, \sigma) \|_{L^p(\mathbb{R})} \leq \frac{3}{2} \| f \|_{L^p(\mathbb{R})}$,
   (iv) $(J (f, \sigma))^{(r)} = J \left(f^{(r)}, \sigma \right)$ for any $r \in \mathbb{N}$ and $f \in W_r^p (\mathbb{R})$,
   (v) $\| J \left(f, \frac{\sigma}{2} \right) - f \|_{L^p(\mathbb{R})} \to 0$ (as $\sigma \to \infty$) and hence

$$\left\| \left(J \left(f, \frac{\sigma}{2} \right) \right)^{(k)} - f^{(k)} \right\|_{L^p(\mathbb{R})} \to 0 \ as \ \sigma \to \infty,$$

for $f \in W_r^p (\mathbb{R})$ and $1 \leq k \leq r$,
   (vi) $\left\| f - J \left(f, \frac{\sigma}{2} \right) \right\|_{L^p(\mathbb{R})} \leq \frac{5\pi}{4} \frac{4^r}{\sigma^r} \| f^{(r)} \|_{L^p(\mathbb{R})}$ for $f \in W_r^p (\mathbb{R})$.

Theorem 5. *Let $p \in P^{Log}$, $\sigma > 0$, $r \in \mathbb{N}$ and $f \in W_r^{p(\cdot)}$. Then*

$$A_\sigma (f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} \tag{12}$$

holds with a positive constant depend on p, r.

Theorem 6. *Let $p \in P^{Log}$, $\sigma > 0$, $k \in \mathbb{N}$, $r \in \{0\} \cup \mathbb{N}$ and $f \in W_r^{p(\cdot)}$. Then*

$$A_\sigma (f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} \Omega_k \left(f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)}. \tag{13}$$

with positive constants depend on p, k, r.

Theorem 7. *Let $p \in P^{Log}$, $\sigma > 0$ and $g_\sigma \in \mathcal{G}_{\sigma, p(\cdot)}$. Then, Bernstein's inequality*

$$\| (g_\sigma)^{(r)} \|_{p(\cdot)} \lesssim \sigma^r \|g_\sigma\|_{p(\cdot)}$$

holds with a positive constant depend on p, r.

Definition 5. *[47, p.161] For $r, k \in \mathbb{N}$, $\sigma > 0$, we define*

$$g(\sigma, r, x) = \left(\frac{1}{x} \sin \frac{\sigma x}{2r} \right)^{2r}, \text{ and}$$

$$G(\sigma, r, k, \zeta) = \sum_{v=1}^{k} (-1)^{k-v} \frac{1}{v} \binom{k}{v} g\left(\sigma, r, \frac{\zeta}{v}\right).$$

For $r \geq \frac{1}{2}(k+2)$ we set

$$\gamma_{r,\sigma} := \int_{\mathbb{R}} \left(\frac{1}{t} \sin \frac{\sigma t}{2r} \right)^{2r} dt.$$

Let us introduce the Bernstein's singular integral ([47, p.161])

$$D_{\sigma,k} f(x) := \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} f(u) G(\sigma, r, k, u - x) \, dt \tag{14}$$

for $r, k \in \mathbb{N}$, $\sigma > 0$, and measurable complex valued f satisfying $\int_{\mathbb{R}} \frac{|f(u)|}{1+u^{2r}} du < \infty$.

Remark 5. *It is well known that, if $r, k \in \mathbb{N}$, $\sigma \in (0, \infty)$, $r \geq \frac{1}{2}(k+2)$, then $D_{\sigma,k} f \in \mathcal{G}_\sigma (L^p(\mathbb{R}))$ for $p \geq 1$.([47, p.161]).*

Lemma 6. *If $r \in \mathbb{N}$, $\sigma \in (0, \infty)$, then,*
   (i) we have

$$\gamma_{r,\sigma} = \frac{\sigma^{2r-1}}{(2r)^{2r-1}} \int_{\mathbb{R}} \left(\frac{\sin v}{v} \right)^{2r} dv$$

   (ii) (see, e.g. [13, (5)])

$$\int_{\mathbb{R}} \left(\frac{\sin v}{v} \right)^{2r} dv = \frac{2\pi}{(2r-1)! 2^{2r}} \left\{ (2r)^{2r-1} - \binom{2r}{1}(2r-2)^{2r-1} + \binom{2r}{2}(2r-4)^{2r-1} - \cdots \right\}$$

   (iii) and, as a result,

$$\gamma_{r,\sigma} = \frac{\sigma^{2r-1}}{(2r)^{2r-1}} b_r$$

where b_r is the right hand side of equality in (ii), having r terms.

Define $\lceil a \rceil := \min \{n \in \mathbb{N} : n \geq a\}$ and $\lfloor \sigma \rfloor := \max \{n \in \mathbb{Z} : n \leq \sigma\}$. We will take $r := \lceil \frac{1}{2}(k+2) \rceil$ in the next Theorems.

Theorem 8. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$, $f \in W_k^{p(\cdot)}$, then*

$$\|f - D_{\sigma,k}f\|_{p(\cdot)} \lesssim \frac{1}{\sigma^k} \left\|f^{(k)}\right\|_{p(\cdot)} \tag{15}$$

holds with a positive constant depend on p, k.

Theorem 9. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$. If $f \in L^{p(\cdot)}$, then*

$$\|D_{\sigma,k}f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p, k.

Theorem 10. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$, $f \in L^{p(\cdot)}$, then*

$$\|f - D_{\sigma,k}f\|_{p(\cdot)} \lesssim \Omega_k \left(f, \frac{1}{\sigma}\right)_{p(\cdot)}$$

holds with a positive constant depend on p, k.

Corollary 2. *By Theorem 9, if $r, k \in \mathbb{N}$, $\sigma \in (0, \infty)$, $r \geq \frac{1}{2}(k+2)$, then $D_{\sigma,k}f \in \mathcal{G}_{\sigma,p(\cdot)}$ for $p \in P^{Log}$ and $f \in L^{p(\cdot)}$.*

Theorem 11. *Let $r \in \mathbb{N}$, $p \in P^{Log}$, $\sigma > 0$ and $f \in W_r^{p(\cdot)}$. Then for all $k = 0, 1, \ldots, r$, there exist positive constants depending only on k, r and $p(\cdot)$ such that*

$$\left\|f^{(k)} - (g_\sigma^*)^{(k)}\right\|_{p(\cdot)} \lesssim \frac{1}{\sigma^{r-k}} A_\sigma \left(f^{(r)}\right)_{p(\cdot)}$$

holds for any $g_\sigma^* \in \mathcal{G}_{\sigma,p(\cdot)}$ satisfying $A_\sigma(f)_{p(\cdot)} = \|f - g_\sigma^*\|_{p(\cdot)}$.

Theorem 12. *Let $r, s \in \mathbb{N}$, $p \in P^{Log}$ and $f \in W_r^{p(\cdot)}$. Then there exists a $\Phi \in \mathcal{G}_{2\sigma,p(\cdot)}$ such that for all $k = 0, 1, \ldots, r$ inequalities*

$$\left\|f^{(k)} - \Phi^{(k)}\right\|_{p(\cdot)} \lesssim \frac{1}{\sigma^{r-k}} \Omega_s \left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}$$

are hold with a positive constant depending only on k, r and $p(\cdot)$.

