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A New Study on the Fixed Point Sets of Proinov-Type Contractions via Rational Forms

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Abstract: In this paper, we presented some new weaker conditions on the Proinov-type contractions which guarantees that a self-mapping T has a unique fixed point in terms of rational forms. Our main results improved the conclusions provided by Andreea Fulga (On $(\psi,\varphi)-$Rational Contractions) in which the continuity assumption can either be reduced to orbital continuity, $k-$continuity, continuity of T^k, T-orbital lower semi-continuity or even it can be removed. Meanwhile, the assumption of monotonicity on auxiliary functions is also removed from our main results. Moreover, based on the obtained fixed point results and the property of symmetry, we propose several Proinov-type contractions for a pair of self-mappings (P,Q) which will ensure the existence of the unique common fixed point of a pair of self-mappings (P,Q). Finally, we obtained some results related to fixed figures such as fixed circles or fixed discs which are symmetrical under the effect of self mappings on metric spaces, we proposed some new types of $(\psi,\varphi)_c-$rational contractions and obtained the corresponding fixed figure theorems on metric spaces. Several examples are provided to indicate the validity of the results presented.

Keywords: fixed point; $(\psi,\varphi)-$rational-contraction; $(\psi,\varphi)_c-$rational-contraction; fixed circle; fixed disc

MSC: 47H10; 54H25

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1. Introduction

In recent decades, metric fixed point theory has always been a hot-topic in the field of mathematical analysis. Thousands of well-known results have been published since Banach [1] initiated the study of metric fixed point theory. Among those published results, many conclusions are either equivalent to or cover existing ones. Under this circumstance, it is necessary to examine the newly obtained results and make an equivalent classification. Undoubtedly, one of the most interesting and impressive results comes from Proinov's work (see, [2]). Proinov derived a self-mapping T on a complete metric space satisfying a general contraction of the form $\psi(d(Tx,Ty)) \leq \varphi(d(x,y))$ and stated some metric fixed point theorems that cover many of earlier results in this field of research. He also showed that the recently illustrated results of Wardowski [3] and Jleli-Samet [4] are in fact equivalent to the special cases of Skof's theorem [5].

One of main results provided by Proinov is stated as follows.

Theorem 1. *([2], Theorem 3.6) Let (X, d) be a metric space and $T : X \mapsto X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)),$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$, where $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ satisfy the following conditions:
(1) ψ is nondecreasing;
(2) $\varphi(s) < \psi(s)$ for any $s > 0$;
(3) $\lim\limits_{s \to s_0^+} \sup \varphi(s) < \psi(s_0+)$ for any $s_0 > 0$.
Then T admits a unique fixed point.

The main advantage of Proinov-type contraction is that it possess a wide family of auxiliary functions on which very weak constraints are imposed. Consequently, many mathematicians persist to study this class of contractions (see, [6]). Among the follow-up work, I.M. Olaru and N.A. Secelean [7] provided a new novel fixed point theorem for a new kind of (φ, ψ)-contraction in which the involved auxiliary functions φ, ψ satisfy certain weaker conditions. Additionally, they also demonstrated that the previous fixed point results due to Wardowski [3], Turinici [8], Piri and Kumam [9], Secelean [10] and Proinov [2] and others are consequences of their main result.

On the other hand, Andreea Fulga [11] observed that the concerns of Proinov [2] are valid for fixed point theorems via Proinov-type contraction involving rational forms in the context of complete metric spaces which also extended and unified some earlier results. In general, fixed point theory for rational contractions is also a vital research direction which has attracted much attention and produced a bundle of papers (see [12,13] and references therein).

In this presented paper, we will generalize the fixed point results for $(\psi, \varphi)-$rational contractions mentioned in [11] in which some weaker conditions than the ones presented in [7] are imposed on the auxiliary functions. We claim that the continuity assumption on T can be either reduced to orbital continuity, $k-$continuity, continuity of T^k, T-orbital lower semi-continuity or even be removed. Moreover, the assumption of monotonicity on auxiliary functions is removed from our main results. Finally, we will also propose some new types of $(\psi, \varphi)_c-$rational contractions and obtain some corresponding fixed circle and common fixed circle (resp. fixed disc and common fixed disc) theorems on metric spaces. Several examples are provided to indicate the validity of the results presented.

2. The Contractive Condition and a Class of Auxiliary Functions

We start by introducing a new family of auxiliary functions as follows.
Let us consider the pair of functions $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ satisfying the following two conditions:

(A_1): for every $s \geq t > 0$, one has $\psi(s) > \varphi(t)$;
(A_2): $\lim\limits_{s \to \epsilon^+} \psi(s) > \lim\limits_{s \to \epsilon^+} \sup \varphi(s) > 0$ for each $\epsilon > 0$ or,
(A_2'): $\lim\limits_{s \to \epsilon^+} \inf \psi(s) > \lim\limits_{s \to \epsilon^+} \sup \varphi(s) > 0$ for each $\epsilon > 0$.

Denote by $\mathcal{F}$ the family of all pairs of functions (ψ, φ) which satisfy conditions: (A_1) and (A_2) (or (A_2')).

It is easy to check that this family is nonempty, even when considering non-continuous functions. Here are some examples of pairs (ψ, φ) belonging to $\mathcal{F}$:

- $\psi(s) = s$ and $\varphi(s) = \lambda s$, $\lambda \in (0, 1)$.
- $\psi(s) = s$ and $\varphi(s) = s - \tau$, $\tau > 0$.
- $\psi(s) = e^s$ and $\varphi(s) = s + 1$.
- $\psi(s) = \ln(1 + s)$ and $\varphi(s) = k \ln(1 + s), k \in (0, 1)$.
- $\psi(s) = \begin{cases} 1, & s \in (0, 1], \\ 2s, & s \in (1, \infty), \end{cases}$ and $\varphi(s) = \begin{cases} \frac{s^2}{2}, & s \in (0, 1], \\ s, & s \in (1, \infty). \end{cases}$

Proposition 1. *The condition* (A_2) *or* (A_2') *implies the condition* (A_3) *stated in [7] as follows:*

$$(A_3): \quad \liminf_{s\to\epsilon^+}(\psi(s)-\varphi(s))>0 \quad \textit{for each} \quad \epsilon>0.$$

Proof. Suppose that (A_2) holds but (A_3) is false. Then there is $\epsilon_0>0$ such that the following holds:

$$\liminf_{s\to\epsilon_0^+}(\psi(s)-\varphi(s))\leq 0.$$

Let us define $\delta_0=-\liminf_{s\to\epsilon_0^+}(\psi(s)-\varphi(s))\geq 0$, that is, we are assuming the following:

$$\liminf_{s\to\epsilon_0^+}(\psi(s)-\varphi(s))=-\delta_0\leq 0.$$

As this limit inferior is $-\delta_0$, then there exists a sequence $\{s_n\}_{n\in\mathbb{N}}\subset(\epsilon_0,\infty)$ such that the following holds:

$$\begin{aligned}
&\epsilon_0<s_{n+1}<s_n \quad \text{for all} \quad n\in\mathbb{N},\\
&s_n\to\epsilon_0 \quad \text{and}\\
&\lim_{n\to\infty}(\psi(s_n)-\varphi(s_n))=-\delta_0\leq 0.
\end{aligned}$$

Since $s_n\to\epsilon_0^+$ and considering (A_2) holds, then the limit $\lim_{s\to\epsilon_0^+}\psi(s)$ exists, and it is equal to the following:

$$\lim_{s\to\epsilon_0^+}\psi(s)=\lim_{n\to\infty}\psi(s_n).$$

Then by taking limits in the following expression,

$$\varphi(s_n)=\psi(s_n)-(\psi(s_n)-\varphi(s_n)),$$

we deduce the following:

$$\begin{aligned}
\lim_{n\to\infty}\varphi(s_n)&=\lim_{n\to\infty}\psi(s_n)-(\psi(s_n)-\varphi(s_n))\\
&=\lim_{n\to\infty}\psi(s_n)-\lim_{n\to\infty}(\psi(s_n)-\varphi(s_n))\\
&=\lim_{s\to\epsilon_0^+}\psi(s)-(-\delta_0)\\
&=\lim_{s\to\epsilon_0^+}\psi(s)+\delta_0.
\end{aligned}$$

As a result, we have the following:

$$\limsup_{s\to\epsilon_0^+}\varphi(s)\geq\lim_{n\to\infty}\varphi(s_n)\geq\lim_{s\to\epsilon_0^+}\psi(s)+\delta_0\geq\lim_{s\to\epsilon_0^+}\psi(s),$$

which contradicts the condition (A_2).
Applying similar arguments again, with (A_2) replaced by (A_2'), the conclusion holds true. For brevity, we omit the rest of the proof. □

Next, we will show that condition (A_3) can be equivalently stated in an alternative way by using series of non-negative terms.

Lemma 1. *Let* $\psi,\varphi(0,\infty)\to\mathbb{R}$ *be two functions satisfying the following condition stated in [11]:*

(f_0) $\psi(s)>\varphi(s)$ *for any* $s\in(0,1)$.

Then the following conditions are equivalent:

(A_3) $\liminf_{s\to\epsilon^+}(\psi(s)-\varphi(s))>0$ *for any* $\epsilon\in(0,\infty)$;

(A_3') *for each non-increasing sequence* $\{t_n\}\subset(0,\infty)$ *such that* $\{t_n\}\to\epsilon^+\in(0,\infty)$ *the series of positive terms* $\sum_{n\geq 1}(\psi(t_n)-\varphi(t_n))$ *diverges;*

(A_3'') *for each strictly decreasing sequence* $\{t_n\}\subset(0,\infty)$ *such that* $\{t_n\}\to\epsilon^+\in(0,\infty)$ *the series of positive terms* $\sum_{n\geq 1}(\psi(t_n)-\varphi(t_n))$ *diverges.*

Proof. $[(A_3)\Rightarrow(A_3')]$ Let $\{t_n\}\subset(0,\infty)$ be a non-increasing sequence such that $\{t_n\}\to\epsilon^+\in(0,\infty)$. Let us consider the real number

$$\varepsilon_0=\liminf_{s\to\epsilon^+}(\psi(s)-\varphi(s))>0,$$

which is strictly positive by (A_3). Therefore,

$$0<\varepsilon_0<\liminf_{s\to\epsilon^+}(\psi(s)-\varphi(s))\leq\liminf_{n\to\infty}(\psi(t_n)-\varphi(t_n)).$$

Hence, the series of positive terms $\sum_{n\geq 1}(\psi(t_n)-\varphi(t_n))$ diverges.

$[(A_3')\Rightarrow(A_3'')]$ It is apparent.

$[(A_3'')\Rightarrow(A_3)]$ Reasoning by contradiction, suppose that there exists $\epsilon\in(0,\infty)$ such that

$$\liminf_{s\to\epsilon^+}(\psi(s)-\varphi(s))=0.$$

Then one can find a sequence $\{t_n\}\subset(0,\infty)$ such that:

$$\{t_n\}\to\epsilon^+,\quad t_n>\epsilon\quad\text{for all } n\in\mathbb{N},\quad\text{and}\quad\liminf_{n\to\infty}(\psi(t_n)-\varphi(t_n))=0.$$

Without loss of generality, we assume that $\{t_n\}$ is strictly decreasing. Then there exists $n_1\in\mathbb{N}$ such that:

$$\psi(t_{n_1})-\varphi(t_{n_1})<\frac{1}{2}.$$

Similarly, we can also find $n_2>n_1$ such that:

$$\psi(t_{n_2})-\varphi(t_{n_2})<\frac{1}{2^2}.$$

By induction, we can find a partial subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that:

$$\psi(t_{n_k})-\varphi(t_{n_k})<\frac{1}{2^k}\quad\text{for all } k\geq 1.$$

Thus, the series $\sum_{k\geq 1}(\psi(t_{n_k})-\varphi(t_{n_k}))$ converges and $\{t_{n_k}\}_{k\in\mathbb{N}}\to\epsilon$. This contradicts the condition (A_3''). □

Corollary 1. *If we replace the condition* (p_0) *in Lemma 1 by*

(A_1) *for every* $r\geq t>0$, *one has* $\psi(r)>\varphi(t)$;

then Lemma 1 remains true.

