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# Efficient Solution of Fractional-Order SIR Epidemic Model of Childhood Diseases With Optimal Homotopy Asymptotic Method

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**ABSTRACT** In providing an accurate approximate analytical solution to the non-linear system of fractional-order susceptible-infected-recovered epidemic model (FOSIREM) of childhood disease has been a challenge, because no norm to guarantees the convergence of the infinite series solution. We compute an accurate approximate analytical solution using the optimal homotopy asymptotic method (OHAM). The fractional differential equations operator (FDEO) is given as conformable derivative operator (CDO). We show the basic idea of the proposed method, the CDO sense, equilibrium points, local asymptotic stability, reproduction number, and the convergence analysis of the proposed method. Numerical results and comparisons with other approximate analytical methods are given to validate the efficiency of the method. The proposed method speedily converges to the exact solution as the fractional-order derivative approaches 1, proved as an excellent tool for solving, and predicting the model.

**INDEX TERMS** Infectious childhood disease, non-linear model, approximate analytical solution, conformable derivative operator, SIR-model, reproduction number.

## I. INTRODUCTION

The study of infectious diseases shows that childhood diseases are the most grievous infectious disease which includes tuberculosis, severe acute respiratory syndrome (SARS), Lyme disease, infectious mononucleosis, salmonella infections, and many more. Lyme disease is a widespread infection disease engendered by a bacterium transported by a particular deer tick. When bitten by an infected tick, there is a chance that the individual-level will develop the symptoms of Lyme disease, including rash, fever, body aches, bull's-eye rash, and sometimes more severe symptoms involving the nervous system and joints. Besides, the introduction and production of vaccines for curing childhood diseases have been a gift to humankind. It defends children from infectious diseases, the main aim of the world health organization (WHO). Since vaccination is believed the most efficient technique against childhood diseases, forming a model that could predict the optimal vaccine coverage level is required to contain the spread of the diseases. The mathematical model plays a significant role in apprehending the spread of childhood disease and provides different methods to control its spread. Several authors have studied childhood disease: for instance, the recent research on childhood diseases vaccination [1]–[3]. In 2007 Makinde proposed a standard Susceptible-Infected-Recovered model, given as

$$\frac{dS}{dt} = (1-p)\pi - \beta\frac{si}{n} - \pi s, \tag{1}$$

$$\frac{dI}{dt} = \beta\frac{si}{n} - (\gamma+\pi)i, \tag{2}$$

$$\frac{dR}{dt} = p\pi + \gamma i - \pi r. \tag{3}$$

The authors [4] rearranged model Eqs.(1-3) using the relation $\frac{s}{n} = S$, $\frac{i}{n} = I$, and $\frac{r}{n} = R$ in a new (SIR) model of the form:

$$\frac{dS}{dt} = (1-p)\pi - \beta SI - \pi S, \tag{4}$$

$$\frac{dI}{dt} = \beta SI - (\gamma+\pi)I, \tag{5}$$

$$\frac{dR}{dt} = p\pi + \gamma I - \pi R. \tag{6}$$

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The SIR model is a standardized compartmental model that describes several epidemiological diseases [5, 6]. The medium by which many childhood diseases spread through a population conforms to this model. The model has a susceptible group (S), an infected group (I), and a recovered group (R), denoting vaccinated as well as recovered people with permanent immunity. This model above shows that vaccination is 100 per cent and the natural death rates $\pi$ in the classes are not equal to births, so the population size N is realistic, not constant. Individuals are born at a constant birth rate $\mu$ with a very low childhood disease mortality rate. We denote the fraction of Individuals vaccinated at birth each year as p (with $0 < p < 1$) and considering that the rest of the population is susceptible. A susceptible individual will move into the infected group through contact with an infected individual, approximated by an average contact rate $\gamma$. An infected individual recovers at a rate $\beta$and enters the recovered group. The recovered group also contains people who are also vaccinated.

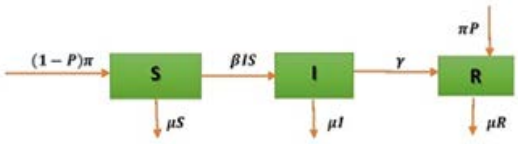


**FIGURE 1.** Flow chart for the SIR model.

The previous studies review that fractional calculus provides more exact models of several applications and shows the behaviour of the dynamic system in sciences than traditional calculus [7]–[10]. A system of non-linear fractional-order reaction-diffusion equations was used to model the superdiffusive spread of modern epidemics because compared with integer-order is well capable of capturing the memory-like effect examined in the non-linear dynamic system [11]. Other recent study includes a fractional-order Brusselator reaction-diffusion model in a triple collision and enzymatic reactions system [12], fractional-order mathematical modeling of novel Corona Virus (COVID-19) [13], and fractional-order analytical and qualitative investigation of (COVID-19) mathematical model [14]. The motivation behind the usefulness of fractional-order differential equations (FODEs) is that FODEs are naturally related to systems that involve memory-like effect which has applications in many systems. Also, they demonstrate the actual behaviour of infection disease but at a prolonged rate. Studying the latest literature review of the childhood disease model includes numerical approximation technique for the FOSIREM [15], homotopy perturbation method for FOSIREM [16], q-homotopy analysis transform method for FOSIREM [17], and solutions of fractional-order model of childhood diseases with constant vaccination strategy [18]. The previous approximate analytical solutions in literature do not possess a norm for the convergence of the infinite series solution, which prompted Marinca (2009) to introduced OHAM, which contains the criteria for convergence of the series solution and is efficient for solving non-linear model [19]. In OHAM, no perturbation or linearization is required. OHAM method does not require any additional parameters, which delay convergence and computational time. Recent articles include OHAM-least square for solving non-linear fractional-order gradient-based dynamic system from an optimization problem [20], an approximate analytical solution of non-linear fractional-order constrained optimization problem using OHAM [21], a new OHAM for fractional optimal control problems [22], approximate solution of two-term fractional-order diffusion, wave-diffusion, and telegraph models arising in mathematical physics using OHAM [23]. We implement OHAM to solve and predicts the model.

We arrange the rest of the paper as follows: Section II discusses a brief introduction to the conformable derivative operator. Section III describes the basic idea of the optimal homotopy asymptotic method, the convergence analysis of the technique, equilibrium points, reproduction number and local asymptotic stability of the model. The numerical results and discussion are presented in section IV. Finally, we present the conclusion in section V.

## II. PRELIMINARIES

### A. CONFORMABLE DERIVATIVE OPERATOR

The earlier studies review that fractional calculus offers more information about non-linear dynamic systems and shows the behaviour of the dynamic system than traditional calculus [24], [25]. There exist several definitions regarding the fractional derivatives, and some basic definitions include Riesz, Riemann–Liouville, Hadamard, Grünwald–Letnikov, Caputo–Fabrizio, and Atangana–Baleanu in the literature. Furthermore, many researchers studied new fractional operators with local, non-local, singular, and non-singular kernels [26]–[30]. The conformable derivative operator was introduced in [31] based on the concept of the local derivative with fractional components. This derivative allows for many extensions of some fundamental theorems in calculus (i.e., the product rule, Rolle's theorem, chain rule, mean value theorem). It can be found many authors focus on using conformable derivative operator to solve a real-life problem [32]–[35]. The CDO conserves many features of classical-order derivatives [36]–[38]. We include here more reasons for using fractional derivative and conformable derivative operator.

(a) Fractional derivative operators gave a more useful information of a non-linear real problem than traditional-order.
(b) It has rendered a new dimension and gave information in between two different integer-order.
(c) More also, as a non-local operator in sense, it considers the fact that the future state not only depends upon the present state but also upon all the history of its previous states.
(d) We considered using conformable derivative operator, because conformable derivative operator has not been

formulated with OHAM approach to study the SIR model of infectious disease for behaviour, performance, and mathematical representation.

(e) To help researchers/authors gain more information on this powerful mathematical tool and widening its application.

(f) The conformable derivative operator is employed to enlarge the stability region of the non-linear dynamical system.

(g) To give a better understanding of newly defined derivative and integral conformable derivative operator in fractional calculus.

(h) To utilize the simplicity and effectiveness of the latest conformable derivative operator.

(i) Conformable derivative operator appeared in more than thousands plus articles and its gaining popularities till today.