Definition 6. *Set $\sigma, \eta > 0$, $f \in L^1(\mathbb{R})$, $\Theta_\eta f(x, y) := f(x + \eta y)$ and*

$$B_\sigma f(x, t) := \int_{\mathbb{R}} \Theta_{\frac{2}{\sigma}} f(x, y) h(y, t) \, dy.$$

Remark 6. *The following theorem was poved in [31] for $\sigma = 2$ with three minor mistypes. For the sake of completeness here we will prove it when $\sigma > 0$.*

Theorem 13. *Suppose that $h(y, t)$, $y, t \in \mathbb{R}$, is positive measurable function with respect to y and*

$$\int_{\mathbb{R}} h(y, t) \, dy \lesssim 1, \quad \int_{\mathbb{R}} \left|y h_y'(y, t)\right| dy \lesssim 1$$

with constants independent of t. If $\sigma > 0$ and $f \in L^1(\mathbb{R})$, then

$$\sup_{t>0} |B_\sigma f(\cdot, t)| \lesssim Mf(\cdot)$$

for $t > 0$ and a.e. on $\mathbb{R}$ where Mf is the Hardy-Littlewood maximal function of f.

6. Proof of the Results

Let $C(A)$ be the class of continuous functions defined on A. For $r \in \mathbb{N}$, we define $C^r(A)$ consisting of every member $f \in C(A)$ such that the derivative $f^{(k)}$ exists and is continuous on A for $k = 1, ..., r$. We set $C^\infty(A) := \{f \in C^r(A)$ for any $r \in \mathbb{N}\}$. We denote by $C_c(A)$, the collection of real valued continuous functions on A and support of f is compact set in A. We define $C_c^r(A) := C^r(A) \cap C_c(A)$ for $r \in \mathbb{N}$ and $C_c^\infty(A) := C^\infty(A) \cap C_c(A)$. Let $L^p(A)$, $1 \le p \le \infty$ be the classical Lebesgue space of functions on A.

Definition 7. *([17]) Let $\mathbb{N} : = \{1, 2, 3, \cdots\}$ be natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.*
(a) A family Q of measurable sets $E \subset \mathbb{R}$ is called locally N-finite ($N \in \mathbb{N}$) if

$$\sum_{E \in Q} \chi_E(x) \le N$$

almost everywhere in $\mathbb{R}$ where χ_U is the characteristic function of the set U.
(b) A family Q of open bounded sets $U \subset \mathbb{R}$ is locally 1-finite if and only if the sets $U \in Q$ are pairwise disjoint.
(c) Let $U \subset \mathbb{R}$ be a measurable set and

$$A_U f := \frac{1}{|U|} \int_U |f(t)| \, dt.$$

(d) For a family Q of open sets $U \subset \mathbb{R}$ we define averaging operator by

$$T_Q : L_{loc}^1 \to L^0,$$

$$T_Q f(x) := \sum_{U \in Q} \chi_U(x) A_U f, \quad x \in \mathbb{R},$$

where L^0 is the set of measurable functions on $\mathbb{R}$.
(e) For a measurable set $A \subset \mathbb{R}$, symbol $|A|$ will represent the Lebesgue measure of A.

Theorem 14. *([17]) Suppose that $p \in P^{Log}$, and $f \in L^{p(\cdot)}$. If Q is 1-finite family of open bounded subsets of $\mathbb{R}$ having Lebesgue measure 1, then, the averaging operator T_Q is uniformly bounded in $L^{p(\cdot)}$, namely,*

$$\|T_Q f\|_{p(\cdot)} \le \mathbf{c}_4 \|f\|_{p(\cdot)}$$

holds with a positive constant $\mathbf{c}_4$ depending only on p.

We define $\langle f, g \rangle := \int_\mathbb{R} f(x)g(x)dx$ when integral exists. We will need the following Propositions.

Proposition 1. *([17]) Let $p \in P^{Log}$. Then*

$$\frac{1}{12\mathbf{c}_4} \|f\|_{p(\cdot)} \leq \sup_{g \in L^{p'(\cdot)} \cap C_0^\infty : \|g\|_{p'(\cdot)} \leq 1} \langle |f|, |g| \rangle \leq 2 \|f\|_{p(\cdot)}$$

holds for all $f \in L^{p(\cdot)}$.

Proposition 2. *(a) $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are dense subsets of $L^p(\mathbb{R})$, $1 \leq p < \infty$.(Theorems 17.10 and 23.59 of [49, p. 415 and p. 575]).*
   (b) $C_c(\mathbb{R})$ contained $L^\infty(\mathbb{R})$ but not dense (Remark 17.11 of [49, p.416]) in $L^\infty(\mathbb{R})$.

Theorem 15. *Let $p \in P^{Log}$. In this case,*
   (a) if $f \in L^{p(\cdot)}$, then, the function $F_f := F_{f,G}$ defined in (5) is a bounded, uniformly continuous function on $\mathbb{R}$,
   (b) if $r \in \mathbb{N}$, and $f \in W_r^{p(\cdot)}$, then, $\frac{d^k}{du^k}(F_f)$ exists and

$$\frac{d^k}{du^k}(F_f) = F_{f^{(k)}}$$

for $k \in \{1, ..., r\}$.

Proof. (a) Since C_0^∞ is a dense subset of $L^{p(\cdot)}$, we consider functions $f \in C_0^\infty$ and its corresponding $F_{f,G}$ given in (5). For any $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ so that

$$|f(x + u_1) - f(x + u_2)| < \frac{\varepsilon}{1 + |sptG|}$$

for any $u_1, u_2 \in \mathbb{R}$ with $|u_1 - u_2| < \delta$, where $sptG$ is the support of the function $G \in L^{p'(\cdot)} \cap C_0^\infty$. Then, there holds inequality

$$|F_{f,G}(u_1) - F_{f,G}(u_2)| \leq \int_{\mathbb{R}} |f(x + u_1) - f(x + u_2)| \, |G(x)| \, dx$$

$$= \int_{sptG} |f(x + u_1) - f(x + u_2)| \, |G(x)| \, dx$$

$$\leq \sup_{x, u_1, u_2 \in sptG} |f(x + u_1) - f(x + u_2)| \, \|G\|_{1, sptG}$$

$$\leq \frac{\varepsilon}{1 + |sptG|} (1 + |sptG|) \|G\|_{p'(\cdot)} \leq \varepsilon$$

for any $u_1, u_2 \in \mathbb{R}$ with $|u_1 - u_2| < \delta$. Thus conclusion of Theorem 15 follows. For the general case $f \in L^{p(\cdot)}$ there exists an $g \in C_0^\infty$ so that

$$\|f - g\|_{p(\cdot)} < \frac{\xi}{4(1 + |sptG|)\mathbf{c}_0}$$

for any $\xi > 0$. Therefore

$$|F_{f,G}(u_1) - F_{f,G}(u_2)| = |F_{f,G}(u_1) - F_{g,G}(u_1)| + |F_{g,G}(u_1) - F_{g,G}(u_2)| +$$

$$+ |F_{g,G}(u_2) - F_{f,G}(u_2)| = |F_{f-g,G}(u_1)| + \frac{\xi}{2} + |F_{g-f,G}(u_2)|$$

$$\leq 2 \left(1 + |sptG| \right) \mathbf{c}_0 \left\| f - g \right\|_{p(\cdot),\omega} + \frac{\xi}{2} < \xi.$$

As a result $F_{f,G}$ is uniformly continuous on $\mathbb{R}$.