Proof. It follows from the fact that (A_1) implies (f_0) (use $r=t$). □

Proposition 2. *If $f : (0, \infty) \to \mathbb{R}$ is a function and $\{t_n\} \subset (0, \infty)$ is a non-increasing sequence such that $\{f(t_n)\} \to -\infty$, then there is $\alpha \in (0, \infty)$ and a partial subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ of $\{t_n\}_{n \in \mathbb{N}}$ such that:*

$$t_{n_k} > t_{n_{k+1}} > \alpha \quad \textit{for all } k \in \mathbb{N}, \quad \{t_{n_k}\}_{k \in \mathbb{N}} \to \alpha \quad \textit{and} \quad \{f(t_{n_k})\}_{k \in \mathbb{N}} \to -\infty.$$

Proof. Since $\{t_n\}$ is non-increasing and bounded from below, it is convergent. Let $\alpha \in (0, \infty)$ be its limit, that is, assume that $\{t_n\} \to \alpha$ and $t_n \geq t_{n+1} \geq \alpha$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $t_n = \alpha$ for all $n \geq n_0$, that is, the sequence $\{t_n\}$ is almost constant. However, this is impossible because $\{f(t_n)\} \to -\infty$. Therefore, $t_n > \alpha$ for all $n \in \mathbb{N}$. In such a case, the sequence $\{t_n\}$ has a strictly decreasing partial subsequence $\{t_{n_k}\}$ such that $t_{n_k} > t_{n_{k+1}} > \alpha$ for all $k \in \mathbb{N}$. As it is a partial subsequence of $\{t_n\}$, we conclude that $\{t_{n_k}\}_{k \in \mathbb{N}} \to \alpha$ and $\{f(t_{n_k})\}_{k \in \mathbb{N}} \to -\infty$. □

I.M. Olaru and N.A. Secelean [7] introduced a notion of the property (P) in order to ensure that the fixed point theory will be able to be developed under these conditions.

Definition 1 ([7])**.** *A mapping $\psi : (0, \infty) \mapsto \mathbb{R}$ is said to satisfy property (P) if, for every non-increasing sequence $\{t_n\}$ of positive numbers such that $\psi(t_n) \to -\infty$, then $\lim_{n \to \infty} t_n = 0$.*

We must clarify that when there is no any non-increasing sequence $\{t_n\} \subset (0, \infty)$ such that $\{f(t_n)\} \to -\infty$, we accept that the function f satisfies the property (P^*). Property (P^*) can be stated in a more convenient way for proving some results.

Proposition 3. *A function $f : (0, \infty) \to \mathbb{R}$ satisfies the property (P) if and only if there is a sequence $\{t_n\} \subset (0, \infty)$ converging to $\alpha \in (0, \infty)$ such that $t_n > t_{n+1} > \alpha > 0$ for all $n \in \mathbb{N}$ and $\{f(t_n)\} \to -\infty$, then $\alpha = 0$.*

Proof. The condition is clearly necessary. To prove that it is also sufficient, suppose that there is a non-increasing sequence $\{t_n\} \subset (0, \infty)$ such that $\{f(t_n)\} \to -\infty$. By Proposition 2, there is $\alpha \in (0, \infty)$ and a partial subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ of $\{t_n\}_{n \in \mathbb{N}}$ such that:

$$t_{n_k} > t_{n_{k+1}} > \alpha > 0 \quad \text{for all } k \in \mathbb{N}, \quad \{t_{n_k}\}_{k \in \mathbb{N}} \to \alpha \quad \text{and} \quad \{f(t_{n_k})\}_{k \in \mathbb{N}} \to -\infty.$$

Using the assumption, we deduce that $\alpha = 0$, so $t_n \to 0$ and f satisfies the property (P). □

The following lemma shows some examples of functions satisfying the property (P).

Lemma 2 ([7])**.** *Let $\psi : (0, \infty) \mapsto \mathbb{R}$ be a mapping and $\{t_n\}$ be a sequence of positive real numbers such that $\psi(t_n) \to -\infty$. If one of the following conditions holds:*
(i) ψ is nondecreasing;
(ii) ψ is right-continuous and $\{t_n\}$ is non-increasing;
(iii) ψ is lower semi-continuous and $\{t_n\}$ is non-increasing.
Then $\lim_{n \to \infty} t_n = 0$.

Remark 1. *Lemma 2 gives some classes of mappings satisfying property (P). However, there exist mappings satisfying property (P), but which do not satisfy any of the conditions of Lemma 2. For more details, we refer the readers to Example 3 in [7].*

Next, we present the following elementary definitions and lemmas which will be used in the sequel.

Definition 2 ([14])**.** *A mapping T on a metric space (X, d) is said to be orbitally continuous if, for any sequence $\{y_n\}$ in $O_x(T)$, $y_n \to u$ implies $Ty_n \to Tu$ as $n \to +\infty$, where $O_x(T) = \{T^n x : n \geq 0\}$ is the orbit of T at x.*

It is easy to observe that a continuous mapping is orbitally continuous, but not conversely.

Definition 3 ([15]). *A self-mapping T of a metric space (X, d) is called $k-$continuous, $k = 1, 2, 3, \dots$, if $T^k x_n \to Tt$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1} x_n \to t$.*

Remark 2. *It was shown in [15] that the continuity of T^k and k-continuity of T are independent conditions when $k > 1$ It is also easy to see that $1-$continuity is equivalent to continuity and continuity $\Rightarrow 2-$continuity $\Rightarrow 3-$continuity $\Rightarrow \dots$, but not conversely.*

Definition 4 ([16]). *Let (X, d) be a metric space and $T : X \mapsto X$. A mapping $f : X \mapsto \mathbb{R}$ is said to be T-orbitally lower semi-continuous at $z \in X$ if $\{x_n\}$ is a sequence in $O_x(T)$ for some $x \in X$, $\lim\limits_{n\to\infty} x_n = z$ implies $f(z) \le \lim\limits_{n\to\infty} \inf f(x_n)$.*

Proposition 4 ([17]). *Let (X, d) be a metric space, $T : X \mapsto X$ and $z \in X$. If T is orbitally continuous at z or T is k-continuous at z for some $k \neq 1$, then the function $f(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at z.*

It is noted that the T-orbital lower semi-continuity of $f(x) = d(x, Tx)$ is weaker than both orbital continuity and k-continuity of T (see Example 1 in [17]).

Lemma 3 ([2]). *Let $\psi : (0, \infty) \mapsto \mathbb{R}$. Then the following conditions are equivalent:*
(i) $\inf\limits_{t>\varepsilon} \psi(t) > -\infty$ for every $\varepsilon > 0$;
(ii) $\lim\limits_{t\to\varepsilon^+} \psi(t) > -\infty$ for every $\varepsilon > 0$;
(iii) $\lim\limits_{n\to\infty} \psi(t_n) = -\infty$ implies $\lim\limits_{n\to\infty} t_n = 0$.

Lemma 4 ([18]). *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X which is not Cauchy and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

$$\lim_{n\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon + .$$

3. Fixed Point Results for (ψ, φ)-Rational Contractions

3.1. New Fixed Point Results

To begin with we first present several types of $(\psi, \varphi)-$rational contractions and provide the existence of the unique fixed point for such contractions.

Definition 5 ([11]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is a (ψ, φ)-rational contraction type A if, for every distinct $x, y \in X$ such that $d(Tx, Ty) > 0$, the following inequality*

$$\psi(d(Tx, Ty)) \le \varphi(M_1(x, y)) \tag{1}$$

holds, where $M_1(x, y)$ is defined by

$$M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)}\}$$

and $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings.

Theorem 2. *Let (X, d) be a complete metric space and ψ, φ be two functions such that $(\psi, \varphi) \in \mathcal{F}$. Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type A such that either T is orbitally continuous or $k-$continuous or T^k is continuous for some integer $k > 1$ or $x \mapsto d(x, Tx)$ is $T-$orbitally lower semi-continuous. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Proof. First of all, we notice that from condition (A_1) and (1), we can deduce that T satisfies

$$d(Tx,Ty) < M_1(x,y), \quad \forall x,y \in X, \quad Tx \neq Ty. \tag{2}$$

To show the existence of the unique fixed point of T, let us start with an arbitrary $x_0 \in X$. We define the sequence $\{x_n\}$ by $T^n x_0 = x_n$, for all $n \in \mathbb{N} \cup \{0\}$ and denote $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Indeed, on the contrary, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T.
So, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $d_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Under this consideration, for $x = x_{n-1}, y = x_n$, we have:

$$\begin{aligned} M_1(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), \\ &\quad \frac{d(x_{n-1},Tx_{n-1}) \cdot d(x_n,Tx_n)}{d(x_{n-1},x_n)}\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n) \cdot d(x_n,x_{n+1})}{d(x_{n-1},x_n)}\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

It follows from (2) that

$$d(Tx_{n-1},Tx_n) = d(x_n,x_{n+1}) < M_1(x_{n-1},x_n) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\},$$

which leads to $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$.
Hence, the sequence $\{d_n\}$ is decreasing and there exists $d \geq 0$ such that $\lim\limits_{n\to\infty} d_n = d$ and $d_n > d$ for all $n \in \mathbb{N} \cup \{0\}$.
Next, we will prove that $d = 0$.
Supposing that $d > 0$, from (1) and condition (A_1), we have:

$$\psi(d_n) \leq \varphi(d_{n-1}) \leq \psi(d_{n-1}), \quad \text{for all} \quad n \in \mathbb{N}.$$

It follows that the sequence $\{\psi(d_n)\}$ is strictly decreasing and since it is bounded below, we can conclude that $\{\psi(d_n)\}$ is a convergent sequence and so is the sequence $\{\varphi(d_{n-1})\}$.
Thus, keeping in mind condition (A_2) or (A_2'), we have

$$\lim_{s\to d^+} \psi(s) = \lim_{n\to\infty} \psi(d_n) = \lim_{n\to\infty} \varphi(d_n) \leq \limsup_{n\to\infty} \varphi(d_n) \leq \limsup_{s\to d^+} \varphi(s).$$

or

$$\liminf_{s\to d^+} \psi(s) \leq \lim_{s\to d^+} \psi(s) = \lim_{n\to\infty} \psi(d_n) = \lim_{n\to\infty} \varphi(d_n) \leq \limsup_{n\to\infty} \varphi(d_n) \leq \limsup_{s\to d^+} \varphi(s).$$

which leads to a contradiction to (A_2) or (A_2'). Therefore, $d = 0$ and

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3}$$

Now, we aim to prove that $\{x_n\}$ is a Cauchy sequence.
Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. From Lemma 4, one can find $\varepsilon \in (0,\infty)$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that:

$$\lim_{k\to\infty} d(x_{m_k+1},x_{n_k+1}) = \lim_{k\to\infty} d(x_{m_k},x_{n_k}) = \varepsilon+, \tag{4}$$

with $d(x_{m_k+1},x_{n_k+1}) > \varepsilon$, for all $k > 1$.
Hence, from (1), we have:

$$\psi(d(x_{m_k+1},x_{n_k+1})) \leq \varphi(M_1(x_{m_k},x_{n_k})) \leq \psi(M_1(x_{m_k},x_{n_k})),$$

where

$$
\begin{aligned}
M_1(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{m_k}, Tx_{m_k}) \cdot d(x_{n_k}, Tx_{n_k})}{d(x_{m_k}, x_{n_k})}\} \\
&= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k}, x_{m_k+1}) \cdot d(x_{n_k}, x_{n_k+1})}{d(x_{m_k}, x_{n_k})}\}.
\end{aligned}
$$

From (3),(4), we have $\lim_{k\to\infty} M_1(x_{m_k}, x_{n_k}) = \varepsilon+$ and it follows from (2) that:

$$
\begin{aligned}
\liminf_{s\to\varepsilon^+} \psi(s) &\le \liminf_{k\to\infty} \psi(d(x_{m_k+1}, x_{n_k+1})) \\
&\le \lim_{k\to\infty} \psi(d(x_{m_k+1}, x_{n_k+1})) \\
&= \lim_{s\to\varepsilon^+} \psi(s) \\
&\le \limsup_{k\to\infty} \varphi(M_1(x_{m_k}, x_{n_k})) \\
&\le \limsup_{s\to\varepsilon^+} \varphi(s).
\end{aligned}
$$