*Definition 1*: A (left) fractional derivative starting from s of a function $z{:}[s,\infty)\to\Re$ of order $\alpha\in(m-1,m)$, $m\in N$ is defined by

$$T_s^{\alpha}z(t)=z^{(\alpha)}(t)=\lim_{\epsilon\to 0}\frac{z^{(m-1)}\left(t+\epsilon(t-s)^{(m-\alpha)}-z^{(m-1)}t\right)}{\epsilon},\quad t>s, \tag{7}$$

$$T_s^{\alpha}z(s)=\lim_{x\to s^{+}}T_s^{\alpha}z(t), \tag{8}$$

Provided the limits exist and $z(t)$ is $(m-1)-$differentiable at $t>s$.

The (right) fractional derivative terminating at $s$ of a function $z:(-\infty,s]\to\Re$ of order $\alpha\in(m-1,m)$, $m\in N$ is defined by

$${}_s^{\alpha}Tz(t)=z^{(\alpha)}(t)=(-1)^m\times\lim_{\epsilon\to 0}\frac{z^{(m-1)}\left(t+\epsilon(s-t)^{(m-\alpha)}-z^{(m-1)}t\right)}{\epsilon},\quad t<s, \tag{9}$$

$${}_s^{\alpha}Tz(s)=\lim_{x\to s^{-}}{}_s^{\alpha}Tz(t), \tag{10}$$

Provided the limits exist and $z(t)$ is $(m-1)$-differentiable at $t<s$.

If $T_s^{\alpha}z(t)$ exists on $t>s$, then we say that $z$ is left $\alpha-$differentiable on $t>s$ whereas $z$ is right $\alpha-$differentiable on $t<s$ if ${}_s^{\alpha}Tz(t)$ exist on $t<s$.

*Definition 2*: A (left) fractional integral starting from $s$ a function $z:[s,\infty)\to\Re$ of order $\alpha\in(m-1,m)$, $m\in N$ is defined by

$$I_s^{\alpha}z(t)=\frac{1}{(m-1)!}\int_s^t\frac{(t-x)^{m-1}z(x)}{(x-s)^{m-\alpha}}dx,\quad \alpha>0,\ t>s, \tag{11}$$

$$I_s^{0}z(x)=z(x), \tag{12}$$

and we can define the (right) fractional integral terminating at $s$ of a function $z:(-\infty,s]\to\Re$ of order $\alpha\in(m-1,m)$, $m\in N$ as follows:

$${}_s^{\alpha}Iz(t)=\frac{1}{(m-1)!}\int_t^s\frac{(x-t)^{m-1}z(x)}{(s-x)^{m-\alpha}}dx,\quad \alpha>0,\ t<s, \tag{13}$$

$${}_s^{0}Iz(x)=z(x). \tag{14}$$

It is worth mentioning here that $I_s^{\alpha}I_s^{\beta}z(t)\neq I_s^{\beta}I_s^{\alpha}z(t)$ and ${}_s^{\alpha}I_s^{\beta}Iz(t)\neq_s^{\beta}I_s^{\alpha}Iz(t)$.

*Lemma 1*: If $\alpha\in(m-1,m)$, $m\in N$ and $z{:}[s,\infty)\to\Re$ is $(m-1)$-differentiable, then

(1) $T_s^{\alpha}I_s^{\alpha}z(t)=z(t)$,

(2) $I_s^{\alpha}T_s^{\alpha}z(t)=z(t)-\sum_{k=0}^{m-1}z^{(k)}(s)\frac{(t-s)^k}{k!}$, $t>s$.

*Lemma 2*: If $\alpha\in(m-1,m)$, $m\in N$ and $z{:}[-\infty,s)\to\Re$ is $(m-1)$-differentiable, then

(1) ${}_s^{\alpha}T_s^{\alpha}Iz(t)=z(t)$,

(2) ${}_s^{\alpha}I_s^{\alpha}Tz(t)=z(t)-\sum_{k=0}^{m-1}(-1)z^{(k)}(s)\frac{(s-t)^k}{k!}$, $t<s$.

## III. BASIC IDEA OF OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

Using the conformable derivative operator system definition 1, we have the following form:

$$T^{a}(\varkappa_k(t))+N_k(\varkappa_k(t))+L_k(\varkappa_k(t))=0,\quad t\in\varphi i=1,2\ldots m \tag{15}$$

with initial conditions

$$\varkappa_k(b)=a_i. \tag{16}$$

where $T^a$ is the CDO, $L_k$ is a linear operator, $N_k$ is a non-linear operator, t is an independent variable, $\varkappa_k(t)$ is an unknown function, $\varphi$ is the problem domain. According to OHAM, one can construct a homotopy map $H_k(\mathrm{F}_k(t,\mathcal{P}))$ : $\varphi\times[0,1]\to\varphi$ that satisfies Eq.(15) can be constructed using OHAM as [39], [40].

$$(1-\mathcal{P})[T^{a}(\mathrm{F}_k(t,\mathcal{P}))]=H_k(\mathcal{P})[T^{a}\mathrm{F}_k(t,\mathcal{P})+N_k\mathrm{F}_k(t,\mathcal{P})+L_k\mathrm{F}_k(t,\mathcal{P})], \tag{17}$$

where embedding parameter $(\mathcal{P})$ is $0\leq\mathcal{P}\leq 1$, auxiliary function $H_k(\mathcal{P})\forall\mathcal{P}\neq 0$, unknown function $(\mathrm{F}_k(t,\mathcal{P}))$ and $H(0)=0$, when $\mathcal{P}=0$ and $\mathcal{P}=1$, it holds that $\mathrm{F}_k(t,0)=\varkappa_{k,0}(t)$, and $\mathrm{F}_k(t,1)=\varkappa_k(t)$ respectively. Thus as $\mathcal{P}$ moves from 0 to 1, the solution $\mathrm{F}_k(t,\mathcal{P})$ approach from $\varkappa_{k,0}(t)$ to $\varkappa_k(t)$, where initial guess $\varkappa_{k,0}(t)$ satisfies the linear operator generated from Eq.(17) for $\mathcal{P}=0$.

$$T^{\alpha}(\varkappa_{k,0}(t))=0.\quad \varkappa_{k,0}(b)=0, \tag{18}$$

The $H_k(\mathcal{P})$ is given as

$$H_k(\mathcal{P})=\sum_{j=1}^{n}\mathcal{P}^{j}C_j, \tag{19}$$

where $C_j^s$ can be known later. We get approximate solution by expanding $\mathrm{F}_k(t,\mathcal{P},C_j)$ in Taylor's series in terms of $\mathcal{P}$,

$$\mathrm{F}_k(t,\mathcal{P},C_j)=\varkappa_{k,0}(t)+\sum_{k\geq 1}\varkappa_{i,k}(t,C_j)\mathcal{P}^{i}\quad j=1,2,\ldots,n \tag{20}$$

using above in Eq.(17) with collections of the coefficient like the power of $\mathcal{P}$ gives the governing equations $\varkappa_{i,0}(t)$ in a linear form in Eq.(18). Then $1^{st}$ problems are given as

$$T^{\alpha}(\varkappa_{k,1}(t)) = C_1 N_0(\varkappa_{k,0}(t)), \quad \varkappa_{k,1}(b) = 0, \tag{21}$$

the general governing equation $\varkappa_{k,i}(t)$ is

$$\begin{aligned} T^{\alpha}(\varkappa_{k,i}(t)) - T^{\alpha}(_{k,i-1}(t)) = {} & C_i N_{k,0}(\varkappa_{k,0}(t)) \\ & + \sum_{m=1}^{i-1} C_{j,m}[T^{\alpha}(\varkappa_{k,i-m}(t)) \\ & + N_{k,i-m}(\varkappa_{k,i-1}(t))], \end{aligned} \tag{22}$$

$$\varkappa_{k,i}(b) = 0 \quad i = 2, 3, \ldots m \tag{23}$$

where $N_{k,m}(\varkappa_0(t), \varkappa_{k,1}(t) \ldots, \varkappa_{k,m}(t))$ is the coefficient of $\mathcal{P}^m$, produce by expanding $N_k(\mathrm{F}_k(t, \mathcal{P}, C_j))$ in series relating to $\mathcal{P}$

$$\begin{aligned} N_k(\mathrm{F}_k(t, \mathcal{P}, C_j)) = {} & N_{k,0}(\varkappa_{k,0}(t)) \\ & + \sum\nolimits_{m\geq 1} N_{k,m}(\varkappa_0, \varkappa_1, \ldots \varkappa_m)\mathcal{P}^m \end{aligned} \tag{24}$$