(b) is follow from definitions. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1. Let $0 \leq f, g \in L^{p(\cdot)}$. In this case there exists a constant $C > 0$ such that

$$\left\| F_{f,G} \right\|_{C(\mathbb{R})} \leq C \left\| F_{g,G}\left(u\right) \right\|_{C(\mathbb{R})} = C \left\| \int_{\mathbb{R}} g\left(u+x\right) \left| G\left(x\right) \right| dx \right\|_{C(\mathbb{R})}$$

$$= C \sup_{u \in \mathbb{R}} \int_{\mathbb{R}} g\left(u+x\right) \left| G\left(x\right) \right| dx = C \sup_{u \in sptG} \int_{sptG} g\left(u+x\right) \left| G\left(x\right) \right| dx$$

$$\leq C \sup_{u \in sptG} \left\| g\left(u + \cdot\right) \right\|_{1,sptG} \left\| G \right\|_{\infty} \leq C \left(1 + |sptG| \right) \mathbf{c}_0 \left\| g \right\|_{p(\cdot)}.$$

On the other hand, for any $\varepsilon > 0$ and appropriately chosen $\tilde{G}_\varepsilon \in L^{p'(\cdot)}$ with

$$\int_{\mathbb{R}} g\left(x\right) \tilde{G}_\varepsilon\left(x\right) dx \geq \frac{1}{12\mathbf{c}_4} \left\| g \right\|_{p(\cdot)} - \varepsilon, \qquad \left\| \tilde{G}_\varepsilon \right\|_{p'(\cdot)} \leq 1,$$

(see Proposition 1), one can find

$$\left\| F_{f,G} \right\|_{C(\mathbb{R})} \geq \left| F_{f,G}\left(0\right) \right| \geq \int_{\mathbb{R}} f\left(x\right) \left| G\left(x\right) \right| dx > \frac{1}{12\mathbf{c}_4} \left\| f \right\|_{p(\cdot)} - \varepsilon.$$

In the last inequality we take as $\varepsilon \to 0^+$ and obtain

$$\left\| F_{f,G} \right\|_{C(\mathbb{R})} > \frac{1}{12\mathbf{c}_4} \left\| f \right\|_{p(\cdot)}.$$

Combining these inequalities we get

$$\left\| f \right\|_{p(\cdot)} < 12\mathbf{c}_4 \left\| F_{f,G} \right\|_{C(\mathbb{R})} \leq 12\mathbf{c}_4 C \left\| F_{g,G} \right\|_{C(\mathbb{R})}$$

$$\leq 12\mathbf{c}_4 C \left(1 + |sptG| \right) \mathbf{c}_0 \left\| g \right\|_{p(\cdot)}.$$

For general case $f, g \in L^{p(\cdot)}$ we obtain

$$\left\| f \right\|_{p(\cdot)} < 24\mathbf{c}_4 \left(1 + |sptG| \right) \mathbf{c}_0 C \left\| g \right\|_{p(\cdot)} \qquad\qquad (16)$$

and proof is finished. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 7. _Note that, in (16) constant depend on $|sptG|$ and $\left\| G \right\|_\infty$ but it is possible to avoid dependence on $|sptG|$ and $\left\| G \right\|_\infty$. To do so, we can change the definition of F_f with_

$$F_f\left(u\right) := \int_{\mathbb{R}} \mathcal{S}_{1,u} f(x) \left| G\left(x\right) \right| dx, \quad u \in \mathbb{R},$$

where $G \in L^{p'(\cdot)} \cap C_0^\infty$ and $\left\| G \right\|_{p'(\cdot)} \leq 1$. Now, boundedness of $\mathcal{S}_{1,u} f$ in $L^{p(\cdot)}$ for any $u \in \mathbb{R}$, and the same procedure give (16) with a constant does not depend on $|sptG|$ and $\left\| G \right\|_\infty$. Hence, constants in other results can be free of dependence on $|sptG|$ and $\left\| G \right\|_\infty$.

Proof of Lemma 1. Let $0 < h \leq \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then, using (16) we get

$$\left\| (I - T_h) f \right\|_{p(\cdot)} < 24 \mathbf{c}_4 \left\| F_{(I-T_h)f,G} \right\|_{C(\mathbb{R})} \leq 24 \cdot 72 \mathbf{c}_4 \left\| F_{(I-T_\delta)f,G} \right\|_{C(\mathbb{R})}$$

$$\leq 1728 \mathbf{c}_4 \left(1 + |sptG| \right) \mathbf{c}_0 \left\| (I - T_\delta) f \right\|_{p(\cdot)}.$$

$\square$

Proof of Lemma 2. If $f \in L^{p(\cdot)}$, then, using generalized Minkowski's integral inequality and Lemma 1 we obtain

$$\left\| (I - \mathfrak{R}_\delta) f \right\|_{p(\cdot)} = \left\| \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\frac{1}{h} \int_0^h \left(f(x+t) - f(x) \right) dt \right) dh \right\|_{p(\cdot)}$$

$$= \left\| \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(T_h f(x) - f(x) \right) dh \right\|_{p(\cdot)} \leq \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\| T_\delta f - f \right\|_{p(\cdot)} dh$$

$$\leq 1728 \mathbf{c}_4 \left(1 + |sptG| \right) \mathbf{c}_0 \left\| T_\delta f - f \right\|_{p(\cdot)} \frac{2}{\delta} \int_{\delta/2}^{\delta} dh$$

$$= 1728 \mathbf{c}_4 \left(1 + |sptG| \right) \mathbf{c}_0 \left\| (I - T_\delta) f \right\|_{p(\cdot)}.$$

$\square$

Proof of Lemma 4. Using

$$\left\| F_{\delta(\mathfrak{R}_\delta f)',G} \right\|_{C(\mathbb{R})} = \left\| \delta \left(F_{(\mathfrak{R}_\delta f),G} \right)' \right\|_{C(\mathbb{R})} = \delta \left\| \left(\mathfrak{R}_\delta (F_{f,G}) \right)' \right\|_{C(\mathbb{R})}$$

$$\leq \cdots \leq 2 \left(37 + 146 \ln 2^{36} \right) \left\| (I - T_\delta)(F_{f,G}) \right\|_{C(\mathbb{R})}$$

$$= 2 \left(37 + 146 \ln 2^{36} \right) \left\| \left(F_{(I-T_\delta)f,G} \right) \right\|_{C(\mathbb{R})}$$

we conclude from Transference Result that

$$\delta \left\| (\mathfrak{R}_\delta f)' \right\|_{p(\cdot)} \leq \mathbf{c}_5 \left\| (I - T_\delta) f \right\|_{p(\cdot)}.$$

with $\mathbf{c}_5 := 24 \mathbf{c}_4 \left(1 + |sptG| \right) \mathbf{c}_0 \left(37 + 146 \ln 2^{36} \right)$. $\square$