This contradicts conditions (A_2) and (A_2'), then $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$, that is T is a Picard operator.
Suppose that T is orbitally continuous. Since $\{x_n\}$ converges to x^*, the orbital continuity implies that $Tx_n \to Tx^*$. This yields $Tx^* = x^*$. Therefore, x^* is a fixed point of T.
Suppose that T is $k-$continuous for some integer $k > 1$. Since $T^{k-1}x_n \to x^*$, the $k-$continuity of T implies that $T^k x_n \to Tx^*$. Hence, $x^* = Tx^*$ as $T^k x_n \to x^*$. Therefore, x^* is a fixed point of T.
Suppose that T^k is continuous for some integer $k > 1$, then $\lim_{n\to\infty} T^k x_n = T^k x^*$. This yields $T^k x^* = x^*$ as $x_n \to T^k x^*$, that is x^* is a fixed point of T^k.
If we assume that $Tx^* \neq x^*$, we have for any $j = 0, 1, 2, \dots k-1$ that $T^{k-j-1}x^* \neq T^{k-j}x^*$. Taking $x = T^{k-j-1}x^*, y = T^{k-j}x^*$ in $M_1(x, y)$, we have:

$$
\begin{aligned}
M_1(T^{k-j-1}x^*, T^{k-j}x^*) &= \max\{d(T^{k-j-1}x^*, T^{k-j}x^*), d(T^{k-j}x^*, T^{k-j+1}x^*), \\
&\qquad \frac{d(T^{k-j-1}x^*, T^{k-j}x^*) \cdot d(T^{k-j}x^*, T^{k-j+1}x^*)}{d(T^{k-j-1}x^*, T^{k-j}x^*)}\} \\
&= \max\{d(T^{k-j-1}x^*, T^{k-j}x^*), d(T^{k-j}x^*, T^{k-j+1}x^*)\}.
\end{aligned}
$$

From (2), we have

$$
\begin{aligned}
d(T^{k-j}x^*, T^{k-j+1}x^*) &< M_1(T^{k-j-1}x^*, T^{k-j}x^*) \\
&= \max\{d(T^{k-j-1}x^*, T^{k-j}x^*), d(T^{k-j}x^*, T^{k-j+1}x^*)\},
\end{aligned}
$$

which leads to:

$$
d(T^{k-j}x^*, T^{k-j+1}x^*) < d(T^{k-m-1}x^*, T^{k-m}x^*),
$$

for every $m = j, j+1, \dots k-1$. Taking in the above inequality $j = 0$ and $m = k-1$, we have:

$$
d(x^*, Tx^*) = d(T^k x^*, T^{k+1}x^*) < d(x^*, Tx^*),
$$

which is a contradiction. Consequently, $Tx^* = x^*$, and x^* is a fixed point of T.
Furthermore, if $x \mapsto d(x, Tx)$ is $T-$orbitally lower semi-continuous, then we have:

$$d(x^*, Tx^*) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

which implies that $Tx^* = x^*$ and x^* is a fixed point of T.
For the uniqueness of the fixed point, we may assume that there exists another fixed point of T, $x' \in X$ such that $x^* \neq x'$. Since $d(Tx^*, Tx') > 0$, from (1), we have:

$$\begin{aligned}
\psi(d(x^*, x')) &= \psi(d(Tx^*, Tx') \\
&\leq \varphi(M_1(x^*, x')) \\
&= \max\{d(x^*, x'), d(x^*, Tx^*), d(x', Tx'), \frac{d(x^*, Tx^*) \cdot d(x', Tx')}{d(x*, x')}\} \\
&= \max\{d(x^*, x'), 0\} \\
&= d(x^*, x').
\end{aligned}$$

From (2), we have

$$d(x^*, x') < M_1(x^*, x') = d(x^*, x'),$$

which is a contradiction. Hence, $x^* = x'$. □

Replacing $M_1(x, y)$ by $M_1'(x, y)$ in the following theorem, we can deduce that the conclusion of Theorem 2 also remains true without any continuity assumption.

Theorem 3. *Let (X, d) be a complete metric space and ψ, φ be two functions such that $(\psi, \varphi) \in \mathcal{F}$. Suppose that $T : X \mapsto X$ satisfies that*

$$\psi(d(Tx, Ty)) \leq \varphi(M_1'(x, y)), \quad \forall x, y \in X, \quad Tx \neq Ty,$$

where $M_1'(x, y)$ is defined by

$$M_1'(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) \cdot d(y, Ty)}{d(Tx, Ty)}\}.$$

Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Applying arguments similar to the proof in the front part of Theorem 2, we can also obtain that the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ is a Picard sequence.
From Proposition 2.3 in [19], we can conclude that the Pciard sequence $\{x_n\}$ mentioned above is infinite, that is, $x_m \neq x_n$ for all $m \neq n$.
To prove x^* is a fixed point of T, assume, by contradiction, that $x^* \neq Tx^*$. Since the sequence $\{x_n\}$ is infinite, there exists $n_0 \in \mathbb{N}$ such that $x_n \neq x^*$ and $x_n \neq Tx^*$ for all $n \geq n_0$.
By (1), we have:

$$\psi(d(Tx_n, Tx^*)) \leq \varphi(d(M_1'(x_n, x^*)),$$

which implies that

$$
\begin{aligned}
d(Tx_n, Tx^*) < M_1'(x_n, x^*) \\
&= \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \\
&\quad \frac{d(x_n, Tx_n) \cdot d(x^*, Tx^*)}{d(Tx_n, Tx^*)}\} \\
&= \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \\
&\quad \frac{d(x_n, x_{n+1}) \cdot d(x^*, Tx^*)}{d(x_{n+1}, Tx^*)}\}.
\end{aligned}
$$

So, we have:

$$
d(Tx_n, Tx^*) < M_1'(x_n, x^*) = d(x^*, Tx^*) \quad \text{for } n \text{ sufficiently large.}
$$

Furthermore, then:

$$
\psi(d(Tx_n, Tx^*)) \leq \varphi(d(x^*, Tx^*)) \leq \psi(d(x^*, Tx^*)) \quad \text{for } n \text{ sufficiently large.}
$$

Taking limits in the above inequality as $n \to \infty$, we have:

$$
\psi(d(x^*, Tx^*)) < \psi(d(x^*, Tx^*)),
$$

which is a contradiction. Hence, $x^* = Tx^*$, that is x^* is a fixed point of T.
Using the same arguments as in Theorem 2, we have that this fixed point is unique. For brevity, we omit the rest of the proof. □

Remark 3. *Comparing the assumptions of Theorem 2 (or Theorem 3) and Theorem 4 in [11], we can find that Theorem 2 (or Theorem 3) weakens the conditions of Theorem 4 in the following aspects: (i) (f_0): $\varphi(s) < \psi(s)$ for any $s > 0$ is reduced to a certain weak form stated as condition (A_1)(Since (A_1) implies (f_0) by taking $s = t$). (ii) Condition (f_1) can be removed; (iii) Continuity assumption can be reduced to orbital continuity, $k-$continuity, continuity of T^k, T-orbital lower semi-continuity (or be removed).*

In the following result, we explore the property (P).

Theorem 4. *Let (X, d) be a complete metric space and ψ, φ be two functions satisfying the following conditions:*

(A_1): for every $s \geq t > 0$, one has $\psi(s) > \varphi(t)$;
(A_3): $\liminf_{s \to \epsilon^+}(\psi(s) - \varphi(s)) > 0$ for every $\epsilon > 0$;
(A_4): at least one of the functions of the pair (ψ, φ) satisfies property (P).

Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type A such that either T is orbitally continuous or $k-$continuous or T^k is continuous for some integer $k > 1$ or $x \mapsto d(x, Tx)$ is $T-$orbitally lower semi-continuous. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Using the discussion similar to the proof in the front part of Theorem 2, we can also obtain two sequences $\{x_n\}$, $\{d_n\}$ defined by $x_n = T^n x_0$ and $d_n = d(x_n, x_{n+1})$. Moreover, the sequence $\{d_n\}$ is decreasing and convergent to some $d > 0$ that is $\lim_{n \to \infty} d_n = d > 0$.
To prove that $d = 0$, from (1), we have:

$$
\psi(d_n) \leq \varphi(d_{n-1}), \quad \text{for all} \quad n \in \mathbb{N}.
$$

Further, we also have:

$$
\psi(d_n) - \psi(d_{n-1}) \leq \varphi(d_{n-1}) - \psi(d_{n-1}), \quad \text{for all} \quad n \in \mathbb{N}.
$$

Therefore,

$$\sum_{k=1}^{n}(\psi(d_k)-\psi(d_{k-1}))\leq\sum_{k=1}^{n}(\varphi(d_{k-1})-\psi(d_{k-1})).$$

So,

$$\psi(d_n)\leq\psi(d_0)+\sum_{k=1}^{n}(\varphi(d_{k-1})-\psi(d_{k-1})).$$

From Lemma 1, it follows that $\lim\limits_{n\to\infty}\psi(d_n)=-\infty$.
Since $d_n<d_{n-1}$, we deduce from condition (A_1) that $\varphi(d_n)<\psi(d_{n-1})$ for all $n\in\mathbb{N}$, hence, $\lim\limits_{n\to\infty}\varphi(d_n)=-\infty$. Therefore, $\lim\limits_{n\to\infty}d_n=0$.
Continuing the proof along the line of the proof of Theorem 2, we can demonstrate that T admits a unique fixed point and the iterative sequence $\{T^n x\}$ converges to x^* for every $x\in X$. For brevity, we omit the rest of the proof. □

Similarly, replacing $M_1(x,y)$ by $M'_1(x,y)$, we can deduce that the conclusion of Theorem 4 also remains true without any continuity assumption.

Theorem 5. *Let (X,d) be a complete metric space and ψ,φ be two functions satisfying the following conditions:*
(A_1): for every $s\geq t>0$, one has $\psi(s)>\varphi(t)$;
(A_3): $\liminf\limits_{s\to\epsilon^+}(\psi(s)-\varphi(s))>0$ for every $\epsilon>0$;
(A_4): at least one of the functions of the pair (ψ,φ) satisfies property (P).
Suppose that $T:X\mapsto X$ satisfy that

$$\psi(d(Tx,Ty))\leq\varphi(M'_1(x,y)),\quad\forall x,y\in X,\quad Tx\neq Ty,$$

where $M'_1(x,y)$ is defined by

$$M'_1(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)\cdot d(y,Ty)}{d(Tx,Ty)}\}.$$

Then T admits a unique fixed point $x^*\in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x\in X$.

Proof. The conclusion can be immediately drawn by combing the proofs of Theorem 3 and Theorem 4. □

Here are two examples to support Theorems 2 and 4.

Example 1. *Let $X=[0,1]$ be endowed with the usual distance $d(x,y)=|x-y|$ for every $x,y\in X$ and $T:X\mapsto X$ be given by $Tx=\frac{-x^2+2x+2}{6}$ and two functions $\psi,\varphi:(0,\infty)\mapsto\mathbb{N}$, $\psi(s)=\frac{s}{2}$ and $\varphi(s)=\frac{s}{4}$. A trivial verification shows that conditions $(A_1)-(A_2)$ are satisfied. It remains to check that T is a $(\psi,\varphi)-$rational contraction type A. Since*

$$d(Tx,Ty)=|\frac{-x^2+2x+2}{6}-\frac{-y^2+2y+2}{6}|=\frac{1}{6}|(x-y)(-x-y+2)|=\frac{1}{6}|(x-y)||(-x-y+2)|,$$

and $|-x-y+2|<3$ for every $x,y\in[0,1]$, we have

$$\psi(d(Tx,Ty)=\frac{1}{12}|(x-y||-x-y+2|\leq\frac{1}{4}|x-y|=\frac{1}{4}d(x,y)\leq\frac{1}{4}M_1(x,y),$$

which shows that T is a $(\psi,\varphi)-$rational contraction type A. Hence, Theorem 2 guarantees that T is a Picard operator its unique fixed point being $x=\sqrt{6}-2$.