The convergence of series solution Eq.(24) relies on $C_j^s$. If its convergent at $\mathcal{P} = 1$ gives solution to Eq.(15) as

$$\begin{aligned} \varkappa_k(t, C_j) = \varkappa_{k,0}(t) + \sum\nolimits_{k\geq 1}^{m} \varkappa_{i,k}(t, C_j), \\ j = 1, 2, \ldots, n \end{aligned} \tag{25}$$

using Eq.(25) in Eqs.(15-16), we have an expression for the governing equation as

$$R_k(t, C_j) = T^{\alpha}(\varkappa_k(t, C_j)) + N(\varkappa_k(t, C_j)) + L_k(\varkappa_k(t, C_j)) \tag{26}$$

If

$$R_k(t, C_j) = 0, \tag{27}$$

then $\widetilde{\varkappa}_k(t, C_j)$ is the exact solution. Typically, such an instance does not arise. We implement the Galerkin method to find the optimal values $C_j^s$ as given below

$$\mathcal{P}_k = \frac{\partial \widetilde{\varkappa}_k(t, C_j)}{\partial C_j} = 0 \quad k = 1, 2, \ldots m \tag{28}$$

minimize the functional

$$\Delta_k(C_j) = \int_a^b \mathcal{P}_k \times R_k(t, C_j) dt \tag{29}$$

Error norm $L_\infty$

$$L_\infty = ||Z^{exact} - Z_N||_\infty \approx max_i |Z_i^{exact} - (Z_N)_i| \tag{30}$$

### A. CONVERGENCE ANALYSIS

*Theorem 1.* Suppose the series $\varkappa_k(t, C_j) = \varkappa_{k,0}(t) + \sum_{i=1}^{m} \varkappa_{k,i}(t, C_j), for\ j = 1, 2, \ldots, n$ converges where $\varkappa_k(t, C_j)$ is governed by Eq.(25) under the definitions Eq.(22) and Eq.(23), becomes Eq.(15) and Eq.(16) solutions.

Proof: If we assume $\sum_{m=1}^{\infty} \varkappa_{k,m}(t, C_j)$ for $k = 1, 2..n$, converges to $\varkappa_k(t, C_j)$ then

$$\lim_{m\to\infty} \widetilde{\varkappa}_{k,m}(t, C_j) = 0 \quad \forall k = 1, 2 \ldots n. \tag{31}$$

from Eq.(22), we can write

$$\begin{aligned} & \sum\nolimits_{i=1}^{\infty} [C_i N_{k,0}(x_{k,0}(t)) \\ & \quad + \sum\nolimits_{m=1}^{i-1} C_{j,m}[T^{\alpha}(\varkappa_{k,i-m}(t)) + N_{k,i-m}(\varkappa_{k,i-1}(t))] \\ & = \sum\nolimits_{k=1}^{\infty} [T^{\alpha}(\varkappa_{i,k}(t)) - T^{\alpha}(\varkappa_{i,k-1}(t))], \end{aligned} \tag{32}$$

$$= \lim_{n\to\infty} \sum\nolimits_{k=1}^{n} T^{\alpha}(\varkappa_{i,k}(t)) - T^{\alpha}(\varkappa_{i,k-1}(t)), \tag{33}$$

$$\begin{aligned} & = T^{\alpha}\varkappa_{11}(t) + (T^{\alpha}\varkappa_{22}(t) - T^{\alpha}\varkappa_{21}(t)) + .. \\ & \quad + (T^{\alpha}\varkappa_{nn}(t) - T^{\alpha}x_{n(n-1)}(t)), \end{aligned} \tag{34}$$

$$= T^{\alpha}[\lim_{n\to\infty} \sum\nolimits_{m=1}^{n} \varkappa_{nn}(t)] = T^{\alpha}[\lim_{n\to\infty} \varkappa_{nn}(t)] = 0. \tag{35}$$

equating the RHS of Eq.(35) with equation below

$$\begin{aligned} 0 = {} & \sum\nolimits_{m=1}^{\infty} T^{\alpha}\varkappa_{k(m-1)} + \sum\nolimits_{m=1}^{\infty} N\varkappa_{k(m-1)} \\ & + \sum\nolimits_{m=1}^{\infty} L_k(t\varkappa_{k(m-1)}), \end{aligned} \tag{36}$$

$$\begin{aligned} 0 = {} & \sum\nolimits_{m=1}^{\infty} [T^{\alpha}\varkappa_{k(m-1)} + N\varkappa_{k(m-1)} \\ & + L_k(t, \mu, \varkappa_{k(m-1)})], \end{aligned} \tag{37}$$

$$\begin{aligned} & T^{\alpha}\varkappa_k(t, C_j) + N\varkappa_k(t, C_j) \\ & \quad + L_k(\varkappa_k(t, C_j)) = 0 \quad \forall k = 1, 2..n. \end{aligned} \tag{38}$$

If the $C_j$ is chosen properly, then Eq.(38) leads to the solution of Eqs.(15-16).

### B. EQUILIBRIUM POINTS, REPRODUCTION NUMBER AND LOCAL ASYMPTOTIC STABILITY

This section includes the possible fixed points of the model Eqs.(4-6). Two possible equilibrium points are calculated, i.e., Disease-free equilibrium (DFE) and endemic equlibrium (EE). The steady-state solution of the model is given below by considering the rate of change for a time becomes zero:

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0.$$

Using the above equation, model Eqs. (4-6) becomes:

$$\begin{aligned} 0 &= (1-p)\pi - \beta SI - \pi S, \\ 0 &= \beta SI - (\gamma + \pi)I, \\ 0 &= p\pi + \gamma I - \pi R. \end{aligned}$$

From a steady-state systems above, DFE is obtained as

$$\Psi_{DFE} = (S_0, I_0, R_0) = (0, -p + 1, p).$$

Similarly, EE of the model Eqs. (4-6) is given by: $\Psi_{EE} = (S_*, I_*, R_*)$, where

$$S_* = -\frac{\beta\pi p - (\beta - \gamma)\pi + \pi^2}{\beta\gamma + \beta\pi},$$

$$I_* = \frac{\gamma + \pi}{\beta},$$

and

$$R_* = \frac{\beta\pi p + \beta\gamma - \gamma^2 - \gamma\pi}{\beta\gamma + \beta\pi}.$$

**TABLE 1. Parameters values.**

| Parameter | Description |
|---|---|
| $N_1 = 1$ | the initial population of S(t), who are susceptible |
| $N_2 = 0.5$ | the initial population of I(t), who are infective |
| $N_3 = 0$ | the initial population of R(t), who are recover |
| $\beta = 0.8$ | rate of change of infective to recover the population |
| $\gamma = 0.03$ | rate of change of susceptible to infective population |
| $\mu = 0.4$ | constant birth rate |
| $\rho = 0.9$ | individual vaccinated at birth |
| $\pi = 0.4$ | natural death rates |

The basic reproduction number $\Re_0$ is calculated by the next-generation technique. The $F$ and $V$ matrices at DFE $\Psi_0$ is given as follow:

$$F = \beta SI,$$

and

$$V = (\gamma + \pi)I.$$

Taking the product of $F$ and $V$ inverse, we have the reproduction number in the form

$$\Re_0 = \left(\frac{\beta(p-1)}{\gamma + \pi}\right).$$

The DFE locally asymptotically stable (LAS) if $\Re_0 < 1$, but unstable if $\Re_0 > 1$. This $\Re_0$ is the product of the transmission rate, the mean infectious time $S_0$ and fits with the epidemiological definition $\Re_0$. Note that $\Re_0$ it is independent of the fraction dying from the disease. From the dynamics of the system, if $\Re_0 < 1$ the number of infectious individuals declines gradually to 0, whereas if $\Re_0 > 1$, then this number first inclines (before tending to zero), thus $R_0 = 1$ acting as a sharp threshold between the disease dying out or causing a pandemic.