Proof of Theorem 4. For $r = 1, 2, 3, \ldots$ we consider the operator

$$\mathcal{A}_\delta^r := I - (I - \mathfrak{R}_\delta^r)^r = \sum_{j=0}^{r-1} (-1)^{r-j+1} \binom{r}{j} \mathfrak{R}_\delta^{r(r-j)}.$$

From the identity $I - \mathfrak{R}_\delta^r = (I - \mathfrak{R}_\delta) \sum_{j=0}^{r-1} \mathfrak{R}_\delta^j$ we find

$$\left\| (I - \mathfrak{R}_\delta^r) g \right\|_{p(\cdot)} \leq \left(\sum_{j=0}^{r-1} \mathbf{c}_6^j \right) \left\| (I - \mathfrak{R}_\delta) g \right\|_{p(\cdot)}$$

with $\mathbf{c}_6 := 24\mathbf{c}_4 \left(1 + |sptG|\right) \mathbf{c}_0$. Therefore

$$\left\| \left(I - \mathfrak{R}_\delta^r\right) g \right\|_{p(\cdot)} \leq \left(1728\mathbf{c}_4 \left(1 + |sptG|\right) \mathbf{c}_0 \sum_{j=0}^{r-1} \mathbf{c}_6^j \right) \left\| \left(I - T_\delta\right) g \right\|_{p(\cdot)} \qquad (17)$$

$$= \mathbf{c}_7 \left\| \left(I - T_\delta\right) g \right\|_{p(\cdot)}$$

when $0 < \delta < \infty$, $p \in P$ and $g \in L^{p(\cdot)}$. Since $\left\| f - \mathcal{A}_\delta^r f \right\|_{p(\cdot)} = \left\| \left(I - \mathfrak{R}_\delta^r\right)^r f \right\|_{p(\cdot)}$, recursive procedure gives

$$\left\| f - \mathcal{A}_\delta^r f \right\|_{p(\cdot)} = \left\| \left(I - \mathfrak{R}_\delta^r\right)^r f \right\|_{p(\cdot)} \leq \cdots \leq \mathbf{c}_7^r \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)}.$$

On the other hand, using Lemmas 5 and 4,

$$\delta^r \left\| \frac{d^r}{dx^r} \mathfrak{R}_\delta^r f \right\|_{p(\cdot)} = \delta^{r-1} \delta \left\| \frac{d}{dx} \mathfrak{R}_\delta \frac{d^{r-1}}{dx^{r-1}} \mathfrak{R}_\delta^{r-1} f \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_5 \delta^{r-1} \left\| \left(I - T_\delta\right) \frac{d^{r-1}}{dx^{r-1}} \mathfrak{R}_\delta^{r-1} f \right\|_{p(\cdot)} \leq \cdots \leq$$

$$\leq \mathbf{c}_5^{r-1} \delta \left\| \frac{d}{dx} \mathfrak{R}_\delta \left(I - T_\delta\right)^{r-1} f \right\|_{p(\cdot)} \leq \mathbf{c}_5^r \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)}.$$

Thus

$$K_r \left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)} \right)_{p(\cdot)} \leq \left\| f - \mathcal{A}_\delta^r f \right\|_{p(\cdot)} + \delta^r \left\| \frac{d^r}{dx^r} \mathcal{A}_\delta^r f(x) \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_7^r \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)} + \sum_{j=0}^{r-1} \left| \binom{r}{j} \right| \delta^r \left\| \frac{d^r}{dx^r} \mathfrak{R}_\delta^{r(r-j)} f(x) \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_7^r \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)} + \mathbf{c}_5^r \sum_{j=0}^{r-1} \left| \binom{r}{j} \right| \left\| \left(I - T_\delta\right)^r \mathfrak{R}_\delta^{(r-j)} f \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_7^r \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)} + \mathbf{c}_5^r \sum_{j=0}^{r-1} \left| \binom{r}{j} \right| \mathbf{c}_6^{r-j} \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_8 \left\| \left(I - T_\delta\right)^r f \right\|_{p(\cdot)}$$

where

$$\mathbf{c}_8 := \max \left\{ \mathbf{c}_7^r, \mathbf{c}_5^r \sum_{j=0}^{r-1} \left| \binom{r}{j} \right| \mathbf{c}_6^{r-j} \right\}.$$

For the reverse of the last inequality, when $g \in W_r^{p(\cdot)}$, we get

$$\Omega_r \left(f, \delta \right)_{p(\cdot)} \leq \left(1 + \mathbf{c}_6\right)^r \left\| f - g \right\|_{p(\cdot)} + \Omega_r \left(g, \delta \right)_{p(\cdot)}$$

$$\leq \left(1 + \mathbf{c}_6\right)^r \left\| f - g \right\|_{p(\cdot)} + 2^{-r} \mathbf{c}_6^r \delta^r \left\| g^{(r)} \right\|_{p(\cdot)}, \qquad (18)$$

and taking infimum on $g \in W^r_{p(\cdot)}$ in (18) we obtain

$$\Omega_r \left(f, \delta \right)_{p(\cdot)} \leq \left(1 + \mathbf{c}_6 \right)^r K_r \left(f, \delta; L^{p(\cdot)}, W^{p(\cdot)}_r \right)_{p(\cdot)}.$$

$\square$

Proof of Theorem 5. The following inequality

$$A_\sigma \left(f \right)_{C(\mathbb{R})} \leq \left\| f - J \left(f, \frac{\sigma}{2} \right) \right\|_{C(\mathbb{R})} \leq \frac{5\pi}{4} \frac{4^r}{\sigma^r} \| f^{(r)} \|_{C(\mathbb{R})}, \quad \forall f \in C^r(\mathbb{R})$$

known (see (vi) of Remark 4). Now using TR we find

$$\left\| f - J \left(f, \frac{\sigma}{2} \right) \right\|_{p(\cdot)} \leq \frac{5\pi}{2} \frac{4^r \mathbf{c}_6}{\sigma^r} \| f^{(r)} \|_{p(\cdot)}, \quad \forall f \in W^{p(\cdot)}_r. \tag{19}$$

Let $r = 1$. Suppose that

$$A_\sigma \left(f' \right)_{p(\cdot)} = \| f' - g^*_\sigma(f') \|_{p(\cdot)}, \quad g^*_\sigma(f') \in \mathcal{G}_{\sigma, p(\cdot)}$$

and

$$F \left(x \right) := \int_0^x g^*_\sigma(f') \left(t \right) dt.$$

Then $F \in \mathcal{G}_\sigma$ ([26, p.397]). Setting

$$\varphi \left(x \right) = f \left(x \right) - F \left(x \right)$$

one has

$$\| \varphi' \|_{p(\cdot)} = \| f' - g^*_\sigma(f') \|_{p(\cdot)} = A_\sigma \left(f' \right)_{p(\cdot)}.$$