Example 2. *Let $X = [0,2]$ be endowed with the usual distance $d(x,y) = |x-y|$ for every $x,y \in X$ and $T : X \mapsto X$ be given by $T(x) = \begin{cases} \frac{x^2}{x+1}, & if \quad x \in [0,1] \\ \frac{3}{4}, & if \quad x \in (1,2] \end{cases}$. It is clear that T is not continuous. Let us consider $\psi, \varphi : (0,\infty) \mapsto \mathbb{R}$ defined by $\psi(t) = \ln t$, $\varphi(t) = \ln \frac{t}{1+t}$ for all $t > 0$. It is easy to verify that the pair (φ, ψ) satisfies conditions $(A_1), (A_3), (A_4)$.*
We need to check whether T satisfies the contraction condition or not.
Let $x, y \in X$ be such that $Tx \neq Ty$. Then $x \neq y$, say $x < y$. The following cases can occur:
Case 1. $0 \leq x < y \leq 1$. Then

$$\begin{aligned} d(Tx,Ty)) &= \frac{(y-x)(x+y+xy)}{1+x+y+xy}. \\ M_1'(x,y) &= \{y-x, \frac{x}{1+x}, \frac{y}{1+y}, \frac{xy}{(y-x)(x+y+xy)}\}. \end{aligned}$$

Let $\alpha = \frac{xy}{(y-x)(x+y+xy)}$. Then

$$\begin{aligned} \frac{\alpha}{1+\alpha} &= \frac{xy}{xy+(y-x)(x+y+xy)} \\ &\geq \frac{xy}{(y-x)(x+y+xy)} \\ &> d(Tx,Ty). \quad (Since\ \frac{xy}{(y-x)(x+y+xy)} < \frac{1}{d(Tx,Ty)}) \end{aligned}$$

Hence, we have $\psi(d(Tx,Ty)) \leq \varphi(\alpha) \leq \varphi(M_1'(x,y))$.
Case 2. $0 \leq x < 1 < y \leq 3$. Then

$$\begin{aligned} d(Tx,Ty) &= \frac{4x^2-3x-3}{4(1+x)} < \frac{4x^2-3x}{4(1+x)} < \frac{x}{4(1+x)}. \\ M_1'(x,y) &= \{y-x, \frac{x}{1+x}, y-\frac{1}{4}, |\frac{4x^2y-3x^2}{4x^2+3x-4xy-4y+3}|\} \end{aligned}$$

Let $\beta = \frac{x}{1+x}$. Then

$$\frac{\beta}{1+\beta} = \frac{x}{1+2x} > \frac{x}{2(1+x)}.$$

Hence, we have $\psi(d(Tx,Ty)) \leq \varphi(\beta) \leq \varphi(M_1'(x,y))$. Therefore, the contraction condition is fulfilled. Theorem 4 shows that T is a Picard operator and 0 is a unique fixed point of T.

Next, we will proceed to introduce the notions of other types of rational contractions and present the corresponding fixed point theorems as follows.

Definition 6 ([11])**.** *Let (X,d) be a metric space. A mapping $T : X \to X$ is a (ψ,φ)-rational contraction type B if, for every distinct $x, y \in X$ such that $d(Tx,Ty) > 0$, the following inequality is satisfied:*

$$\psi(d(Tx,Ty)) \leq \varphi(M_2(x,y)), \tag{5}$$

where $M_2(x,y)$ is defined by

$$M_2(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Ty)\cdot(1+d(x,Tx))}{1+d(x,y)}\}$$

and $\psi, \varphi : (0,\infty) \mapsto \mathbb{R}$ are two mappings.

Theorem 6. *Let (X, d) be a complete metric space and ψ, φ be two mappings such that $(\psi, \varphi) \in \mathcal{F}$. Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type B. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Proof. Our proof starts with the observation from condition (A_1) and (5) that T satisfies

$$d(Tx, Ty) < M_2(x, y), \quad \forall x, y \in X, \quad Tx \neq Ty. \tag{6}$$

Fix an arbitrary $x_0 \in X$ and define the sequence $\{x_n\}$ by $T^n x_0 = x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Applying analysis similar to that in the proof of Theorem 2, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then $d_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Taking $x = x_{n-1}, y = x_n$ in $M_2(x, y)$, we have

$$\begin{aligned} M_2(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\qquad \frac{d(x_n, Tx_n) \cdot (d(x_{n-1}, Tx_{n-1}) + 1)}{1 + d(x_{n-1}, x_n)}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) \cdot (d(x_{n-1}, x_n) + 1)}{1 + d(x_{n-1}, x_n)}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Consequently, from (5), we have:

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) < M_2(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

which leads to $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$.
Hence, the sequence $\{d_n\}$ is decreasing and there exists $d \geq 0$ such that $\lim_{n\to\infty} d_n = d$.
Proceeding the same arguments as in the proof of Theorem 2, we can obtain that is T is a Picard operator, that is, the iterative sequence $\{T^n x_0\}$ converges to $x^* \in X$.
Next, we will claim that x^* is a fixed point of T.
Conversely, suppose that $d(Tx^*, x^*) > 0$. Since $x_n \to x^*$ as $n \to \infty$, we can find $n_0 \in \mathbb{N}$ such that $d(Tx^*, x_{n+1}) = d(Tx^*, Tx_n) > 0$ for all $n \geq n_0$.
Now, by (5), for $n \geq n_0$, we have:

$$\psi(d(Tx^*, Tx_n)) \leq \varphi(M_2(x^*, x_n)), \tag{7}$$

where,

$$M_2(x^*, x_n) = \max\{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) \cdot (1 + d(x^*, Tx^*))}{1 + d(x^*, x_n)}\}.$$

Further, from condition (A_1) and (7), we have:

$$d(Tx^*, Tx_n) < M_2(x^*, x_n).$$

So, we have:

$$d(Tx_n, Tx^*) < M_2(x_n, x^*) = d(x^*, Tx^*) \quad \text{for } n \text{ sufficiently large},$$

and then

$$\psi(d(Tx_n, Tx^*)) \leq \varphi(d(x^*, Tx^*)) < \psi(d(x^*, Tx^*)) \quad \text{for } n \text{ sufficiently large}.$$

Taking limits in the above inequality as $n \to \infty$, we have:

$$\psi(d(x^*, Tx^*)) < \psi(d(x^*, Tx^*)),$$

which is a contradiction. Hence, $Tx^* = x^*$.
The uniqueness of the fixed point of T can be obtained following with the lines of the proof in Theorem 2. For brevity, we omit the rest of the arguments. □

Theorem 7. *Let (X, d) be a complete metric space and ψ, φ be two functions satisfying the following conditions:*

(A_1): for every $s \geq t > 0$, one has $\psi(s) > \varphi(t)$;

(A_3): $\liminf\limits_{s \to \epsilon^+}(\psi(s) - \varphi(s)) > 0$ for every $\epsilon > 0$;

(A_4): at least one of the functions of the pair (ψ, φ) satisfies property (P).

Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type B. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. This conclusion can be obtained by applying a demonstration similar to the proof in Theorem 4 with (ψ, φ)-rational contraction type A being replaced by (ψ, φ)-rational contraction type B. □

Here is a example to support the validity of Theorem 7.

Example 3. *Let $X = [0, 1]$ and d be the usual distance on X. Let $T : X \mapsto X$ defined by $Tx = \frac{x+1}{2}$ and $\varphi, \psi : (0, \infty) \mapsto \mathbb{R}$ defined by $\psi(t) = \ln t$ and $\varphi(t) = \ln \frac{t}{1+t}$ for $t > 0$.*
We first check that (φ, ψ) satisfies conditions (A_1), (A_3) and (A_4).
(A_1) Let $r \geq t > 0$. Then

$$\psi(r) = \ln r \geq \ln t > \ln \frac{t}{1+t} = \varphi(t).$$

(A_3) Let $t > 0$, then $\lim\limits_{s \to t+} \inf(\psi(s) - \varphi(s)) = \ln(1 + t) > 0$.
Moreover, ψ also satisfies property (P).
Next, it remains to check that T is a (φ, ψ)-contraction type (B).
Indeed, if $x < y$ (and it is analogues for the case $x > y$), Then

$$d(Tx, Ty) = \frac{y - x}{2} \leq \frac{y - x}{1 + y - x}$$

Thus, we have:

$$\begin{aligned}
\psi(d(Tx, Ty)) &= \ln \frac{y - x}{2} \\
&\leq \ln \frac{y - x}{1 + y - x} \\
&= \ln \frac{d(x, y)}{1 + d(x, y)} \\
&= \varphi(d(x, y)) \\
&\leq \varphi(M_2(x, y)).
\end{aligned}$$

Therefore, T satisfies the contraction condition and by Theorem 7, we have that T has a unique fixed point 1.

Now, we will claim in the following corollaries that Theorems 6 and 7 in [11] can also be deduced from Theorem 7.

Corollary 2. *(Theorem 6, [11]) Let (X, d) be a complete metric space and $T : X \mapsto X$ be a $(\psi, \varphi)-$rational contraction type B. Assume that:*

(f_0) $\varphi(s) < \psi(s)$, for all $s > 0$;

(f_1') ψ is non-decreasing and lower semi-continuous;

(f_4) $\lim\limits_{s\to s_0^+} \sup \varphi(s) < \psi(s_0+)$.
Then T admits exactly one fixed point.

Proof. Repeating the previous arguments in Theorem 2, we get a decreasing positive sequence $\{d_n\}$ such that $\lim\limits_{n\to\infty} \psi(d_n) = -\infty$.
From the lower semi-continuity of ψ and Lemma 2, we can conclude that ψ satisfies property (P).
From assumption (f_0) and monotonicity of ψ, we also obtain that for some $r \geq t > 0$,

$$\psi(r) \geq \psi(t) > \varphi(t).$$

This yields that (ψ, φ) satisfies condition (A_1).
Additionally, due to the monotonicity of ψ, we have that $\lim\limits_{s\searrow s_0} \psi(s) = \lim\limits_{s\searrow s_0} \inf \psi(s)$ for any $s_0 > 0$.
Thus, using (f_4), we have:

$$\begin{aligned}\lim_{s\searrow s_0} \inf(\psi(s) - \varphi(s)) &\geq \lim_{s\searrow s_0} \inf \psi(s) - \lim_{s\searrow s_0} \sup \varphi(s)\\ &> \lim_{s\searrow s_0} \inf \psi(s) - \lim_{s\searrow s_0} \psi(s)\\ &= 0.\end{aligned}$$

Hence, (ψ, φ) satisfies condition (A_3).
Therefore, Theorem 7 can guarantee the validity of Corollary 2. □

Corollary 3. *(Theorem 7, [11]) A $(\psi, \varphi)-$rational contraction type B on the complete metric space (X, d) has a unique fixed point presuming that the following conditions are satisfied:*
(f_0) *$\varphi(s) < \psi(s)$, for all $s > 0$;*
(f_1) *$\lim\limits_{s>s_0} \psi(s) > -\infty$, for any $s_0 > 0$;*
(f_4) *$\lim\limits_{s\to s_0^+} \sup \varphi(s) < \lim\limits_{s\to s_0} \inf \psi(s)$;*
(f_5) *$\varphi(s_0) < \lim\limits_{s\to s_0} \inf \psi(s)$ for any $s_0 > 0$.*

Proof. Firstly, from Lemma 3 and condition (f_1), we can obtain that ψ satisfies property (P). In addition, from condition (f_4), we have:

$$\lim_{s\to s_0^+} \sup \varphi(s) < \lim_{s\to s_0} \inf \psi(s) \leq \lim_{s\to s_0^+} \inf \psi(s),$$

which together with Proposition 1 implies that condition (A_3) holds true.
Moreover, it is easy to check that conditions (f_0) and (f_5) imply condition (A_1). Therefore, Theorem 7 can guarantee the validity of Corollary 3. □

Definition 7 ([11])**.** *Let (X, d) be a metric space. A mapping $T : X \to X$ is a (ψ, φ)-rational contraction type C if, for every distinct $x, y \in X$ when $\max\{d(x, Ty), d(y, Tx)\} \neq 0$, then $d(Tx, Ty) > 0$ and the following condition is satisfied:*

$$\psi(d(Tx, Ty)) \leq \varphi(\frac{d(x, Tx) \cdot d(x, Ty) + d(y, Ty) \cdot d(y, Tx)}{\max\{d(x, Ty), d(y, Tx)\}}); \tag{8}$$

if $\max\{d(x, Ty), d(y, Tx)\} = 0$, then $d(Tx, Ty) = 0$ and $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings.