## IV. NUMERICAL RESULTS AND DISCUSSIONS

$$T^{\alpha}S = (1-p)\pi - \beta SI - \pi S, \tag{39}$$
$$T^{\alpha}I = \beta SI - (\gamma + \pi)I, \tag{40}$$
$$T^{\alpha}R = p\pi + \gamma I - \pi R. \tag{41}$$

where $0 < \alpha \leq 1$, while $p$, $\beta$, $\pi$, and $\gamma$ are positive constant parameters and the given initial conditions are $S(0) = N_1$, $I(0) = N_2$, and $R(0) = N_3$.

the OHAM method procedure is given as follows

$$L_1[\mathrm{F}_1(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_1(t,\mathcal{P}), \tag{42}$$
$$L_2[\mathrm{F}_2(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_2(t,\mathcal{P}), \tag{43}$$
$$L_3[\mathrm{F}_3(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_3(t,\mathcal{P}), \tag{44}$$

$$N_1[\mathrm{F}_1(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_1(t,\mathcal{P}) - (1-p)\pi + \beta\mathrm{F}_1(t,\mathcal{P})\mathrm{F}_2(t,\mathcal{P}) - \pi\mathrm{F}_1(t,\mathcal{P}), \tag{45}$$
$$N_2[\mathrm{F}_2(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_2(t\mathcal{P}) - \beta(\mathrm{F}_1(t,\mathcal{P})\mathrm{F}_2(t,\mathcal{P}) - (\gamma + \pi)\mathrm{F}_2(t\mathcal{P}), \tag{46}$$
$$N_3[\mathrm{F}_3(t,\mathcal{P})] = T^{\alpha}\mathrm{F}_3(t\mathcal{P}) - p\pi - \gamma(\mathrm{F}_2(t,\mathcal{P}) + \pi\mathrm{F}_3(t,\mathcal{P}). \tag{47}$$

using Eq.(17)

$$(1-\mathcal{P})T^{\alpha}\mathrm{F}_1(t,\mathcal{P}) = H_k(\mathcal{P})[T^{\alpha}\mathrm{F}_1(t,\mathcal{P}) - (1-p)\pi +, \beta\mathrm{F}_1(t,\mathcal{P})_2(t,\mathcal{P}) - \pi\mathrm{F}_1(t,\mathcal{P})], \tag{48}$$
$$(1-\mathcal{P})T^{\alpha}\mathrm{F}_2(t,\mathcal{P}) = H_k(\mathcal{P})[T^{\alpha}\mathrm{F}_2(t\mathcal{P}) - \beta(\mathrm{F}_1(t,\mathcal{P})\mathrm{F}_2(t,\mathcal{P}) - (\gamma + \pi)\mathrm{F}_2(t\mathcal{P})], \tag{49}$$
$$(1-\mathcal{P})T^{\alpha}\mathrm{F}_3(t,\mathcal{P}) = H_k(\mathcal{P})[T^{\alpha}\mathrm{F}_3(t\mathcal{P}) - p\pi - \gamma(\mathrm{F}_2(t,\mathcal{P}) + \pi\mathrm{F}_3(t,\mathcal{P}). \tag{50}$$

where

$$\mathrm{F}_1(t,\mathcal{P}) = s_0(t) + \sum\nolimits_{j\leq 1} s_{1,j}(t)\mathcal{P}^j, \tag{51}$$
$$\mathrm{F}_2(t,\mathcal{P}) = i_0(t) + \sum\nolimits_{j\leq 1} i_{1,j}(t)\mathcal{P}^j, \tag{52}$$
$$\mathrm{F}_3(t,\mathcal{P}) = r_0(t) + \sum\nolimits_{j\leq 1} r_{1,j}(t)\mathcal{P}^j. \tag{53}$$
$$H_k(\mathcal{P}) = \mathcal{P}C_1 + \mathcal{P}^2C_2 + \mathcal{P}^3C_3 + \ldots k = 1, 2\ldots m. \tag{54}$$

substitute $\mathrm{F}_1(t,\mathcal{P})$, $\mathrm{F}_2(t,\mathcal{P})$, $\mathrm{F}_3(t,\mathcal{P})$ and $H_k(\mathcal{P})$ into Eqs.(51-54), and equating the coefficient of likes power of $\mathcal{P}$, gives linear FDEs as,

$$\mathcal{P}^0 : T^{\alpha}S_0(t) = 0, \tag{55}$$
$$\mathcal{P}^0 : T^{\alpha}I_0(t) = 0, \tag{56}$$
$$\mathcal{P}^0 : T^{\alpha}R_0(t) = 0. \tag{57}$$
$$\mathcal{P}^1 : T^{\alpha}S_1(t) = T^{\alpha}s_0(t)C_1 - T^{\alpha}s_0(t) - 0.4s_0(t)C_1 - 0.8s_0(t)C_1i_0(t)C_1 + 0.04C_1 = 0, \tag{58}$$
$$\mathcal{P}^1 : T^{\alpha}S_1(t) = T^{\alpha}i_0(t)C_1 - T^{\alpha}i_0(t) + 0.43i_0(t)C_1 + 0.8s_0(t)i_0(t)C_1 = 0, \tag{59}$$
$$\mathcal{P}^1 : T^{\alpha}R_1(t) = T^{\alpha}r_0(t)C_1 - T^{\alpha}r_0(t) + 0.4r_0(t)C_1 - 0.03i_0(t)C_1 - 0.36C_1 = 0, \tag{60}$$
$$\mathcal{P}^2 : T^{\alpha}S_2(t) = T^{\alpha}s_0(t)C_2 - T^{\alpha}s_1(t)C_1 - T^{\alpha}s_1(t) - 0.4s_0(t)C_2 - 0.4s_1(t)C_1 - 0.8s_0(t)i_0(t)C_2 - 0.8s_1(t)i_0(t)C_1 - 0.8s_0(t)i_1(t)C_1 + 0.04C_2 = 0, \tag{61}$$
$$\mathcal{P}^2 : T^{\beta}I_2(t) = T^{\beta}i_0(t)C_2 + T^{\beta}u_1(t)C_1 - T^{\beta}u_1(t) + 0.43i_0(t)C_2 - 0.8s_1(t)C_1 + i_0(t)C_10.43i_1(t)C_1 - 0.8s_0(t)i_0(t)C_2 - 0.8s_0(t)i_1(t)C_1 = 0, \tag{62}$$

**TABLE 2.** Control-convergence parameters $C_k$ at different values of $\alpha$.

| VARIABLE | S(T) | S(T) | I(T) | I(T) |
|---|---|---|---|---|
| $\alpha$ | $C_1$ | $C_2$ | $C_1$ | $C_2$ |
| 1.0 | 1.32856988 | -7.28341868 | 1.58108712 | 0.70156516 |
| 0.95 | 1.22834561 | -7.45351356 | 1.45105722 | 0.65135517 |
| 0.85 | 1.12821465 | -7.58353753 | 1.38168232 | 0.61144215 |
| 0.75 | 1.08456929 | -7.68245878 | 1.32129183 | 0.57136408 |
| 0.65 | 1.06234513 | -7.74843198 | 1.27182345 | 0.51094562 |
| 0.55 | 1.02983471 | -7.84239812 | 1.24345123 | 0.46341223 |

**TABLE 3.** Continuation of control-convergence parameters $C_k$ of Table 2.

| VARIABLE | R(T) | R(T) |
|---|---|---|
| $\alpha$ | $C_1$ | $C_1$ |
| 1 | 1.123481639 | 1.4827017992 |
| 0.95 | 1.073271522 | 1.4226016121 |
| 0.85 | 1.023581727 | 1.3825017131 |
| 0.75 | 0.872471528 | 1.292701312 |
| 0.65 | 0.82345562 | 1.228632142 |
| 0.55 | 0.78342176 | 1.192378991 |

$$\begin{aligned}\mathcal{P}^2 : T^{\beta}R_2(t) &= T^{\beta}r_0(t)C_2 + T^{\beta}r_1(t)C_1 - T^{\beta}r_1(t)\\ &\quad + 0.4r_1(t)C_1 - 0.03i_1(t)C_1\\ &\quad - 0.03i_0(t)C_2 + 0.4r_0(t)C_2 - 0.36C_2 = 0.\end{aligned} \tag{63}$$