Thus

$$A_\sigma \left(f \right)_{p(\cdot)} = A_\sigma \left(f - F \right)_{p(\cdot)} \overset{(19)}{\leq} 10\pi \mathbf{c}_6 \frac{1}{\sigma} \left\| (f - F)' \right\|_{p(\cdot)}$$

$$= \frac{10\pi \mathbf{c}_6}{\sigma} \| f' - F' \|_{p(\cdot)} = \frac{10\pi \mathbf{c}_6}{\sigma} \| f' - g^*_\sigma(f') \|_{p(\cdot)}$$

$$= 10\pi \mathbf{c}_6 \frac{1}{\sigma} A_\sigma \left(f' \right)_{p(\cdot)}.$$

Now, result follows from the last inequality:

$$A_\sigma \left(f \right)_{p(\cdot)} \leq 10\pi \mathbf{c}_6 \frac{1}{\sigma} A_\sigma \left(f' \right)_{p(\cdot)} \leq \cdots \leq \left(10\pi \mathbf{c}_6 \right)^r \frac{1}{\sigma^r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)}.$$

$\square$

Proof of Theorem 6. Let $p \in P^{Log}$, $\sigma > 0$, $k \in \mathbb{N}$, $r \in \{0\} \cup \mathbb{N}$ and $f \in W^{p(\cdot)}_r$. First we consider the case $r = 0$. For every $g \in W^{p(\cdot)}_k$ we find

$$A_\sigma \left(f \right)_{p(\cdot)} \leq A_\sigma \left(f - g \right)_{p(\cdot)} + A_\sigma \left(g \right)_{p(\cdot)}$$

$$\leq \| f - g \|_{p(\cdot)} + \frac{5\pi}{2} \frac{4^k \mathbf{c}_6}{\sigma^k} \| f^{(k)} \|_{p(\cdot)}.$$

Taking infimum on g in the last inequality

$$A_\sigma\left(f\right)_{p(\cdot)} \le \frac{5\pi}{2}4^k\mathbf{c}_6 K_k\left(f,\delta;L^{p(\cdot)},W_k^{p(\cdot)}\right)_{p(\cdot)}.$$

Now using (11)

$$A_\sigma\left(f\right)_{p(\cdot)} \le \mathbf{c}_8\frac{5\pi}{2}4^k\mathbf{c}_6\Omega_k\left(f,\frac{1}{\sigma}\right)_{p(\cdot)}.$$

In the second stage we consider the case $r \in \mathbb{N}$. In this case

$$A_\sigma\left(f\right)_{p(\cdot)} \le \left(10\pi\mathbf{c}_6\right)^r\frac{1}{\sigma^r}A_\sigma\left(f^{(r)}\right)_{p(\cdot)}$$

$$\le 5\pi\mathbf{c}_8\left(10\right)^r\pi^r\mathbf{c}_6^{r+1}2^{2k-1}\frac{1}{\sigma^r}\Omega_k\left(f^{(r)},\frac{1}{\sigma}\right)_{p(\cdot)}.$$

$\square$

Proof of Theorem 7. Let $p \in P^{Log}$, $\sigma > 0$ and $g_\sigma \in \mathcal{G}_{\sigma,p(\cdot)}$. Then, Bernstein's inequality

$$\|\left(g_\sigma\right)^{(r)}\|_{C(\mathbb{R})} \le \sigma^r\|g_\sigma\|_{C(\mathbb{R})}, \quad \forall g_\sigma \in \mathcal{G}_{\sigma,\infty}$$

and TR gives

$$\|\left(g_\sigma\right)^{(r)}\|_{p(\cdot)} \le \mathbf{c}_6\sigma^r\|g_\sigma\|_{p(\cdot)}, \quad \forall g_\sigma \in \mathcal{G}_{\sigma,p(\cdot)}.$$

$\square$

Proof of Theorem 8. Define for $k \in \mathbb{N}$ the classical modulus of smoothness of function $f \in C\left(\mathbb{R}\right)$ of step $\delta > 0$ by

$$\omega_k\left(f,\delta\right)_{C(\mathbb{R})} := \sup_{|h|\le\delta}\left\|\Delta_t^k f\right\|_{C(\mathbb{R})}$$

where $\Delta_t^k f\left(\cdot\right) := \left(I - \tilde{T}_h\right)^k f\left(\cdot\right)$, $\tilde{T}_h f\left(\cdot\right) := f\left(\cdot + h\right)$ and I is the identity operator. From (14), one can write

$$\|f - D_{\sigma,k}f\|_{C(\mathbb{R})} = \left\|\frac{\left(-1\right)^k}{\gamma_{r,\sigma}}\int_{\mathbb{R}}\sum_{v=0}^k\left(-1\right)^{k-v}\binom{k}{v}f\left(x + vt\right)g\left(\sigma,r,t\right)dt\right\|_{C(\mathbb{R})}$$

$$\le \frac{1}{\sigma^{2r-1}\frac{b_r}{(2r)^{2r-1}}}\int_{\mathbb{R}}\left\|\Delta_t^k f\left(x\right)\right\|_{C(\mathbb{R})}g\left(\sigma,r,t\right)dt \le \frac{\left(2r\right)^{2r-1}}{b_r\sigma^{2r-1}}\int_{\mathbb{R}}\omega_k\left(f,t\right)_{C(\mathbb{R})}g\left(\sigma,r,t\right)dt$$

$$\le \frac{\left(2r\right)^{2r-1}\sigma^k}{b_r\sigma^{2r-1}}\omega_k\left(f,\frac{1}{\sigma}\right)_{C(\mathbb{R})}\int_{\mathbb{R}}\left(t + \frac{1}{\sigma}\right)^k g\left(\sigma,r,t\right)dt$$

$$\le \frac{\left(2r\right)^{2r-1}\sigma^k}{b_r\sigma^{2r-1}}\frac{1}{\sigma^k}\left\|f^{(k)}\right\|_{C(\mathbb{R})}\int_{\mathbb{R}}\left(t + \frac{1}{\sigma}\right)^k g\left(\sigma,r,t\right)dt$$

$$\leq \frac{(2r)^{2r-1}}{b_r \sigma^{2r-1}} \left\| f^{(k)} \right\|_{C(\mathbb{R})} \left\{ \frac{2^k}{\sigma^k} \int\limits_{|t| \leq \frac{1}{\sigma}} |g(\sigma,r,t)|\, dt + 2^k \int\limits_{|t| \geq \frac{1}{\sigma}} |t|^k\, |g(\sigma,r,t)|\, dt \right\}.$$

Using $r = \left\lceil \frac{1}{2}(k+2) \right\rceil$

$$\frac{(2r)^{2r-1} 2^k}{b_r \sigma^{2r-1}} \int_{|t| \geq 1/\sigma} |t|^k \left(\frac{1}{t} \sin \frac{\sigma t}{2r} \right)^{2r} dt$$

$$\leq \frac{(2r)^{2r-1} 2^k}{b_r \sigma^{2r-1}} \int_{|t| \geq 1/\sigma} \left(\frac{1}{t} \sin \frac{\sigma t}{2r} \right)^{2r-k} dt$$

$$\leq \frac{(2r)^{2r-1} 2^k}{b_r \sigma^{2r-1}} \frac{\sigma^{2r-k+1}}{(2r)^{2r-k+1}} \int_{\mathbb{R}} \left(\frac{\sin u}{u} \right)^2 dt = \frac{1}{\sigma^k} \frac{2^{2k} r^k}{b_r} \pi.$$