Theorem 8. *Let (X, d) be a complete metric space and ψ, φ be two functions such that $(\psi, \varphi) \in \mathcal{F}$. Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type C. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Proof. It follows from condition (A_1) and (8) that T satisfies

$$d(Tx, Ty) < \frac{d(x, Tx) \cdot d(x, Ty) + d(y, Ty) \cdot d(y, Tx)}{\max\{d(x, Ty), d(y, Tx)\}}, \tag{9}$$

for $\forall x, y \in X$, when $\max\{d(x, Ty), d(y, Tx)\} \neq 0$.
Let $x_0 \in X$ be fixed. Define the sequence $\{x_n\}$ by $T^n x_0 = x_n$ for all $n \in \mathbb{N} \cup \{0\}$ and denote $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, by similar reasoning, we have $x_n \neq x_{n+1}, d_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Therefore, since $d(x_n, x_{n+1}) > 0$ for every $n \in \mathbb{N} \cup \{0\}$, taking $x = x_{n-1}, y = x_n$ in (8), we have:

$$\begin{aligned}
\psi(d(Tx_{n-1}, Tx_n)) &\leq \varphi\left(\frac{d(x_{n-1}, Tx_{n-1}) \cdot d(x_{n-1}, Tx_n) + d(x_n, Tx_n) \cdot d(x_n, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}}\right) \\
&= \varphi\left(\frac{d(x_{n-1}, x_n) \cdot d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1}) \cdot d(x_n, x_n)}{\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}}\right) \\
&= \varphi(d(x_{n-1}, x_n)).
\end{aligned}$$

Consequently, by (9), we have:

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Hence, the sequence $\{d_n\}$ is decreasing and there exists $d \geq 0$ such that $\lim_{n\to\infty} \psi(d_n) = -\infty$.
Analysis similar to that in the proof of Theorem 2 shows that $d = 0$.
The rest of the proof to show that T is a Picard operator and T admits a unique fixed point $x^* \in X$ can run as the discussion in the proof of Theorem 2 and Theorem 3. We omit it for brevity. □

Theorem 9. *Let (X, d) be a complete metric space and ψ, φ be two functions satisfying the following conditions:*

(A_1): for every $s \geq t > 0$, one has $\psi(s) > \varphi(t)$;
(A_3): $\liminf_{s\to\epsilon^+}(\psi(s) - \varphi(s)) > 0$ for every $\epsilon > 0$;
(A_4): at least one of the functions of the pair (ψ, φ) satisfies property (P).
Suppose that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type C. Then T admits a unique fixed point $x^* \in X$ and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. This conclusion can be drawn by applying a demonstration similar to the proof in Theorem 4 with (ψ, φ)-rational contraction type A being replaced by (ψ, φ)-rational contraction type C. □

Example 4. *Let $X = \{a, b, c, d\}$ and $d : X \times X \to [0, \infty)$,*

d(x,y)	a	b	c	d
a	0	2	4	6
b	2	0	2	4
c	4	2	0	6
d	6	4	2	0

Let $T : X \to X$ be a self-mapping defined by $Ta = c$, $Tb = Tc = b$, $Td = a$ and $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ be two functions defined by $\psi(t) = \ln(1 + t)$, $\varphi(t) = \frac{5}{6}\ln(1 + t)$. (Since ψ is continuous, then $(\psi, \varphi) \in \mathcal{F}$).
We claim that $T : X \mapsto X$ is a (ψ, φ)-rational contraction type C. Indeed, denoting $C(x, y) = \frac{d(x,Tx)\cdot d(x,Ty)+d(y,Ty)\cdot d(y,Tx)}{\max\{d(x,Ty),d(y,Tx)\}}$, we have:

1. *for* $x = a, y = b$,
$$\begin{aligned} d(Ta, Tb) = &\ \ d(c, b) = 2, \\ C(a, b) = &\ \ \tfrac{d(a,Ta)\cdot d(a,Tb) + d(b,Tb)\cdot d(b,Ta)}{\max\{d(a,Tb), d(b,Ta)\}} \\ = &\ \ \tfrac{d(a,c)\cdot d(a,b) + d(b,b)\cdot d(b,c)}{\max\{d(a,b), d(b,c)\}} \\ = &\ \ 4, \end{aligned}$$
and
$$\phi(d(Ta, Tb)) = ln(3) < \frac{5}{6} ln(5) = \psi(C(a, b)).$$

2. *for* $x = a, y = c$,
$$\begin{aligned} d(Ta, Tc) = &\ \ d(c, b) = 2, \\ C(a, c) = &\ \ \tfrac{d(a,Ta)\cdot d(a,Tc) + d(c,Tc)\cdot d(c,Ta)}{\max\{d(a,Tc), d(c,Ta)\}} \\ = &\ \ \tfrac{d(a,c)\cdot d(a,b) + d(c,b)\cdot d(c,c)}{\max\{d(a,b), d(c,c)\}} \\ = &\ \ 4, \end{aligned}$$
and
$$\phi(d(Ta, Tc)) = ln(3) < \frac{5}{6} ln(5) = \psi(C(a, c)).$$

3. *for* $x = a, y = d$,
$$\begin{aligned} d(Ta, Td) = &\ \ d(c, a) = 4, \\ C(a, d) = &\ \ \tfrac{d(a,Ta)\cdot d(a,Td) + d(d,Td)\cdot d(d,Ta)}{\max\{d(a,Td), d(d,Ta)\}} \\ = &\ \ \tfrac{d(a,c)\cdot d(a,a) + d(d,a)\cdot d(d,c)}{\max\{d(a,a), d(d,c)\}} \\ = &\ \ 6, \end{aligned}$$
and
$$\phi(d(Ta, Td)) = ln(5) < \frac{5}{6} ln(7) = \psi(C(a, d)).$$

4. *for* $x = b, y = c$,
$$\begin{aligned} d(Tb, Tc) = &\ \ d(b, b) = 0, \\ C(b, c) = &\ \ \tfrac{d(b,Tb)\cdot d(b,Tc) + d(c,Tc)\cdot d(c,Tb)}{\max\{d(b,Tc), d(c,Tb)\}} \\ = &\ \ \tfrac{d(b,b)\cdot d(b,b) + d(c,b)\cdot d(c,b)}{\max\{d(b,b), d(c,b)\}} \\ = &\ \ 2, \end{aligned}$$
and
$$\phi(d(Tb, Tc)) = ln(1) < \frac{5}{6} ln(3) = \psi(C(b, c)).$$

5. *for* $x = b, y = d$,
$$\begin{aligned} d(Tb, Td) = &\ \ d(b, a) = 2, \\ C(b, d) = &\ \ \tfrac{d(b,Tb)\cdot d(b,Td) + d(d,Td)\cdot d(d,Tb)}{\max\{d(b,Td), d(d,Tb)\}} \\ = &\ \ \tfrac{d(b,b)\cdot d(b,a) + d(d,a)\cdot d(d,b)}{\max\{d(b,a), d(d,b)\}} \\ = &\ \ 6, \end{aligned}$$
and
$$\phi(d(Tb, Td)) = ln(3) < \frac{5}{6} ln(7) = \psi(C(b, d)).$$

6. *for $x = c, y = d$,*

$$\begin{aligned} d(Tc,Td) = &\ d(b,a) = 2, \\ C(c,d) = &\ \tfrac{d(c,Tc)\cdot d(c,Td) + d(d,Td)\cdot d(d,Tc)}{\max\{d(c,Td), d(d,Tc)\}} \\ = &\ \tfrac{d(c,b)\cdot d(c,a) + d(d,a)\cdot d(d,b)}{\max\{d(c,a), d(d,b)\}} \\ = &\ 8, \end{aligned}$$

and

$$\phi(d(Tc,Td)) = ln(3) < \frac{5}{6} ln(9) = \psi(C(c,d)).$$

Consequently, by Theorem 9, the mapping T has a unique fixed point; that is $x = b$.

The following corollary shows that Theorem 8 in [11] is a consequence of Theorem 9.

Corollary 4. *(Theorem 8, [11]) Let (X,d) be a complete metric space and $T : X \mapsto X$ be a $(\psi,\varphi)-$rational contraction type C. Assume that:*
(f_0) $\varphi(s) < \psi(s)$, for all $s > 0$;
(f_1'') ψ is non-decreasing and $\lim\limits_{s \to s_0^+} \sup \varphi(s) < \psi(s_0+)$, for any $s_0 > 0$.
Then T admits exactly one fixed point.

Proof. By similar reasoning, we can find a decreasing positive sequence $\{d_n\}$ such that $\lim\limits_{n\to\infty} \psi(d_n) = -\infty$.
From the monotonicity of ψ and Lemma 2, we can conclude that ψ satisfies property (P).
Using assumption (f_1''), we have

$$\begin{aligned} \lim_{s \searrow s_0} \inf(\psi(s) - \varphi(s)) &\geq \lim_{s \searrow s_0} \inf \psi(s) - \lim_{s \searrow s_0} \sup \varphi(s) \\ &> \lim_{s \searrow s_0} \inf \psi(s) - \lim_{s \searrow s_0} \psi(s) \\ &= 0, \end{aligned}$$

where the last equality follows from the monotonicity of ψ.
Hence, (ψ,φ) satisfies condition (A_3).
From (f_0) and monotonicity of ψ, we also can obtain that for some $r \geq t > 0$,

$$\psi(r) \geq \psi(t) > \varphi(t),$$

which shows that (ψ,φ) satisfies condition (A_1).
Therefore, the conclusion of this corollary can be deduced from Theorem 9. □

3.2. New Common Fixed Point Results

At the end of this section, we now turn to common fixed point problem for some rational contractions mentioned above.

Theorem 10. *Let (X,d) be a complete metric space and ψ,φ be two mappings such that $(\psi,\varphi) \in \mathcal{F}$. Suppose that $P,Q : X \mapsto X$ satisfy that*

$$\psi(d(Px,Qy)) \leq \varphi(M_1''(x,y)), \tag{10}$$

for all $x,y \in X$ with $d(Px,Qy) > 0$, where $M_1'(x,y)$ is defined by

$$M_1''(x,y) = \max\{d(x,y), d(x,Px), d(y,Qy), \frac{d(x,Px)\cdot d(y,Qy)}{d(Px,Qy)}\}.$$

Then P and Q have a unique common fixed point x^* in X.

Proof. First of all, we can see at once that P and Q satisfy

$$d(Px, Qy) < M_1''(x, y), \quad \forall x, y \in X \quad \text{with} \quad d(Px, Qy) > 0, \tag{11}$$

which is clear from condition (A_1) and (10).
Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ as follows:

$$x_1 = Px_0, x_2 = Qx_1, \ldots, x_{2n-1} = Px_{2n}, x_{2n+2} = Qx_{2n+1}, \ldots,$$

for all $n \in \mathbb{N} \cup \{0\}$.
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of P (in the case that n_0 is even) or Q (if n_0 is odd). Moreover, if x_{n_0} is a fixed point of P (or Q) but not a common fixed point of P and Q (this means $d(x_{n_0}, Qx_{n_0}) > 0$), we get $d(Px_{n_0}, Qx_{n_0}) > 0$ and

$$\psi(d(Px_{n_0}, Qx_{n_0})) \leq \varphi(M_1''(x_{n_0}, x_{n_0})),$$

which implies that

$$\begin{aligned}
d(x_{n_0}, Qx_{n_0}) &= d(Px_{n_0}, Qx_{n_0}) \\
&< M_1''(x_{n_0}, x_{n_0}) \\
&= \max\{d(x_{n_0}, x_{n_0}), d(x_{n_0}, Px_{n_0}), d(x_{n_0}, Qx_{n_0}), \\
&\quad \frac{d(x_{n_0}, Px_{n_0}) \cdot d(x_{n_0}, Qx_{n_0})}{d(Px_{n_0}, Qx_{n_0})}\} \\
&= d(x_{n_0}, Qx_{n_0}).
\end{aligned}$$