Using the definition (1-2) and lemma (1-2) on the above equations with the initial condition gives

$$S_0(t) = 1, \tag{64}$$

$$I_0(t) = 0.5, \tag{65}$$

$$R_0(t) = 0. \tag{66}$$

$$S_1(t, C_1) = \frac{19}{25}tC_1 + 1, \tag{67}$$

**TABLE 4.** The number of susceptible (t) individuals in case $\alpha = 1$.

| Time (s) | OHAM | RK4 | HAM | HPM | Abs Error |
|---|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.00000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 0.1 | 0.929379248 | 0.929368237 | 0.930055043 | 0.931207687 | 1.21E-05 |
| 0.2 | 0.863127311 | 0.863115211 | 0.864261318 | 0.866474727 | 1.1011E-05 |
| 0.3 | 0.801222123 | 0.801221102 | 0.802641003 | 0.805828081 | 1.021E-06 |
| 0.4 | 0.743593798 | 0.743483786 | 0.745165172 | 0.749244071 | 0.000110012 |
| 0.5 | 0.690130998 | 0.690110898 | 0.691758404 | 0.696653083 | 2.01E-05 |
| 0.6 | 0.640688362 | 0.640687362 | 0.642306538 | 0.647946778 | 1E-05 |
| 0.7 | 0.595092567 | 0.595096567 | 0.596661972 | 0.602983522 | 3E-05 |
| 0.8 | 0.553150872 | 0.553130872 | 0.554652487 | 0.561597082 | 2E-05 |
| 0.9 | 0.514656453 | 0.514645453 | 0.516087323 | 0.523602396 | 1.1E-05 |
| 1 | 0.479395205 | 0.479384203 | 0.480764135 | 0.488802586 | 1.1002E-05 |

**TABLE 5.** The number of infected (t) individuals in case $\alpha = 1$.

| Time (s) | OHAM | RK4 | HAM | HPM | Abs Error |
|---|---|---|---|---|---|
| 0.0 | 0.50000000 | 0.5000000 | 0.5000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.517303285 | 0.517202274 | 0.518298997 | 0.519798112 | 0.000101011 |
| 0.2 | 0.532157737 | 0.532146537 | 0.534124433 | 0.537103361 | 1.12E-05 |
| 0.3 | 0.544521403 | 0.544421303 | 0.547412868 | 0.551828479 | 0.0001001 |
| 0.4 | 0.554404513 | 0.554404213 | 0.558157009 | 0.563946022 | 3E-07 |
| 0.5 | 0.561862264 | 0.561852234 | 0.566399051 | 0.573482011 | 1.003E-05 |
| 0.6 | 0.566986866 | 0.566885866 | 0.572222272 | 0.580507425 | 0.000101 |
| 0.7 | 0.569900851 | 0.569800751 | 0.57574437 | 0.585131446 | 0.0001001 |
| 0.8 | 0.570748057 | 0.570648046 | 0.577107802 | 0.587491363 | 0.000100011 |
| 0.9 | 0.569687936 | 0.569677635 | 0.576473458 | 0.587745809 | 1.0301E-05 |
| 1 | 0.566888398 | 0.566786357 | 0.574013064 | 0.586067075 | 0.000102041 |

$$I_1(t, C_1) = \frac{37}{200}tC_1 + \frac{1}{2}, \tag{68}$$

$$R_1(t, C_1) = \frac{3}{8}tC_1 - \frac{3}{8}C_1. \tag{69}$$

**TABLE 6.** The number of recovered (t) individuals in case $\alpha = 1$.

| Time (s) | OHAM | RK4 | HAM | HPM | Abs Error |
|---|---|---|---|---|---|
| 0.0 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 0.1 | 3.68E-02 | 3.67E-02 | 3.76E-02 | 3.88E-02 | 1.00E-04 |
| 0.2 | 7.22E-02 | 7.21E-02 | 7.37E-02 | 7.60E-02 | 1.00E-04 |
| 0.3 | 0.10622036 | 0.10621016 | 0.10851398 | 0.11190669 | 1.02E-05 |
| 0.4 | 0.13896145 8 | 0.13886144 8 | 0.14197418 7 | 0.14640987 4 | 1.00E-04 |
| 0.5 | 0.17044422 | 0.17032322 | 0.17415420 7 | 0.17959228 4 | 1.21E-04 |
| 0.6 | 0.20071097 4 | 0.20071056 4 | 0.20509673 2 | 0.21149789 7 | 4.10E-07 |
| 0.7 | 0.22980271 7 | 0.22970261 7 | 0.23484305 8 | 0.24216954 6 | 1.00E-04 |
| 0.8 | 0.25775923 1 | 0.25774823 1 | 0.26343322 6 | 0.27164875 5 | 1.10E-05 |
| 0.9 | 0.28461918 | 0.28451918 | 0.29090613 8 | 0.29997584 9 | 1.00E-04 |
| 1 | 0.31042020 2 | 0.31042010 2 | 0.31729966 8 | 0.32719006 7 | 1.00E-07 |

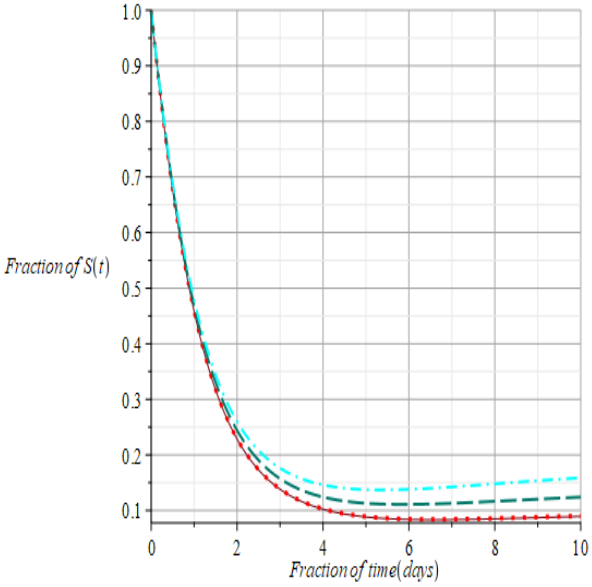


**FIGURE 2.** For the value of $\alpha = 1$ (OHAM = dot, HAM = dash, HPM = dash-dot, and Rk4 = solid) at $S(t)$.

$$S_2(t, C_1, C_2) = \frac{189}{500}C_1^2t^2 + (\frac{19}{25}C_2 + \frac{49}{25}C_1 + \frac{19}{25}C_1^2)t + 1, \quad (70)$$

$$I_2(t, C_1, C_2) = \frac{744}{40000}C_1^2t^2 + (\frac{77}{100}C_1 + \frac{37}{200}C_2 - \frac{37}{200}C_1^2)t + \frac{1}{2}, \quad (71)$$