On the other hand

$$\frac{(2r)^{2r-1} 2^k}{b_r \sigma^{2r-1}} \frac{1}{\sigma^k} \int_{|t| \leq 1/\sigma} \left(\frac{1}{t} \sin \frac{\sigma t}{2r} \right)^{2r} dt$$

$$\leq \frac{(2r)^{2r-1} 2^k}{b_r \sigma^{2r-1}} \frac{1}{\sigma^k} \int_{\mathbb{R}} \left(\frac{1}{t} \sin \frac{\sigma t}{2r} \right)^{2r} dt$$

$$= \frac{(2r)^{2r-1}}{b_r^r \sigma^{2r-1}} \sigma^{2r-1} \frac{b_r}{(2r)^{2r-1}} = \frac{2^k}{\sigma^k}.$$

Thus

$$\| f - D_{\sigma,k} f \|_{C(\mathbb{R})} \leq \left(\frac{2^{2k} r^k}{b_r} + 2^k \right) \frac{1}{\sigma^k} \left\| f^{(k)} \right\|_{C(\mathbb{R})}.$$

From this and TR we get

$$\| f - D_{\sigma,k} f \|_{p(\cdot)} \leq \mathbf{c}_6 \left(\frac{2^{2k} r^k}{b_r} + 2^k \right) \frac{1}{\sigma^k} \left\| f^{(k)} \right\|_{p(\cdot)} = \mathbf{c}_6 \mathbf{c}(k,r) \frac{1}{\sigma^k} \left\| f^{(k)} \right\|_{p(\cdot)}.$$

$\hfill\square$

Proof of Theorem 9. Fixed $\sigma > 0$, we find

$$\| D_{\sigma,k} f \|_{C(\mathbb{R})} = \left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} f(u) G(\sigma,r,k,u-x)\, du \right\|_{C(\mathbb{R})}$$

$$\left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} (-1)^{k-v} \binom{k}{v} f(u)\, g\left(\sigma,r,\frac{u-x}{v} \right) du \right\|_{C(\mathbb{R})}$$

$$\leq \left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} (-1)^{k-v} \binom{k}{v} f(x+vt)\, g(\sigma,r,t)\, v\, dt \right\|_{C(\mathbb{R})}$$

$$\leq \frac{k}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} \left| \binom{k}{v} \right| \| f(x+vt) \|_{C(\mathbb{R})}\, g(\sigma,r,t)\, dt$$

$$\leq \|f\|_{C(\mathbb{R})} \sum_{v=1}^{k} \left| \binom{k}{v} \right| \frac{k}{\gamma_{r,\sigma}} \int_{\mathbb{R}} g\left(\sigma, r, t\right) dt \leq k2^{k} \|f\|_{C(\mathbb{R})}.$$

Now, transference result TR gives

$$\|D_{\sigma,k} f\|_{p(\cdot)} \leq k2^{k} \mathbf{c}_6 \|f\|_{p(\cdot)}.$$

$\square$

Proof of Theorem 10. We can write

$$\|f - D_{\sigma,k} f\|_{p(\cdot)} = \left\| f - \mathcal{A}_{\frac{1}{\sigma}}^{k} f + \mathcal{A}_{\frac{1}{\sigma}}^{k} f - D_{\sigma,k} \mathcal{A}_{\frac{1}{\sigma}}^{k} f + D_{\sigma,k} \mathcal{A}_{\frac{1}{\sigma}}^{k} f - D_{\sigma,k} f \right\|_{p(\cdot)}$$

$$\leq \left\| f - \mathcal{A}_{\frac{1}{\sigma}}^{k} f \right\|_{p(\cdot)} + \left\| \mathcal{A}_{\frac{1}{\sigma}}^{k} f - D_{\sigma,k} \mathcal{A}_{\frac{1}{\sigma}}^{k} f \right\|_{p(\cdot)} + \left\| D_{\sigma,k} (\mathcal{A}_{\frac{1}{\sigma}}^{k} f - f) \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_7^{k} \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)} + \mathbf{c}_6 \mathbf{c}\left(k\right) \frac{1}{\sigma^k} \left\| (\mathcal{A}_{\frac{1}{\sigma}}^{k} f)^{(k)} \right\|_{p(\cdot)} + k2^{k} \mathbf{c}_6 \left\| \mathcal{A}_{\frac{1}{\sigma}}^{k} f - f \right\|_{p(\cdot)}$$

$$\leq \left(\mathbf{c}_7^{k} + \mathbf{c}_6 \mathbf{c}\left(k, r\right) \mathbf{c}_5^{k} \sum_{j=0}^{k-1} \left| \binom{k}{j} \right| \mathbf{c}_6^{k-j} + 2^{k} k \mathbf{c}_6 \mathbf{c}_7^{k} \right) \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)}$$

$$= \mathbf{c}_9 \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)}$$

and the result follows.

$\square$

Proof of Theorem 11. Let $q \in \mathcal{G}_\sigma$ and $A_\sigma \left(f^{(k)} \right)_{p(\cdot)} = \left\| f^{(k)} - q \right\|_{p(\cdot)}$. Then

$$\left\| f^{(k)} - (g_\sigma^*)^{(k)} \right\|_{p(\cdot)} \leq \left\| f^{(k)} - (J\left(f, \sigma\right))^{(k)} \right\|_{p(\cdot)} + \left\| (J\left(f, \sigma\right))^{(k)} - (g_\sigma^*)^{(k)} \right\|_{p(\cdot)}$$

$$\leq \left\| f^{(k)} - q \right\|_{p(\cdot)} + \left\| q - J\left(f^{(k)}, \sigma \right) \right\|_{p(\cdot)} + \left\| (J\left(f, \sigma\right) - g_\sigma^*)^{(k)} \right\|_{p(\cdot)}$$

$$\leq A_\sigma \left(f^{(k)} \right)_{p(\cdot)} + \left\| J\left(q - f^{(k)}, \sigma \right) \right\|_{p(\cdot)} + 2^{k} \mathbf{c}_6 \sigma^{k} \| J\left(f, \sigma\right) - g_\sigma^* \|_{p(\cdot)}$$

$$\leq \left(1 + 3\mathbf{c}_6 \right) A_\sigma \left(f^{(k)} \right)_{p(\cdot)} + 2^{k} \mathbf{c}_6 \sigma^{k} \| J\left(f, \sigma\right) - J\left(g_\sigma^*, \sigma\right) \|_{p(\cdot)}$$

$$\leq \left(1 + 3\mathbf{c}_6 \right) \frac{2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r}{\sigma^{r-k}} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} + 3\mathbf{c}_6^2 2^{k} \frac{2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r}{\sigma^{r-k}} A_\sigma \left(f^{(r)} \right)_{p(\cdot)}$$

$$\leq \left(2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r \right) \left(1 + 3\mathbf{c}_6 + 3\mathbf{c}_6^2 2^{k} \right) \frac{\sigma^{k}}{\sigma^{r}} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} = \mathbf{c}_{10} \sigma^{k-r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)}$$

and the proof of Theorem 11 is completed.