This leads to a contradiction. Thus, we can claim that a fixed point of P or Q is also a common fixed point of the pair (P, Q). Therefore, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.
Let $d_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N} \cup \{0\}$. We first claim that $d_{n+1} < d_n$ for all $n \in \mathbb{N}$. For this purpose, we consider the following two cases:
Case 1: If $n = 2i, i \in \mathbb{N}$, we have $d(Px_{2i}, Qx_{2i+1}) > 0$ and from (10), we have:

$$\psi(d(Px_{2i}, Qx_{2i+1})) \leq \varphi(M_1''(x_{2i}, x_{2i+1})).$$

From (11), we have:

$$\begin{aligned}
d(Px_{2i}, Qx_{2i+1}) &< M_1''(x_{2i}, x_{2i+1}) \\
&= \max\{d(x_{2i}, x_{2i+1}), d(x_{2i}, Px_{2i}), d(x_{2i+1}, Qx_{2i+1}), \\
&\quad \frac{d(x_{2i}, Px_{2i}) \cdot d(x_{2i+1}, Qx_{2i+1})}{d(Px_{2i}, Qx_{2i+1})} \\
&= \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\},
\end{aligned}$$

which leads to $d(Px_{2i}, Qx_{2i+1}) = d(x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1})$.
Hence, for every even natural number n, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing.
Case 2: If $n = 2i + 1, i \in \mathbb{N}$, by the same reasoning, we could obtain a same conclusion when n is an odd natural number.
Therefore, we can find $d \neq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = d$.
The conclusion that $d = 0$ follows by the same methods as in the proof of Theorem 2.
Next, we claim that the sequence $\{x_n\}$ is a Cauchy sequence.
Reasoning by contradiction, if $\{x_{2n}\}$ is not Cauchy, from Lemma 4, we can find $\varepsilon \in (0, \infty)$ and two subsequences $\{x_{2j_k}\}$ and $\{x_{2p_k}\}$ of $\{x_{2n}\}$ such that $d(x_{2p_k+1}, x_{2j_k+2}) \to \varepsilon^+$, $d(x_{2p_k}, x_{2j_k+1}) \to \varepsilon^+$ as $k \to \infty$.
Hence, from (10), we have:

$$\psi(d(x_{2p_k+1}, x_{2j_k+2})) = \psi(d(Px_{2p_k}, Qx_{2j_k+1})) \leq \varphi(M_1''(x_{2p_k}, x_{2j_k+1})) \leq \psi(M_1''(x_{2p_k}, x_{2j_k+1})),$$

where:

$$\begin{aligned} M_1''(x_{2p_k}, x_{2j_k+1}) &= \max\{d(x_{2p_k}, x_{2j_k+1}), d(x_{2p_k}, Px_{2p_k}), d(x_{2j_k+1}, Qx_{2j_k+1}), \\ &\quad \frac{d(x_{2p_k}, Px_{2p_k}) \cdot d(x_{2j_k+1}, Qx_{2j_k+1})}{d(Px_{2p_k}, Qx_{2j_k+1})}\} \\ &= \max\{d(x_{2p_k}, x_{2j_k+1}), d(x_{2j_k+1}, x_{2j_k+2})\}. \end{aligned}$$

Since $\lim\limits_{k\to\infty} M_1''(x_{2p_k}, x_{2j_k+1}) = \varepsilon^+$, and it follows from (10) that:

$$\begin{aligned} \liminf_{s\to\varepsilon^+} \psi(s) &\le \liminf_{k\to\infty} \psi(d(x_{2p_k}, x_{2j_k+1})) \\ &\le \lim_{k\to\infty} \psi(d(x_{2p_k}, x_{2j_k+1})) \\ &= \lim_{s\to\varepsilon^+} \psi(s) \\ &\le \limsup_{k\to\infty} \varphi(M_1''(x_{2p_k}, x_{2j_k+1})) \\ &\le \limsup_{s\to\varepsilon^+} \varphi(s). \end{aligned}$$

This contradicts condition (A_2) and (A_2'), then $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$, that is T is a Picard operator.
In the following, we will claim that x^* is a common fixed point of P and Q.
First, we prove that x^* is a fixed point of Q.
If $d(Px_{2n}, Qx^*) = d(x_{2n+1}, Qx^*) = 0$ for n sufficient large enough, then

$$d(x^*, Qx^*) < d(x^*, Px_{2n}) + d(Px_{2n}, Qx^*) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $d(x^*, Qx^*) = 0$, so that $Qx^* = x^*$.
If $d(Px_{2n}, Qx^*) > 0$ for all $n \in \mathbb{N}$, we have:

$$\psi(d(Px_{2n}, Qx^*)) \le \varphi(M_1'(x_{2n}, x^*)),$$

which implies that:

$$\begin{aligned} d(Px_{2n}, Qx^*) &< M_1''(x_{2n}, x^*) \\ &= \max\{d(x_{2n}, x^*), d(x_{2n}, Px_{2n}), d(x^*, Qx^*), \\ &\quad \frac{d(x_{2n}, Px_{2n}) \cdot d(x^*, Qx^*)}{d(Px_{2n}, Qx^*)}\} \\ &= \max\{d(x_{2n}, x^*), d(x_{2n}, x_{2n+1}), d(x^*, Qx^*), \\ &\quad \frac{d(x_{2n}, x_{2n+1}) \cdot d(x^*, Qx^*)}{d(x_{2n+1}, Qx^*)}\}. \end{aligned}$$

So, we have:

$$d(Px_{2n}, Qx^*) < M_1''(x_{2n}, x^*) = d(x^*, Qx^*) \quad \text{for } n \text{ sufficiently large}$$

and then

$$\psi(d(Px_{2n}, Qx^*)) \le \varphi(d(x^*, Qx^*)) < \psi(d(x^*, Qx^*)) \quad \text{for } n \text{ sufficiently large}$$

Taking limits in the above inequality as $n \to \infty$, we have:

$$\psi(d(x^*, Qx^*)) < \psi(d(x^*, Qx^*)),$$

which is a contradiction, consequently, $d(x^*, Qx^*) = 0$, and x^* is a fixed point of Q.
Assume that x^* is not a fixed point of P. Then $d(Px^*, Qx^*) > 0$ and (10) gives us

$$\psi(d(Px^*, Qx^*)) \le \varphi(M_1''(x^*, x^*)),$$

which implies that

$$\begin{aligned} d(Px^*, Qx^*) &< M_1''(x^*, x^*) \\ &= \max\{d(x^*, x^*), d(x^*, Px^*), d(x^*, Qx^*), \\ &\quad \frac{d(x^*, Px^*) \cdot d(x^*, Qx^*)}{d(Px^*, Qx^*)}\} \\ &= \max\{0, d(x^*, Px^*), 0, 0\} \\ &= d(Px^*, Qx^*), \end{aligned}$$

which is a contradiction. Hence $d(x^*, Px^*) = 0$, that is x^* is a fixed point of P. Therefore, x^* is a common fixed point of P and Q.
Finally, to show the uniqueness of the common fixed point, we suppose that there exists another distinct common fixed point y^* such that $Py^* = Qy^*$.
Since $d(Px^*, Qy^*) > 0$, we have:

$$\psi(d(Px^*, Qy^*)) \le \varphi(M_1''(x^*, y^*)),$$

which implies that

$$\begin{aligned} d(Px^*, Qy^*) &< M_1''(x^*, y^*) \\ &= \max\{d(x^*, y^*), d(x^*, Px^*), d(y^*, Qy^*), \frac{d(x^*, Px^*) \cdot d(y^*, Qy^*)}{d(Px^*, Qy^*)}\} \\ &= d(x^*, y^*). \end{aligned}$$

This leads to a contradiction. Hence, $x^* = y^*$, so the common fixed point x^* is unique. □

Theorem 11. *Let (X, d) be a complete metric space and ψ, φ be two functions satisfying the following conditions:*

(A_1): for every $s \ge t > 0$, one has $\psi(s) > \varphi(t)$;
(A_3): $\liminf\limits_{s \to \epsilon^+} (\psi(s) - \varphi(s)) > 0$ for every $\epsilon > 0$;
(A_4): At least one of the functions of the pair (ψ, φ) satisfies property (P).

Suppose that $P, Q : X \mapsto X$ satisfy the contraction condition (10) Then P and Q have a unique common fixed point x^* in X.

Proof. This conclusion can be drawn by applying a proof similar to Theorem 4 with (ψ, φ)-rational contraction type A being replaced by the (ψ, φ)-rational contraction presented in (10). □

Theorem 12. *Let (X, d) be a complete metric space and ψ, φ be two functions such that $(\psi, \varphi) \in \mathcal{F}$. Suppose that $P, Q : X \mapsto X$ satisfy that*

$$\psi(d(Px, Qy)) \le \varphi(\frac{d(x, Px) \cdot d(x, Qy) + d(y, Qy) \cdot d(y, Px)}{\max\{d(x, Qy), d(y, Px)\}}), \tag{12}$$

for any $x, y \in X$ with $d(Px, Qy) > 0$ when $\max\{d(x, Qy), d(y, Px)\} \ne 0$ and $d(Px, Qy) = 0$ when $\max\{d(x, Qy), d(y, Px)\} = 0$.
Then P and Q have a unique fixed point x^* in X.

Proof. Applying similar arguments as the proof in Theorem 10, one can obtain the conclusion. □

4. Fixed Circle Results for $(\psi, \varphi)_c$-Rational Contractions

In this section, we examine the geometric properties of the fixed point set $Fix(T) = \{x \in X : Tx = x\}$ if it is not a singleton. Recently, the fixed circle (resp. fixed disc) problem was discussed by Özgür et al. [20] in this context. More generally, a geometric figure contained in the set $Fix(T)$ is called a *fixed figure* of T denoted by $\mathfrak{F}$. For example, a fixed ellipse, a fixed Cassini curve and so on. The study of these kind fixed figure problems retain importance both in terms of theoretical mathematical studies and some applied fields (see [21,22] and the references therein).

Here, in the context of the fixed figure problem, we investigate some new geometric properties of the set $Fix(T)$ via some new types of $(\psi, \varphi)_c-$rational contractions.

4.1. New Fixed Circle and Fixed Disc Results

First of all, let us recall some basic definitions related to fixed circle (resp. fixed disc). A circle and a disc are defined on a metric space as follows, respectively:

$$C_{x_0,r} := \{x \in X : d(x, x_0) = r\}.$$

and

$$D_{x_0,r} := \{x \in X : d(x, x_0) \leq r\}.$$

Definition 8 ([20])**.** *Let $C_{x_0,r}$ be a circle on X. If $Tx = x$ for every $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is said to be a fixed circle of T.*

First, exploring some modified versions of the inequalities menstioned in the previous section, we define new types of contractions whose fixed point sets contain a circle and a disc. Similar theorems can be studied for more geometric figures such as an ellipse, a hyperbola, a Cassini curve and an Apollonius circle etc. Now, using a modified version of the number $M_2(x, y)$ defined in (3) we define this new number $N_1(x, y)$ by

$$N_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Ty) \cdot (1 + d(x, Tx))}{1 + d(x, Ty)}\}. \tag{13}$$

For our purpose, we fix the second variable y as $y = x_0$ in (13).

Definition 9. *Let (X, d) be a metric space and $x_0 \in X$. A mapping $T : X \to X$ is called a $(\psi, \varphi)_c$-rational contraction type I with x_0 if, for every $x \in X$ such that $d(x, Tx) > 0$, the following inequality*

$$\psi(d(x, Tx)) \leq \varphi(N_1(x, x_0)) \tag{14}$$

holds, where $N_1(x, x_0)$ is defined by

$$N_1(x, x_0) = \max\{d(x, x_0), d(x, Tx), d(x_0, Tx_0), \frac{d(x_0, Tx_0) \cdot (1 + d(x, Tx))}{1 + d(x, Tx_0)}\}$$

and $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).