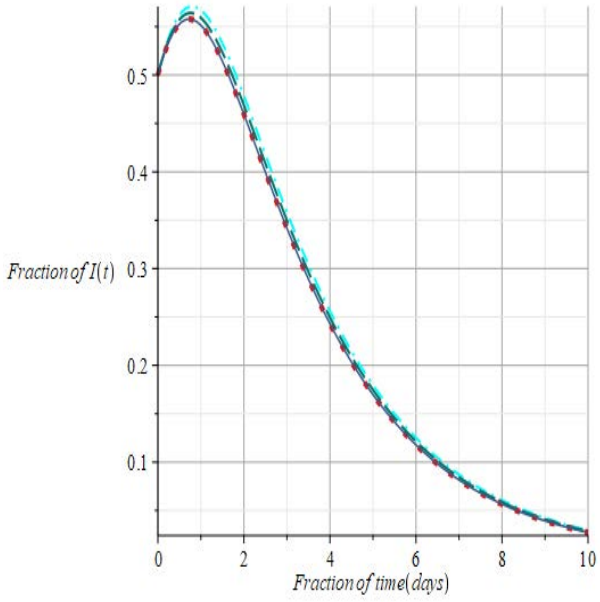


**FIGURE 3.** For the value of $\alpha = 1$ (OHAM = dot, HAM = dash, HPM = dash-dot, and RK4 = solid) at $I(t)$.

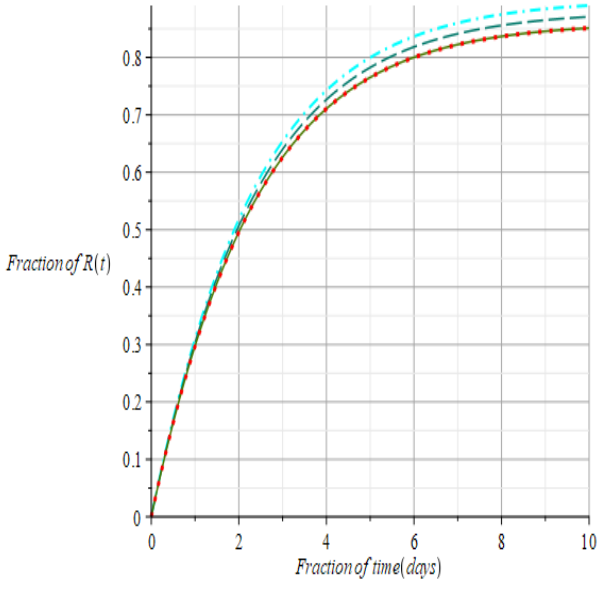


**FIGURE 4.** For the value of $\alpha = 1$ (OHAM = dot, HAM = dash, HPM = dash-dot, and RK4 = solid) at $R(t)$.

$$R_2(t, C_1, C_2) = \frac{-2889}{40000}C_1^2t^2 + (\frac{39}{100}C_1 - \frac{9}{40}C_1^2) + \frac{3}{8}C_2)t + \frac{11889}{40000}C_1^2 - 39C_1\frac{1}{100} - 3C_2\frac{1}{8}. \quad (72)$$

From the $3^{rd}$-order approximate analytical solutions generated by OHAM, for $\alpha = 1$, we have

$$S(t, C_1, C_2) = (0.3780000000C_1^2)^2t^2 + (2.720000000C_1 + 0.7600000000C_2 + 0.7600000000C_1^2)t + 3, \quad (73)$$

**TABLE 7.** The number of each individuals in case $\alpha = 1$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.500000 | 0.00000000 |
| 0.1 | 0.929379248 | 0.517303285 | 3.68E-02 |
| 0.2 | 0.863127311 | 0.532157737 | 7.22E-02 |
| 0.3 | 0.801222123 | 0.544521403 | 0.10622036 |
| 0.4 | 0.743593798 | 0.554404513 | 0.138961458 |
| 0.5 | 0.690130998 | 0.561862264 | 0.17044422 |
| 0.6 | 0.640688362 | 0.566986866 | 0.200710974 |
| 0.7 | 0.595092567 | 0.569900851 | 0.229802717 |
| 0.8 | 0.553150872 | 0.570748057 | 0.257759231 |
| 0.9 | 0.514656453 | 0.569687936 | 0.28461918 |
| 1 | 0.479395205 | 0.566888398 | 0.310420202 |

**TABLE 8.** The number of each individuals in case $\alpha = 0.95$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.927842288 | 0.512346758 | 3.52E-02 |
| 0.2 | 0.860175334 | 0.522381672 | 6.91E-02 |
| 0.3 | 0.796970288 | 0.530128732 | 0.101697177 |
| 0.4 | 0.738149859 | 0.535651526 | 0.13304721 |
| 0.5 | 0.683594945 | 0.539045874 | 0.16319345 |
| 0.6 | 0.633152225 | 0.540432732 | 0.192176089 |
| 0.7 | 0.586640434 | 0.539952213 | 0.220034067 |
| 0.8 | 0.543859051 | 0.537756279 | 0.246805192 |
| 0.9 | 0.504593749 | 0.534004192 | 0.272526232 |
| 1 | 0.468623313 | 0.528857061 | 0.297233004 |

$$I(t, C_1, C_2) = 1.500000000 + 0.1862250000C_1^2t^2 + (0.9550000000C_1 + 0.1850000000C_2 - 0.1850000000C_1^2)t, \quad (74)$$

**TABLE 9.** The number of each individuals in case $\alpha = 0.85$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.926689569 | 0.507437731 | 3.29E-02 |
| 0.2 | 0.857961351 | 0.512785195 | 6.45E-02 |
| 0.3 | 0.793781413 | 0.516116486 | 9.49E-02 |
| 0.4 | 0.734066906 | 0.517532866 | 0.124175837 |
| 0.5 | 0.678692905 | 0.517156022 | 0.152317295 |
| 0.6 | 0.627500122 | 0.515122226 | 0.17937376 |
| 0.7 | 0.580301334 | 0.51157739 | 0.205381091 |
| 0.8 | 0.536890185 | 0.50667157 | 0.230374134 |
| 0.9 | 0.497046722 | 0.500555579 | 0.254386812 |
| 1 | 0.460544394 | 0.493377161 | 0.277452207 |

**TABLE 10.** The number of each individuals in case $\alpha = 0.75$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.926305329 | 0.502575746 | 3.09E-02 |
| 0.2 | 0.857223357 | 0.503365007 | 6.06E-02 |
| 0.3 | 0.792718454 | 0.502474608 | 8.93E-02 |
| 0.4 | 0.732705921 | 0.500027073 | 0.116783026 |
| 0.5 | 0.677058892 | 0.496155081 | 0.143253833 |
| 0.6 | 0.625616088 | 0.490997101 | 0.168705153 |
| 0.7 | 0.578188301 | 0.484693678 | 0.193170278 |
| 0.8 | 0.534567229 | 0.477383691 | 0.216681586 |
| 0.9 | 0.494531046 | 0.469202093 | 0.239270628 |
| 1 | 0.457851421 | 0.460277529 | 0.260968209 |

$$R(t, C_1, C_2) = 1.500000000 + 0.1862250000C_1^2t^2 + (.9550000000C_1 + 0.1850000000C_2 - 0.1850000000C_1^2)t. \quad (75)$$

We determine $C_1$ and $C_2$ in Eqs. (73-75) by using the procedure mentioned in Eqs.(28-29). As given below

$$S(t), \quad C_1 = 1.328569889, \; C_2 = -7.283418688,$$

**TABLE 11.** The number of each individuals in case $\alpha = 0.65$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.916305329 | 0.497760348 | 0.028943990 |
| 0.2 | 0.837223357 | 0.4941178679 | 0.0568003469 |
| 0.3 | 0.742718454 | 0.4891933068 | 0.0836044469 |
| 0.4 | 0.712705921 | 0.4831134178 | 0.1093902152 |
| 0.5 | 0.657058892 | 0.4760069541 | 0.1341903703 |
| 0.6 | 0.615616088 | 0.4680018421 | 0.1580365457 |
| 0.7 | 0.558188301 | 0.4592227164 | 0.1809594649 |
| 0.8 | 0.514567229 | 0.4497887758 | 0.2029890374 |
| 0.9 | 0.474531046 | 0.4398125017 | 0.2241544440 |
| 1 | 0.427851421 | 0.4293984773 | 0.2444842111 |

**TABLE 12.** The number of each individuals in case $\alpha = 0.55$.

| Time (s) | Susceptible(T) | Infected(T) | Recovered(T) |
|---|---|---|---|
| 0.0 | 1.00000000 | 0.50000000 | 0.00000000 |
| 0.1 | 0.906305329 | 0.4931277244 | 0.0269834622 |
| 0.2 | 0.817223357 | 0.4855378494 | 0.0529561641 |
| 0.3 | 0.722718454 | 0.4772795571 | 0.0779504686 |
| 0.4 | 0.702705921 | 0.4684120774 | 0.1019974044 |
| 0.5 | 0.627058892 | 0.4590013361 | 0.1251269079 |
| 0.6 | 0.595616088 | 0.4491173060 | 0.1473679386 |
| 0.7 | 0.518188301 | 0.4388317603 | 0.1687486519 |
| 0.8 | 0.494567229 | 0.4282159989 | 0.1892964891 |
| 0.9 | 0.454531046 | 0.4173394736 | 0.2090382602 |
| 1 | 0.407851421 | 0.4062683356 | 0.2280002133 |

$$I(t), \quad C_1 = 1.581087126, \ C_2 = 0.70156516,$$
$$R(t), \quad C_1 = 1.123481639, \ C_2 = 1.482701799.$$