$\square$

Proof of Theorem 12. Let $g_\sigma^* \in \mathcal{G}_\sigma$, $A_\sigma(f)_{p(\cdot)} = \|f - g_\sigma^*\|_{p(\cdot)}$ and $\Phi = J(f, \sigma)$. Then

$$\|f - J(f, \sigma)\|_{p(\cdot)} \leq \|f - g_\sigma^* + g_\sigma^* - J(f, \sigma)\|_{p(\cdot)}$$

$$= \|f - g_\sigma^* + J(g_\sigma^*, \sigma) - J(f, \sigma)\|_{p(\cdot)}$$

$$\leq A_\sigma(f)_{p(\cdot)} + 3\mathbf{c}_6 \|f - g_\sigma^*\|_{p(\cdot)} = (1 + 3\mathbf{c}_6) A_\sigma(f)_{p(\cdot)}$$

and

$$\|f - J(f, \sigma)\|_{p(\cdot)} \leq (1 + 3\mathbf{c}_6) A_\sigma(f)_{p(\cdot)}$$

$$\leq (1 + 3\mathbf{c}_6) 5\pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1} \frac{1}{\sigma^r} \Omega_s \left(f^{(r)}, 1/\sigma \right)_{p(\cdot)}.$$

Now, from

$$\|f - g_\sigma^*\|_{p(\cdot)} \leq \frac{\pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1}}{\sigma^r} \Omega_s \left(f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)}$$

we obtain

$$\|J(f, \sigma) - g_\sigma^*\|_{p(\cdot)} \leq \frac{\mathbf{c}_{11}}{\sigma^r} \Omega_s \left(f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)}$$

with

$$\mathbf{c}_{11} = \pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1} ((1 + 3\mathbf{c}_6) 5 + 1).$$

Hence

$$\left\| f^{(k)} - (J(f, \sigma))^{(k)} \right\|_{p(\cdot)} \leq \left\| f^{(k)} - (g_\sigma^*)^{(k)} \right\|_{p(\cdot)} + \left\| (J(f, \sigma))^{(k)} - (g_\sigma^*)^{(k)} \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_{10} \sigma^{k-r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} + 2^k \mathbf{c}_6 \sigma^k \frac{\mathbf{c}_{11}}{\sigma^r} \Omega_s \left(f^{(r)}, \frac{1}{\sigma} \right)_{p(\cdot)}$$

$$\leq \left(\mathbf{c}_{10} \frac{5\pi \mathbf{c}_8}{2} 4^s \mathbf{c}_6 + 2^k \mathbf{c}_6 \mathbf{c}_{11} \right) \sigma^{k-r} \Omega_s \left(f^{(r)}, 1/\sigma \right)_{p(\cdot)}$$

and the proof is completed. $\square$

Proof of Theorem 13. Given $x \in \mathbb{R}$, let

$$\Gamma(y) := \int_0^y \Theta_{\frac{2}{\sigma}} f(x, u) \, du, \quad y > 0,$$

and $a, b > 0$. Integration by parts gives

$$\int_{-a}^b \Theta_{\frac{2}{\sigma}} f(x, y) h(y, t) \, dy = \int_{-a}^b h(y, t) \, d\Gamma(y)$$

$$= \Gamma(y) h(y, t) \big|_{-a}^b - \int_{-a}^b h_y'(y, t) \Gamma(y) \, dy.$$

Since $\Gamma(y) \leq |y| Mf(x)$ we obtain

$$\left| \int_{-a}^b \Theta_{\frac{2}{\sigma}} f(x, y) h(y, t) \, dy \right| \leq Mf(x) \left(\int_{-a}^b \left| y h_y'(y, t) \right| dy + h(y, t) \big|_{-a}^b \right).$$

Now

$$\mathbf{c}_{12} \geq \int_{\mathbb{R}} h\left(y, t\right) dy \geq \int_{-a}^{b} h\left(y, t\right) dy = h\left(y, t\right) \mid_{-a}^{b} - \int_{-a}^{b} y h'_y\left(y, t\right) dy$$

gives

$$\left| \int_{-a}^{b} \Theta_{\frac{2}{\sigma}} f\left(x, y\right) h\left(y, t\right) dy \right| \leq \left(\mathbf{c}_{12} + 2\mathbf{c}_{13}\right) M f\left(x\right)$$

for any $t > 0$. The last inequality implies the result. $\qquad \square$

References

[1] Abdullaev, F., Chaichenko, S., Imashqızı, M., Shidlich, A., Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent, *Turk. J. Math.*, 44(1) (2020), 284-299. https://doi.org/10.3906/mat-1911-3

[2] Ackhiezer, N. I., Theory of Approximation, Fizmatlit, Moscow, 1965, English translation of 2nd edition, Frederick Ungar, New York, 1956.

[3] Akgün, R., Approximation of functions of weighted Lebesgue and Smirnov spaces, *Mathematica (Cluj)*, 54(77)(Special) (2012), 25-36.

[4] Akgün, R., Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces, *Proc. A. Razmadze Math. Inst.*, 152 (2010), 1-18.

[5] Akgün, R., Inequalities for one sided approximation in Orlicz spaces, *Hacet. J. Math. Stat.*, 40(2) (2011), 231-240.

[6] Akgün, R., Some convolution inequalities in Musielak Orlicz spaces, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 42(2) (2016), 279-291.

[7] Akgün, R., Ghorbanalizadeh, A., Approximation by integral functions of finite degree in variable exponent Lebesgue spaces on the real axis, *Turk. J. Math.*, 42(4) (2018), 1887-1903. https://doi.org/10.3906/mat-1605-26

[8] Avşar, A. H., Koç, H., Jackson and Stechkin type inequalities of trigonometric approximation in $A_{p,q(.)}^{w,\theta}$, *Turk. J. Math.*, 42(6) (2018), 2979-2993. https://doi.org/10.3906/mat-1712-84

[9] Avşar, A. H., Yildirir, Y. E., On the trigonometric approximation of functions in weighted Lorentz spaces using Cesaro submethod, *Novi Sad J. Math.*, 48(2) (2018), 41-54. https://doi.org/10.30755/NSJOM.06335

[10] Bardaro, C., Butzer, P. L., Stens, R. L., Vinti, G., Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals, *J. Math. Anal. Appl.,* 316(1) (2006), 269-306. https://doi.org/10.1016/j.jmaa.2005.04.042

[11] Bernstein, S. N., Sur la meilleure approximation sur tout l'axe reel des fonctions continues par des fonctions entieres de degre n I, *Dokl. Acad. Sci. URSS (N.S.)*, 51 (1946), 331-334.

[12] Bernstein, S. N., Collected Works, Mir, Vol. I, Izdat. Akad. Nauk SSSR, Moscow, 1952, 11-104.

[13] Butler, R., On the evaluation of $\int_0^\infty \frac{\sin^m(t)}{t^m} dt$ by the trapezoidal rule, *Amer. Math. Monthly*, 67(6) (1960), 566-569.