Proposition 5. *If T is a $(\psi, \varphi)_c-$rational contraction type I with $x_0 \in X$, then $Tx_0 = x_0$.*

Proof. Suppose on the contrary that $Tx_0 \neq x_0$. From Definition 9, we get:

$$\psi(d(x_0, Tx_0)) \leq \varphi(N_1(x_0, x_0)). \tag{15}$$

From condition (A_1) and (15), we have:

$$\begin{aligned} d(x_0, Tx_0) &< N_1(x_0, x_0) \\ &= \max\{d(x_0, x_0), d(x_0, Tx_0), \frac{d(x_0, Tx_0) \cdot (1 + d(x_0, Tx_0))}{1 + d(x_0, Tx_0)}\} \\ &= d(x_0, Tx_0), \end{aligned}$$

which is a contradiction. Hence, $Tx_0 = x_0$. □

Theorem 13. *Let (X, d) be a metric space and T be a $(\psi, \varphi)_c-$rational contraction type I with $x_0 \in X$ and $r = \inf\{d(x, Tx) : x \neq Tx\}$. If $0 < d(x_0, Tx) \leq r$ for all $x \in C_{x_0,r}$, then $C_{x_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{x_0,\rho}$ with $\rho < r$.*

Proof. Let $x \in C_{x_0,r}$. If $Tx \neq x$, by the definition of r, we have $d(Tx, x) \geq r$.
From the assumption on T, we have:

$$\psi(d(Tx, x)) \leq \varphi(N_1(x, x_0)),$$

which together with condition (A_1) and Proposition 5, implies that

$$\begin{aligned} d(Tx, x) &< N_1(x, x_0) \\ &= \max\{d(x, x_0), d(x, Tx), d(x_0, Tx_0), \frac{d(x_0, Tx_0) \cdot (1 + d(x, Tx))}{1 + d(x, Tx_0)}\} \\ &= \max\{r, d(x, Tx), 0, 0\} \\ &= d(x, Tx). \end{aligned}$$

This leads to a contradiction which consequently shows that $Tx = x$, that is $C_{x_0,r}$ is a fixed circle of T.
Similar arguments mentioned above apply to the case that $x \in C_{x_0,\rho}$ with $\rho < r$, we can deduce that T also fixes any circle $C_{x_0,\rho}$ with $\rho < r$. □

Corollary 5. *Let (X, d) be a metric space and T be a $(\psi, \varphi)_c-$rational contraction of type I with $x_0 \in X$ and $r = \inf\{d(x, Tx) : x \neq Tx\}$. If $0 < d(x_0, Tx) \leq r$ for all $x \in C_{x_0,r}$, then T fixes the disc $D_{x_0,r}$.*

Example 5. *Let $\mathbb{R}$ be the usual metric space with the usual metric d defined by $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. We define the self-mapping T_λ as:*

$$T_\lambda(x) = \begin{cases} x - \lambda, & x > 1 + \lambda \\ x, & x \leq 1 + \lambda \end{cases},$$

where λ is a constant with $0 < \lambda < 1$, and consider the functions $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ defined by:

$$\psi(x) = x, \varphi(x) = \lambda x.$$

For all $x \in (1 + \lambda, \infty)$, we obtain $x \neq T_\lambda x$ and

$$\psi(d(x, Tx)) = \psi(\lambda) = \lambda \leq \varphi(N_1(x, 0)) = \varphi(\max\{|x|, \lambda\}) = \varphi(|x|) = \lambda|x|.$$

Thus, T_λ is a $(\psi, \varphi)_c$-rational contraction type I with $x_0 = 0$. Additionally, we find:

$$r = \inf\{d(x, T_\lambda x) : x \neq T_\lambda x, x \in X\} = \inf\{|x - (x - \lambda)| : x > 1 + \lambda\} = \lambda.$$

For all $x \in C_{0,\lambda} = \{-\lambda, \lambda\}$ we have $d(0, Tx) = \lambda$. Thus, the self-mapping T_λ satisfies the conditions of Theorem 13 for the point $x_0 = 0$. Clearly, we have $Fix(T_\lambda) = (-\infty, 1 + \lambda]$ and the disc $D_{0,\lambda} = [-\lambda, \lambda]$ is contained in the set $Fix(T_\lambda)$, that is, $D_{0,\lambda}$ is a fixed disc of T_λ.

On the other hand, T_λ satisfies the conditions of Theorem 13 for any point $x_0 \in (-\infty, 1]$ and so, the disc $D_{x_0,\lambda} = [x_0 - \lambda, x_0 + \lambda]$ is another fixed disc of T_λ.

In the following example, we see that the converse statement of Theorem 13 is not true everywhen.

Example 6. *Let $\mathbb{R}$ be the usual metric space and define the self-mapping T as*

$$Tx = \begin{cases} x - \frac{1}{2}, & x > \frac{1}{2} \\ x, & x \leq \frac{1}{2} \end{cases}.$$

Consider the functions $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ defined by

$$\psi(x) = x, \varphi(x) = \frac{1}{2}x.$$

Then for all $x \in \left(\frac{1}{2}, \infty\right)$, we have:

$$\psi(d(x, Tx)) = \psi\left(\frac{1}{2}\right) = \frac{1}{2}$$

and

$$\varphi(N_1(x, 0)) = \varphi\left(\max\left\{|x|, \frac{1}{2}\right\}\right) = \varphi(|x|) = \frac{1}{2}|x|.$$

Clearly, for all $x \in \left(\frac{1}{2}, 1\right)$, the (14) does not hold. Hence, we deduce that T is not a $(\psi, \varphi)_c$-rational contraction type I with $x_0 = 0$. Notice that we have:

$$\begin{aligned} r &= \inf\{d(x, Tx) : x \neq Tx, x \in X\} \\ &= \inf\left\{\left|x - \left(x - \frac{1}{2}\right)\right| : x > \frac{1}{2}\right\} \\ &= \frac{1}{2} \end{aligned}$$

and the geometric condition $0 < d(0, Tx) \leq \frac{1}{2}$ holds for all $x \in C_{0,\frac{1}{2}}$. Clearly, the set $Fix(T) = \left(-\infty, \frac{1}{2}\right]$ contains the disc $D_{0,\frac{1}{2}} = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Now, using the number $N_1(x, y)$ defined in (13), we give a general uniqueness theorem for a fixed figure contained in the set $Fix(T)$.

Theorem 14. *Let (X, d) be a metric space, T be a self-mapping on X, $\mathfrak{F}$ be a fixed figure of T and $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ be two mappings satisfying condition (A_1). If the following condition*

$$\psi(d(Tx, Ty)) \leq \varphi(N_1(x, y)) \tag{16}$$

is satisfied for all $x \in \mathfrak{F}, y \in X \setminus \mathfrak{F}$ by T, then $\mathfrak{F}$ is the unique fixed figure of T. That is, we have $Fix(T) = \mathfrak{F}$.

Proof. For the uniqueness of a fixed figure $\mathfrak{F}$ of T, suppose on the contrary that there exist two fixed figures $\mathfrak{F}$ and $\mathfrak{F}_1$. By (16), we have:

$$\psi(d(x, y)) = \psi(d(Tx, Ty)) \leq \varphi(N_1(x, y)).$$

Together with condition (A_1), we have

$$\begin{aligned} d(x,y) &< N_1(x,y) \\ &= \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Ty)\cdot(1+d(x,Tx))}{1+d(x,y)}\} \\ &= d(x,y), \end{aligned}$$

which is a contradiction.
Hence, it should be $x = y$. Therefore, $\mathfrak{F}$ is the unique fixed figure of T. □

Corollary 6. *Let (X,d) be a metric space, T be a self-mapping on X, $D_{x_0,r}$ be a fixed disc of T and and $\psi, \varphi : (0,\infty) \mapsto \mathbb{R}$ be two mappings satisfying condition (A_1). If the condition (16) holds for all $x \in D_{x_0,r}$ and $y \in X \setminus D_{x_0,r}$ then the fixed disc $D_{x_0,r}$ is maximal, that is, we have $Fix(T) = D_{x_0,r}$.*

Example 7. *Consider the set $X = \{-1,0,1,2\}$ with the usual metric. Define the self-mapping T as*

$$Tx = \begin{cases} x, & x \in \{-1,0,1\} \\ 0, & x = 2 \end{cases}$$

and consider the functions $\psi, \varphi : (0,\infty) \mapsto \mathbb{R}$ defined by:

$$\psi(x) = x, \varphi(x) = \frac{2}{3}x.$$

It is easy to check that the condition (16) is satisfied for all $x \in D_{0,1} = \{-1,0,1\}$ and $y = 2$. Consequently, the fixed disc $D_{0,1}$ is maximal, that is, we have $Fix(T) = D_{0,1}$. Notice that the circle $C_{0,1}$ is also unique fixed circle of T.

Definition 10. *Let (X,d) be a metric space and $x_0 \in X$. A mapping $T : X \to X$ is a $(\psi,\varphi)_c$-rational contraction type II if, for every $x \in X$ such that $d(x,Tx) > 0$, the following inequality*

$$\psi(d(x,Tx)) \leq \varphi(N_2(x,x_0)) \tag{17}$$

holds, where $N_2(x,x_0)$ is defined by

$$N_2(x,x_0) = \frac{d(x,Tx)\cdot d(x,Tx_0) + d(x_0,Tx_0)\cdot d(x_0,Tx)}{2\max\{d(x,Tx_0), d(x_0,Tx)\}},$$

if $\max\{d(x,Tx_0), d(x_0,Tx)\} = 0$, then $d(x,Tx) = 0$ and $\psi, \varphi : (0,\infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).

Proposition 6. *If T is a $(\psi,\varphi)_c-$rational contraction type II with $x_0 \in X$, then $Tx_0 = x_0$.*

Proof. Suppose on the contrary that $Tx_0 \neq x_0$. From Definition 10, we get:

$$\psi(d(x_0,Tx_0)) \leq \varphi(N_2(x_0,x_0)). \tag{18}$$

From condition (A_1) and (18), we have:

$$\begin{aligned} d(x_0,Tx_0) &< N_2(x_0,x_0) \\ &= d(x_0,Tx_0), \end{aligned}$$

which is a contradiction. Hence, $Tx_0 = x_0$. □

Theorem 15. *Let (X,d) be a metric space and T be a $(\psi,\varphi)_c-$rational contraction type II with $x_0 \in X$ and $r = \inf\{d(x,Tx) : x \neq Tx\}$. If $0 < d(x_0,Tx) \leq r$ for all $x \in C_{x_0,r}$, then $C_{x_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{x_0,\rho}$ with $\rho < r$.*

Proof. Let $x \in C_{x_0,r}$. If $Tx \neq x$, by the definition of r, we have $d(Tx,x) \geq r$.
From the assumption on T, we have:

$$\psi(d(Tx,x)) \leq \varphi(N_2(x,x_0)),$$

which together with condition (A_1) and Proposition 6, implies that

$$\begin{aligned} d(Tx,x) &< N_2(x,x_0) \\ &= \frac{d(x,Tx)\cdot d(x,Tx_0) + d(x_0,Tx_0)\cdot d(x_0,Tx)}{2\max\{d(x,Tx_0),d(x_0,Tx)\}} \\ &= \frac{d(x,Tx)}{2}, \end{aligned}$$

which leads to a contradiction. Hence, $C_{x_0,r}$ is a fixed circle of T.
The conclusion that T fixes every circle $C_{x_0,\rho}$ with $\rho < r$ can be drawn by applying the similar arguments as the case of $C_{x_0,r}$. For brevity, we omit the rest of the proof.
□

Corollary 7. *Let (X,d) be a metric space and T be a $(\psi,\varphi)_c-$rational contraction of type II with $x_0 \in X$ and $r = \inf\{d(x,Tx) : x \neq Tx\}$. If $0 < d(x_0,Tx) \leq r$ for all $x \in C_{x_0,r}$, then T fixes the disc $D_{x_0,r}$.*

Theorem 16. *Let (X,d) be a metric space, T be a self-mapping on X, $\mathcal{F}$ be a fixed figure of T and $\psi,\varphi : (0,\infty) \mapsto \mathbb{R}$ be two mappings satisfying condition (A_1). If the following condition*

$$\psi(d(Tx,Ty)) \leq \varphi(N_2(x,y)) \tag{19}$$

is satisfied for all $x \in \mathfrak{F}, y \in X \setminus \mathfrak{F}$ by T, then $\mathfrak{F}$ is the unique fixed figure of T, that is, we have $Fix(T) = \mathfrak{F}$.

Proof. To prove the uniqueness of the fixed figure $\mathfrak{F}$, suppose on the contrary that there exist another fixed figure $\mathfrak{F}_1$ of T.
Let $x \in \mathfrak{F}$ and $y \in \mathfrak{F}_1$ be arbitrary distinct points.
By (19), we have:

$$\psi(d(x,y)) = \psi(d(Tx,Ty)) \leq \varphi(N_2(x,y)).$$

Together with condition (A_1), we have

$$\begin{aligned} 0 < d(x,y) &< N_2(x,y) \\ &= \frac{d(x,Tx)\cdot d(x,Ty) + d(y,Ty)\cdot d(y,Tx)}{\max\{d(x,Ty),d(y,Tx)\}} \\ &= 0, \end{aligned}$$

which is a contradiction.
Hence, $x = y$. Thus, $\mathfrak{F}$ is the unique fixed figure of T. □

Corollary 8. *Let (X,d) be a metric space, T be a self-mapping on X, $D_{x_0,r}$ be a fixed disc of T and $\psi,\varphi : (0,\infty) \mapsto \mathbb{R}$ be two mappings satisfying condition (A_1). If the condition (19) holds for all $x \in D_{x_0,r}$ and $y \in X \setminus D_{x_0,r}$ then the fixed disc $D_{x_0,r}$ is maximal, that is, we have $Fix(T) = D_{x_0,r}$.*

4.2. New Common Fixed Figure Results

A geometric figure $\mathcal{F}$ contained in the set $Fix(P) \cap Fix(Q)$ is called a *common fixed figure* of a pair of self-mappings (P, Q). Now, we will present some common fixed circle (resp. common fixed disc) theorems for a pair of self-mappings (P, Q) involving $(\psi, \varphi)_c-$rational contractions as follows.