The general approximate analytical solutions are given as

$$S(t) = 0.6672070251t^2 - 0.580213663t + 3, \quad (76)$$

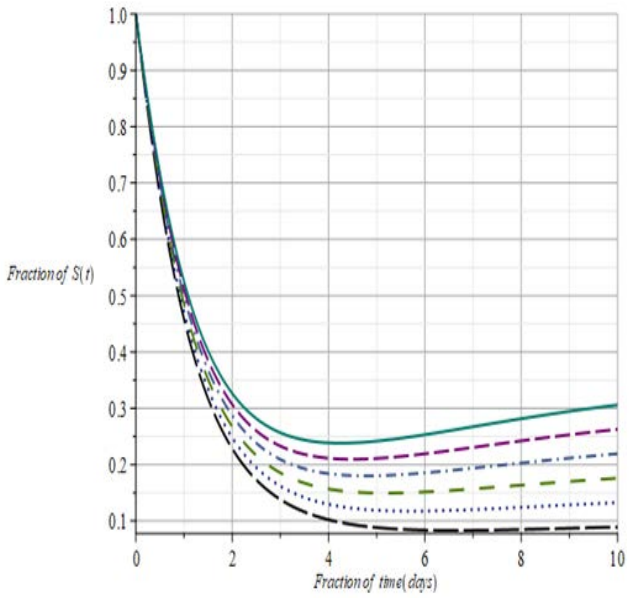


**FIGURE 5.** For different values of $\alpha$ ($\alpha = 1$ Solid, $\alpha = 0.95$ Dash, $\alpha = 0.85$ Dash-dot, $\alpha = 0.75$ Space-dash, $\alpha = 0.65$ Dot, and $\alpha = 0.55$ Long-dash) at $S(t)$.

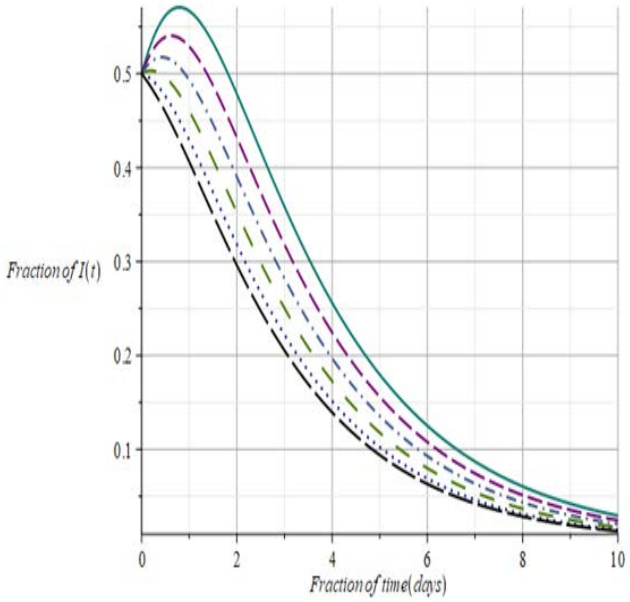


**FIGURE 6.** For different values of $\alpha$ ($\alpha = 1$ Solid, $\alpha = 0.95$ Dash, $\alpha = 0.85$ Dash-dot, $\alpha = 0.75$ Space-dash, $\alpha = 0.65$ Dot, and $\alpha = 0.55$ Long-dash) at $I(t)$.

$$I(t) = 0.4655320522t^2 + +1.177258007t + 1.500000000, \quad (77)$$

$$R(t) = -0.09116318897t^2 + 1.131479155t - 1.040315966. \quad (78)$$

### A. DISCUSSION

In the presented problem, the susceptible group S(t), the infected group I(t), and the recovered group R(t)

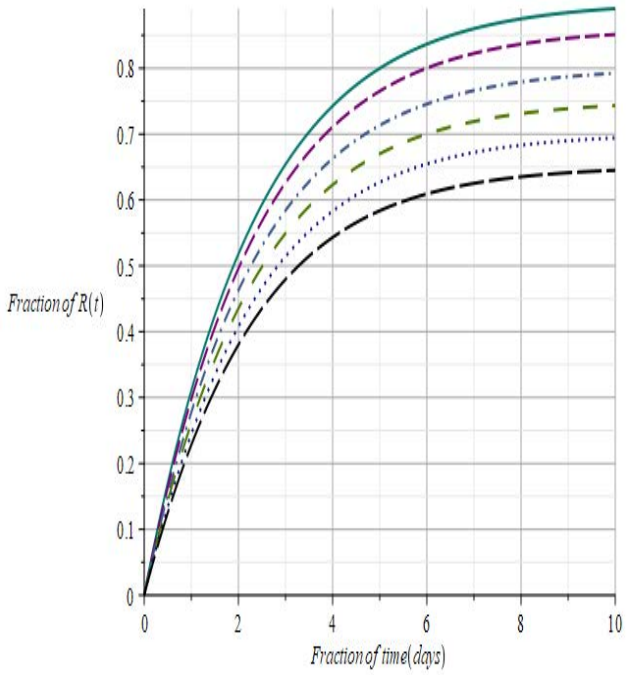


**FIGURE 7.** For different values of $\alpha$ ($\alpha = 1$ Solid, $\alpha = 0.95$ Dash, $\alpha = 0.85$ Dash-dot, $\alpha = 0.75$ Space-dash, $\alpha = 0.65$ Dot, and $\alpha = 0.55$ Long-dash) at $R(t)$.

results have been obtained. The results show that the accurate series solution continually relies on the optimal values of the control-convergence parameters $C_k$ as described in **TABLES** 2-3. The results in **TABLES** (4-6) and **Figures (2-4)** show the behaviour, numerical simulation, and the comparisons between OHAM, HAM, HPM, and RK4 at $\alpha = 1$. **Figures (5-7)** show the number of the susceptible group S(t), the infected group I(t), and the recovered group R(t) results at ($\alpha$ =1, 0.95, 0.85, 0.75, 0.65, and 0.55) for numerical simulation and performance. The fractional-order results in **TABLES** 7-12 show the performance, mathematical values, and the declining behaviour of the spread at a slower rate. We observed that the susceptible group decrease with time while that of the recovered group gradually increases due to the inclusion of the vaccinated susceptible group. The population of are infected group decreases in the period of the epidemic. The results obtained show that when $\alpha \to 1$ the integer-order solution for the system is recovered. The calculations are performed using Maple software 2021a, HP ENVY laptop 13 corei7 8th Gen 16GB.

## V. CONCLUSION

This paper implements the OHAM method to described the fractional-order childhood disease model's behaviours, performance, and mathematical values representation. The model is investigated for accurate approximate analytical solutions using the proposed approach. One can be ascertained that the approach provides an excellent approximate analytical solution of the models as $\alpha \to 1$. The OHAM method agrees with RK4 at ($\alpha = 1$) and performs better than those approximate analytical methods mentioned above. The calculation requires a very short time to complete and consumes a little amount of CPU time, increasing convergence speed. The other methods mentioned above possess no norm for the convergence of the solution and are slow convergence as compared to OHAM. The OHAM technique is reliable, dependable, and efficient for finding an approximate analytical solution and predicting the SIR model.