[14] Cruz-Uribe, D., Fiorenza, A., Approximate identities in variable L_p spaces, *Math. Nachr.*, 280(3) (2007), 256-270.

[15] Cruz-Uribe, D., Fiorenza, A., Variable Lebesgue Spaces, Foundations and Harmonic Analysis, Birkhauser, Applied and Numerical Harmonic Analysis, 2013.

[16] Diening, L., Maximal function on generalized Lebesgue spaces $L^{p(x)}$, *Math. Ineq. & Appl.*, 7(2) (2004); 245-253. https://doi.org/10.7153/mia-07-27

[17] Diening, L., Harjulehto, P., Hästö, P., Růžička, M., Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2011.

[18] Diening, L., Růžička, M., Calderon-Zymund Operators on Generalized Lebesgue Spaces $L^{P(X)}$ and Problems Related to Fluid Dynamics, Preprint, Mathematische Fakültat, Albert-Ludwings-Universität Freiburg, 2002, 1-20.

[19] Dogu, A., Avsar, A. H., Yildirir, Y. E., Some inequalities about convolution and trigonometric approximation in weighted Orlicz spaces, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 44 (1) (2018), 107-115.

[20] Fan, X., Zhao, D., On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, 263(2) (2001), 424-446. doi:10.1006/jmaa.2000.7617

[21] Guven, A., Israfilov, D. M., Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$, *J. Math. Ineq.* 4 (2) (2010), 285-299.

[22] Jafarov, S. Z., Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents, *Ukr. Math. J.*, 66(10) (2015), 1509-1518. https://doi.org/10.1007/s11253-015-1027-y

[23] Jafarov, S. Z., Approximation by trigonometric polynomials in subspace of variable exponent grand Lebesgue spaces, *Global J. Math. Sci.*, 8(2) (2016), 836-843.

[24] Jafarov, S. Z., Ul'yanov type inequalities for moduli of smoothness, *Appl. Math. E-Notes*, 12 (2012), 221-227.

[25] Jafarov, S. Z., S. M. Nikolskii type inequality and estimation between the best approximations of a function in norms of different spaces, *Math. Balkanica*, 21(1-2) (2007), 173-182.

[26] Ibragimov, I. I., Teoriya Priblizheniya Tselymi Funktsiyami(In Russian), The Theory of Approximation by Entire Functions, Elm, Baku, 1979, 468 pp.

[27] Israfilov, D. M., Testici, A., Approximation problems in the Lebesgue spaces with variable exponent, *J. Math. Anal. Appl.*, 459(1) (2018), 112-123. https://doi.org/10.1016/j.jmaa.2017.10.067

[28] Israfilov, D.M., Testici, A., Approximation by Faber-Laurent rational functions in Lebesgue spaces with variable exponent, *Indag. Math.*, 27(4) (2016), 914-922. https://doi.org/10.1016/j.indag.2016.06.001

[29] Israfilov, D. M., Yirtici, E., Convolutions and best approximations in variable exponent Lebesgue spaces, *Math. Reports,* 18(4) (2016), 497-508.

[30] Koc, H., Simultaneous approximation by polynomials in Orlicz spaces generated by quasi-convex Young functions, *Kuwait J. Sci.*, 43(4) (2016), 18-31.

[31] Kokilashvili, V., Nanobashvili, I., Boundedness criteria for the majorants of Fourier integrals summation means in weighted variable exponent Lebesgue spaces and application, *Georgian Math. J.*, 20(4) (2013), 721-727. https://doi.org/10.1515/gmj-2013-0038

[32] Kováčik, Z. O., Rákosnik, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, 41(116)(4) (1991), 592-618.

[33] Musielak, J., Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, Springer, 1983.

[34] Nikolski, S. M., Inequalities for entire functions of finite degree and their application to the theory of differentiable functions of several variables, AMS Translation Series Two, 80 (1969), 1-38 *Trud. Steklov Math. Inst.*, 38 (1951), 211-278.

[35] Orlicz, W., Über konjugierte exponentenfolgen, *Studia Math.* 3 (1931), 200-212.

[36] Paley, R., Wiener, N., Fourier Transforms in the Complex Domain, American Mathematical Society, 1934.

[37] Rajagopal, K. R., Růžička, M., On the modeling elektroreological materials, *Mech. Res. Comm.*, 23(4) (1996), 401-407. https://doi.org/10.1016/0093-6413(96)00038-9

[38] Růžička, M., Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[39] Samko, S., Differentiation and Integration of Variable Order and the Spaces $L^{p(x)}$, In: Operator Theory for Complex and Hypercomplex Analysis (Mexico City, 1994), 203-219, Contemporary Math. 212, American Mathematical Society, Providence, RI, 1998.

[40] Sharapudinov, I. I., Some Questions in the Theory of Approximation in Lebesgue Spaces with Variable Exponent, (In Russian), Itogi Nauki Yug. Rossii Matematicheskii Monografiya, 5, Southern Institute of Mathematics of the Vladikavkaz Sceince Centre of the Russian Academy of Sciences and the Government of the Republic of North Ossetia-Alania, Vladikavkaz, 2012, 267 pp.

[41] Sharapudinov, I. I., Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials, *Izv. Math.*, 77(2) 2013, 407-434. https://doi.org/10.4213/im7808

[42] Sharapudinov, I. I., On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces, *Azerbaijan J. Math.,* 4(1) (2014), 55-72.

[43] Sharapudinov, I. I., Approximation of functions in Lebesgue and Sobolev spaces with variable exponent by Fourier-Haar sums, *Sbornik Math.,* 205(1-2) (2014), 291-306. https://doi.org/10.4213/sm8274

[44] Sharapudinov, I. I., Some problems in approximation theory in the spaces $L^{p(x)}(E)$, (In Russian), *Analysis Math.*, 33(2) (2007), 135-153. https://doi.org/10.1007/s10476-007-0204-0

[45] Sharapudinov, I. I., The basis property of the Haar system in the space $L^{p(t)}([0,1])$ and the principle of localization in the mean, (In Russian), *Mat. Sbornik*, 130(172)(2) (1986), 275-283. https://doi.org/10.1070/SM1987v058n01ABEH003104

[46] Taberski, R., Approximation by entire functions of exponential type, *Demonstratio Math.*, 14 (1981), 151-181.

[47] Taberski, R., On exponential approximation of locally integrable functions, *Annales Sociatatis mathematicae Polonae, Series I: Commentationes mathematicae*, 32 (1992), 159-174.

[48] Volosivets, S. S., Approximation of functions and their conjugates in variable Lebesgue spaces, *Sbornik Math.*, 208(1) (2017), 44-59. https://doi.org/10.1070/SM8636

[49] Yeh, J., Real Analysis: Theory of Measure and Integration, 2nd edition, World Scientific, 2006.

[50] Zhikov, V. V., Averaging of functionals of the calculus of variations and elasticity theory, (In Russian), *Math. USSR-Izvestiya*, 50(4) (1986), 675-710. https://doi.org/10.1070/IM1987v029n01ABEH000958