Definition 11 ([21]). *Let (X, d) be a metric space and $P, Q : X \mapsto X$ be two self-mappings. If $Px = Qx = x$ for all $x \in C_{x_0,r}$ (resp. $x \in D_{x_0,r}$), then $C_{x_0,r}$ (resp. $D_{x_0,r}$) is called the common fixed circle (resp. common fixed disc) of the pair (P, Q).*

Next, we first modify the number $N_1(x, x_0)$ defined in Definition 9 for a pair of self-mappings (P, Q) as follows.

$$N_1'(x, x_0) = \max\{d(x, x_0), d(Px, Qx), d(Px_0, Qx_0), \frac{d(Px_0, Qx_0)(1 + d(Px, Qx))}{1 + d(Px, Qx_0)}\}.$$

Then we define the following numbers to be used in the sequel:

$$\begin{aligned} r_P &= \inf\{d(x, Px), Px \neq x, x \in X\}, \\ r_Q &= \inf\{d(x, Qx), Qx \neq x, x \in X\}, \\ r_{P,Q} &= \inf\{d(Px, Qx), Px \neq Qx, x \in X\}, \\ \mu &= \min\{r_P, r_Q, r_{P,Q}\}. \end{aligned}$$

Proposition 7. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies the following inequality*

$$\psi(d(Px, Qx)) \leq \varphi(N_1'(x, x_0)), \tag{20}$$

for any $x \in X$ such that $d(Px, Qx) > 0$, where $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Then x_0 is a coincidence point of the pair (P, Q), that is $Px_0 = Qx_0$.

Proof. Suppose on the contrary that $Px_0 \neq Qx_0$, then by (20), we have:

$$\psi(d(Px_0, Qx_0)) \leq \varphi(N_1'(x_0, x_0)). \tag{21}$$

By condition (A_1), we have:

$$\begin{aligned} d(Px_0, Qx_0) &< N_1'(x, x_0) \\ &= \max\{d(x_0, x_0), d(Px_0, Qx_0), \frac{d(Px_0, Qx_0) \cdot (1 + d(Px_0, Qx_0))}{1 + d(Px_0, Qx_0)}\} \\ &= d(Px_0, Qx_0), \end{aligned}$$

which is a contradiction. Hence, $Px_0 = Qx_0$, i.e., x_0 is a coincidence point of the pair (P, Q). □

Theorem 17. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies that for any $x \in X$,*

$$d(Px, Qx) > 0 \Rightarrow \psi(d(Px, Qx)) \leq \varphi(N_1'(x, x_0)), \tag{22}$$

where $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Assume that P or Q is a $(\psi, \varphi)_c-$rational contraction type I with x_0, and

$$d(Px, x_0) \leq \mu, \quad d(Qx, x_0) \leq \mu, \quad \text{for all} \quad x \in C_{x_0,\mu}.$$

Then $C_{x_0,\mu}$ is a common fixed circle of the pair (P,Q). Especially, $D_{x_0,\mu}$ is a common fixed disc of the pair (P,Q).
Moreover, if the following condition

$$\psi(d(Px,Qy)) \leq \varphi(N_1'(x,y)) \tag{23}$$

is satisfied for all $x \in C_{x_0,r}, y \in X \setminus C_{x_0,r}$ by P,Q. Then $C_{x_0,r}$ is the unique common fixed circle of P and Q.

Proof. Our proof starts with the observation that x_0 is a coincidence point of the pair (P,Q). If P (or Q) is a $(\psi,\varphi)_c-$rational contraction type I with x_0, then we have $Px_0 = x_0$ (or $Qx_0 = x_0$) and $Px_0 = Qx_0 = x_0$.
Let $\mu = 0$, then we have $C_{x_0,\mu} = \{x_0\}$ and clearly $C_{x_0,\mu}$ is a common fixed circle of the pair (P,Q).
Let $\mu > 0$ and $x \in C_{x_0,\mu}$ be an arbitrary point such that $Px \neq Qx$. Then we have:

$$\psi(d(Px,Qx)) \leq \varphi(N_1'(x,x_0)),$$

which implies that

$$\begin{aligned} d(Px,Qx) &< N_1'(x,x_0) \\ &= \max\{d(x,x_0), d(Px,Qx), d(Px_0,Qx_0), \\ &\quad \frac{d(Px_0,Qx_0)(1+d(Px,Qx))}{1+d(Px,Qx_0)}\} \\ &= d(Px,Qx). \end{aligned}$$

This leads to a contradiction. Therefore, x is a coincidence point of the pair (P,Q).
Hence, if $x^* \in C_{x_0,\mu}$ is a fixed point of P then clearly x^* is also the fixed point of Q and vice versa.
If P (or Q) is a $(\psi,\varphi)_c-$rational contraction type I with x_0, then by Theorem 13, we have $Px = x$ (or $Sx = x$).
Hence, $Px = Sx = x$ for all $x \in C_{x_0,\mu}$, that is, $C_{x_0,\mu}$ is a common fixed circle of the pair (P,Q). Moreover, the same proof remains valid for the case that $x \in D_{x_0,\mu}$, that is, $D_{x_0,\mu}$ is a common fixed disc of the pair (P,Q).
For the uniqueness of the common fixed circle, supposing on the contrary that there exist two fixed circles $C_{x_0,r}$ and $C_{x_1,\rho}$ of P,Q.
Let $x \in C_{x_0,r}$ and $y \in C_{x_1,\rho}$ be arbitrary distinct points.
By (23), we have:

$$\psi(d(x,y)) = \psi(d(Px,Qy)) \leq \varphi(N_1'(x,y)).$$

Together with condition (A_1), we have:

$$\begin{aligned} d(x,y) &< N_1'(x,y) \\ &= \max\{d(x,y), d(Px,Qx), d(Py,Qy), \frac{d(Py,Qy)\cdot(1+d(Px,Qx))}{1+d(Px,Qy)}\} \\ &= d(x,y), \end{aligned}$$

which is a contradiction.
Hence, it should be $x = y$. Therefore, $C_{x_0,r}$ is the unique common fixed circle of P and Q. □

Here is an example to support the validity of Theorem 17.

Example 8. *Let $X = \{-1,0\} \cup [1,\infty)$ be the metric space with usual metric $d(x,y) = |x-y|$ for $x,y \in X$. Let us define self-mappings $P,Q : X \mapsto X$ as follows:*

$$Px = \begin{cases} -1, & x = -1 \\ x^2, & x \in \{0, 1, 3\} \\ x + 1, & otherwise \end{cases}, \text{ and } Qx = \begin{cases} x, & x \in \{-1, 1, 0\} \\ 2x + 2, & x = 3 \\ x + 1, & otherwise \end{cases}.$$

Define $\psi(s) = s$, $\varphi(s) = \frac{7s}{8}$ for all $s > 0$. Clearly, (ψ, φ) satisfies condition (A_1).
An easy computation gives that

$$\begin{aligned} r_P &= \min\{d(x, Px), Px \neq x, x \in X\} = 1, \\ r_Q &= \min\{d(x, Qx), Qx \neq x, x \in X\} = 1, \\ r_{P,Q} &= \min\{d(Px, Qx), Px \neq Qx, x \in X\} = 1, \\ \lambda &= \min\{r_P, r_Q, r_{P,Q}\} = 1. \end{aligned}$$

Fix $x_0 = 0$. It is easy to verify that Q is a $(\psi, \varphi)_c-$rational contraction type I with 0 and

$$d(Px, 0) \leq \mu, \quad d(Qx, 0) \leq \mu, \quad \textit{for all} \quad x \in C_{0,1}.$$

Finally, it remains to check that (P, Q) satisfies the contractive condition (22).
Indeed, it is obvious that only $x = 3$ is such a point that $Px \neq Qx$. Then

$$\psi(d(P3, Q3)) = 1 < \varphi(N_1'(3, 0)) = \frac{7}{8} \max\{3, 1, 0\} = \frac{21}{8}.$$

Therefore, $C_{0,1} = \{-1, 1\}$ is a common fixed circle of P, Q and $D_{0,1} = \{-1, 0, 1\}$ is a common fixed disc of P, Q.

Corollary 9. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies that for any $x \in X$,*

$$d(Px, Qx) > 0 \Rightarrow \psi(d(Px, Qx)) \leq \varphi(N_1'(x, x_0)),$$

where $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Assume that P or Q is a $(\psi, \varphi)_c-$rational contraction type I with x_0, and

$$d(Px, x_0) \leq \mu, \quad d(Qx, x_0) \leq \mu, \quad \textit{for all} \quad x \in D_{x_0,\mu}.$$

Then $D_{x_0,\mu}$ is a common fixed disc of the pair (P, Q).

Again, let us modify the number $N_2(x, x_0)$ defined in Definition 10 for a pair of self-mappings (P, Q) as follows.

$$N_2'(x, x_0) = \frac{d(Px, Qx) \cdot d(Px, Qx_0) + d(Px_0, Qx_0) \cdot d(Px_0, Qx)}{2 \max\{d(Px, Qx_0), d(Px_0, Qx)\}}.$$

Proposition 8. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies the following inequality*

$$\psi(d(Px, Qx)) \leq \varphi(N_2'(x, x_0)), \tag{24}$$

for any $x \in X$ such that $d(Px, Qx) > 0$. If $\max\{d(Px, Qx_0), d(Px_0, Qx)\} = 0$, then $d(Px, Tx) = 0$. In addition, suppose that $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Then x_0 is the coincidence point of the pair (P, Q), that is $Px_0 = Qx_0$.

Proof. Suppose on the contrary that $Px_0 \neq Qx_0$, then by (23), we have:

$$\psi(d(Px_0, Qx_0)) \leq \varphi(N_2'(x_0, x_0)). \tag{25}$$

By condition (A_1) and (25), we have:

$$\begin{aligned} d(Px_0, Qx_0) &< N_2'(x, x_0) \\ &= \frac{d(Px_0, Qx_0) \cdot d(Px_0, Qx_0) + d(Px_0, Qx_0) \cdot d(Px_0, Qx_0)}{2 \max\{d(Px_0, Qx_0), d(Px_0, Qx_0)\}} \\ &= \frac{d(Px_0, Qx_0)}{2}, \end{aligned}$$

which is a contradiction. Hence, $Px_0 = Qx_0$, i.e., x_0 is a coincidence point of the pair (P, Q). □

Theorem 18. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies that for any $x \in X$,*

$$d(Px, Qx) > 0 \Rightarrow \psi(d(Px, Qx)) \leq \varphi(N_2'(x, x_0)),$$

when $\max\{d(Px, Qx_0), d(Px_0, Qx)\} = 0$*, then* $d(Px, Tx) = 0$*.*
Furthermore, $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Assume that P or Q is a $(\psi, \varphi)_c-$rational contraction type II with x_0, and

$$d(Px, x_0) \leq \mu, \quad d(Qx, x_0) \leq \mu, \quad \textit{for all} \quad x \in C_{x_0,\mu}.$$

Then $C_{x_0,\mu}$ is a common fixed circle of the pair (P, Q). Especially, $D_{x_0,\mu}$ is a common fixed disc of the pair (P, Q).
Moreover, if the following condition

$$\psi(d(Px, Qy)) \leq \varphi(N_2'(x, y)) \tag{26}$$

is satisfied for all $x \in C_{x_0,r}, y \in X \setminus C_{x_0,r}$ by P, Q. Then $C_{x_0,r}$ is the unique common fixed circle of P and Q.

Proof. By the similar arguments used in the proof of Theorem 15, we can easily prove it. □

Corollary 10. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that a pair of self-mappings (P, Q) satisfies that for any $x \in X$,*

$$d(Px, Qx) > 0 \Rightarrow \psi(d(Px, Qx)) \leq \varphi(N_2'(x, x_0)),$$

when $\max\{d(Px, Qx_0), d(Px_0, Qx)\} = 0$*, then* $d(Px, Tx) = 0$*.*
Furthermore, $\psi, \varphi : (0, \infty) \mapsto \mathbb{R}$ are two mappings satisfying condition (A_1).
Assume that P or Q is a $(\psi, \varphi)_c-$rational contraction type II with x_0, and

$$d(Px, x_0) \leq \mu, \quad d(Qx, x_0) \leq \mu, \quad \textit{for all} \quad x \in D_{x_0,\mu}.$$

Then $D_{x_0,\mu}$ is a common fixed circle of the pair (P, Q).

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