## REFERENCES

[1] T. Callaghan, M. Motta, S. Sylvester, K. L. Trujillo, and C. C. Blackburn, "Parent psychology and the decision to delay childhood vaccination," *Social Sci. Med.*, vol. 238, Oct. 2019, Art. no. 112407.
[2] G. Hanquet *et al.*, "Effect of childhood pneumococcal conjugate vaccination on invasive disease in older adults of 10 European countries: Implications for adult vaccination," *Thorax*, vol. 74, no. 5, pp. 473–482, May 2019.
[3] M. Butt, R. Mohammed, E. Butt, S. Butt, and J. Xiang, "Why have immunization efforts in Pakistan failed to achieve global standards of vaccination uptake and infectious disease control?" *Risk Manage. Healthcare Policy*, vol. 13, pp. 111–124, Feb. 2020.
[4] F. Haq, M. Shahzad, S. Muhammad, H. A. Wahab, and G. U. Rahman, "Numerical analysis of fractional order epidemic model of childhood diseases," *Discrete Dyn. Nature Soc.*, vol. 2017, pp. 1–7, Jan. 2017.
[5] M. Bohner, S. Streipert, and D. F. M. Torres, "Exact solution to a dynamic SIR model," *Nonlinear Anal., Hybrid Syst.*, vol. 32, pp. 228–238, May 2019.
[6] G. Gaeta, "A simple SIR model with a large set of asymptomatic infectives," 2020, *arXiv:2003.08720*.
[7] A. K. Gupta and S. S. Ray, "On the solitary wave solution of fractional Kudryashov–Sinelshchikov equation describing nonlinear wave processes in a liquid containing gas bubbles," *Appl. Math. Comput.*, vol. 298, pp. 1–12, Apr. 2017.
[8] C. Pan, B. Li, Y. Zhang, N. Geldner, C. Wang, L. Wang, and A. Bertozzi, "Crime modeling with truncated Lévy flights and effects of police patrol," 2016, *arXiv:1601.03415*.
[9] S. Thirumalai and R. Seshadri, "Spectral solutions of fractional differential equation modelling electrohydrodynamics flow in a cylindrical conduit," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 79, Dec. 2019, Art. no. 104931.
[10] C. Huang, H. Li, and J. Cao, "A novel strategy of bifurcation control for a delayed fractional predator–prey model," *Appl. Math. Comput.*, vol. 347, pp. 808–838, Apr. 2019.
[11] V. Vyawahare and P. S. Nataraj, *Fractional-Order Modeling of Nuclear Reactor: From Subdiffusive Neutron Transport to Control-Oriented Models: A Systematic Approach*. Singapore: Springer, 2018.
[12] J. Singh, M. M. Rashidi, D. Kumar, and R. Swroop, "A fractional model of a dynamical Brusselator reaction-diffusion system arising in triple collision and enzymatic reactions," *Nonlinear Eng.*, vol. 5, no. 4, pp. 277–285, Jan. 2016.
[13] S. W. Ahmad, M. Sarwar, K. Shah, A. Ahmadian, and S. Salahshour, "Fractional order mathematical modeling of novel corona virus (COVID-19)," *Math. Methods Appl. Sci.*, vol. 5, no. 4, pp. 277–285, Feb. 2021.
[14] K. Shah, M. Sher, H. Rabai'ah, A. Ahmadian, S. Salahshour, and B. A. Pansera, "Analytical and qualitative investigation of COVID-19 mathematical model under fractional differential operator," *Math. Methods Appl. Sci.*, pp. 1–20, Aug. 2021.
[15] K. M. Owolabi and A. Atangana, "Mathematical analysis and computational experiments for an epidemic system with nonlocal and nonsingular derivative," *Chaos, Solitons Fractals*, vol. 126, pp. 41–49, Sep. 2019.
[16] N. Sene, "SIR epidemic model with Mittag–Leffler fractional derivative," *Chaos, Solitons Fractals*, vol. 137, Aug. 2020, Art. no. 109833.
[17] P. Veeresha, E. Ilhan, D. G. Prakasha, H. M. Baskonus, and W. Gao, "A new numerical investigation of fractional order susceptible-infected-recovered epidemic model of childhood disease," *Alexandria Eng. J.*, vol. 61, no. 2, pp. 1747–1756, 2021.
[18] A. A. M. Arafa, S. Z. Rida, and M. Khalil, "Solutions of fractional order model of childhood diseases with constant vaccination strategy," *Math. Sci. Lett.*, vol. 1, no. 1, pp. 17–23, May 2012.

[19] V. Marinca, N. Herişanu, C. Bota, and B. Marinca, "An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate," *Appl. Math. Lett.*, vol. 22, no. 2, pp. 245–251, Feb. 2009.
[20] O. O. Okundalaye, W. A. M. Othman, and N. Kumaresan, "Optimal homotopy asymptotic method-least square for solving nonlinear fractional-order gradient-based dynamic system from an optimization problem," *Adv. Math. Phys.*, vol. 2020, pp. 1–15, Jul. 2020.
[21] O. O. Okundalaye and W. A. M. Othman, "An approximate analytical solution of non-linear fractional-order constrained optimization problem using optimal homotopy analysis method," *Int. Trans. J. Eng. Manage. Appl. Sci. Tech.*, vol. 12, no. 1, pp. 1–14, 2020.
[22] O. O. Okundalaye and W. A. M. Othman, "A new optimal homotopy asymptotic method for fractional optimal control problems," *Int. J. Differ. Equ.*, vol. 2021, pp. 1–10, May 2021.
[23] S. Sarwar and M. M. Rashidi, "Approximate solution of two-term fractional-order diffusion, wave-diffusion, and telegraph models arising in mathematical physics using optimal homotopy asymptotic method," *Waves Random Complex Media*, vol. 26, no. 3, pp. 365–382, Jul. 2016.
[24] M. Yavuz and N. Özdemir, "New numerical techniques for solving fractional partial differential equations in conformable sense," in *Proc. Conf. Non-Integer Order Calculus Appl.*, 2017, pp. 49–62.
[25] J. Pérez, J. Gómez-Aguilar, D. Baleanu, and F. Tchier, "Chaotic attractors with fractional conformable derivatives in the Liouville–Caputo sense and its dynamical behaviors," *Entropy*, vol. 20, no. 5, p. 384, May 2018.
[26] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Adv. Difference Equ.*, vol. 2017, no. 1, pp. 1–16, Dec. 2017.
[27] D. Kumar, A. R. Seadawy, and A. K. Joardar, "Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology," *Chin. J. Phys.*, vol. 56, no. 1, pp. 75–85, Feb. 2018.
[28] A. Yokuş, "Comparison of Caputo and conformable derivatives for time-fractional Korteweg–de Vries equation via the finite difference method," *Int. J. Mod. Phys. B*, vol. 32, no. 29, Nov. 2018, Art. no. 1850365.
[29] M. Arfan, H. Alrabaiah, M. U. Rahman, Y.-L. Sun, A. S. Hashim, B. A. Pansera, A. Ahmadian, and S. Salahshour, "Investigation of fractal-fractional order model of COVID-19 in Pakistan under Atangana–Baleanu Caputo (ABC) derivative," *Results Phys.*, vol. 24, May 2021, Art. no. 104046.
[30] S. Salahshour, A. Ahmadian, and T. Allahviranloo, "A new fractional dynamic cobweb model based on nonsingular kernel derivatives," *Chaos, Solitons Fractals*, vol. 145, Apr. 2021, Art. no. 110755.
[31] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *J. Comput. Appl. Math.*, vol. 264, pp. 65–70, Jul. 2014.
[32] D. Avci, B. B. İ. Eroğlu, and N. Özdemir, "Conformable fractional wave-like equation on a radial symmetric plate," in *Theory and Applications of Non-Integer Order Systems*. Cham, Switzerland: Springer, 2017, pp. 137–146.
[33] M. Yavuz and N. Özdemir, "A different approach to the European option pricing model with new fractional operator," *Math. Model. Natural Phenomena*, vol. 13, no. 1, p. 12, 2018.
[34] M. Yavuz and N. Özdemir, "European vanilla option pricing model of fractional order without singular kernel," *Fractal Fractional*, vol. 2, no. 1, p. 3, 2018.
[35] B. Xin, W. Peng, Y. Kwon, and Y. Liu, "Modeling, discretization, and hyperchaos detection of conformable derivative approach to a financial system with market confidence and ethics risk," *Adv. Difference Equ.*, vol. 2019, no. 1, pp. 1–14, Dec. 2019.
[36] T. Abdeljawad, "On conformable fractional calculus," *J. Comput. Appl. Math.*, vol. 279, pp. 57–66, May 2015.
[37] Y. Qi and X. Wang, "Asymptotical stability analysis of conformable fractional systems," *J. Taibah Univ. Sci.*, vol. 14, no. 1, pp. 44–49, Jan. 2020.
[38] A. El-Ajou, "A modification to the conformable fractional calculus with some applications," *Alexandria Eng. J.*, vol. 59, no. 4, pp. 2239–2249, Aug. 2020.
[39] S. J. Liao, "The proposed homotopy analysis technique for the solution of nonlinear problems," M.S. thesis, School Math. Sci., Shanghai Jiao Tong Univ., Shanghai, China, 1992.
[40] P. Agarwal, M. Akbar, R. Nawaz, and M. Jleli, "Solutions of system of Volterra integro-differential equations using optimal homotopy asymptotic method," *Math. Methods Appl. Sci.*, vol. 44, no. 3, pp. 2671–2681, 2021.

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