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Determination of Heterogeneity for Manganese Dendrites Using Lacunarity Analysis

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Abstract

The surface patterns of natural and experimental deposits are important as they result from the internal microstructure. For this purpose, lacunarity analysis is applied to determine the heterogeneous nature of deposit surface patterns. In this study, images were digitally moved onto the square mesh to determine the heterogeneous situation of manganese dendrite patterns on the natural magnesite surface. The relation between the lacunarity values of the images and the box size was examined. The lacunarity values corresponding to the box size values were estimated using the gliding-box algorithm. This relation was determined numerically as a power-law function using nonlinear regression method. It has been shown that the system examined with the generated numerical model function can be defined with three specific parameters. As a result, it has been shown that it is possible to describe the relationship between numerical solution-based lacunarity-box size and a third-order nonlinear differential equation. With this study, the lacunarity-box size value on different system images can be determined by using the gliding box algorithm and calculating the coefficient value from the power-law relationship.

1. Introduction

Lacunarity is derived from the Latin word ”lacuna” meaning space or lake in Latin. Geometrical patterns and fractal gaps are specific terms that determine superficial morphological heterogeneity by referring to a measure using the counting method. Since it goes beyond intuitive measures for heterogeneity, lacunarity can quantify additional properties of various patterns, such as ”scale invariance” and heterogeneity [1]-[3]. The earliest description of lacunarity as a geometric term is attributed to Mandelbrot. In 1983, Mandelbrot essentially defined it as an auxiliary element infractal analysis [4]. The geometric texture pattern in an image is scale dependent. It can vary significantly with the size and spatial resolution of the digital image. Any very small image can contain parts of a pattern and be able to characterize the entire pattern, whereas a large image can consist of more than one pattern and accurately describe it as well. Likewise, a pixel in a low spatial resolution image shows signs of many patterns smaller than an integrated pixel size. Spatially, the resolution increases, the image pixels may be smaller. In this case, it may be appropriate to perform lacunarity analysis to generate meaningful information from the image pattern. Lacunarity applications provide flexibility in terms of ease of mathematical operation. Theoretically, however, it should be used with a different scale due to the consistent mean of characterization across tissue patterns [5]. Today, lacunarity analysis is used to characterize data and geometric patterns in various areas such as ecology, physics, medical imaging, urban spatial analysis and etc. It has many applications, especially in multiple fractal analysis [6, 7].

Fractal geometry describes photometric and geometric changes in fractal or non-fractal pattern images using lacunarity analysis. It has also developed a statistical approach that provides separable features over an extremely wide range of image

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transformations [8]-[11]. Accordingly, the numerical determination of the properties of the texture showing the geometric pattern is related to the estimation of the value of the image calculated according to the multi-scale local binary system. The changes are determined by combining the lacunarity parameters. Thus, to distinguish superficial patterns from each other, it is possible to characterize the local distribution of superficial designs using lacunarity analysis [12]. In addition, appropriate numerical methods and software have been developed to calculate the lacunarity value developed by Plotnick et al. [2, 3].

In general, the definition of the morphological image is related to the scale at which it is studied. A pattern that is observed to be homogeneous at a given scale can be heterogeneous when observed at a larger scale. Images of natural and experimental specimens emerge from variations of cellular units, often forming repeating patterns of the same type or pattern from a combination of the cellular units, the pattern of the base unit and the assembly of this group of pixels [11]-[13]. Some of these can be described as fractal. Classification by surface pattern, objectively or by definition, provides a meaningful hint regarding physical properties in many imaging and visualization applications [6, 8, 12].

Geometric pattern gap analysis is a measure of the statistical distribution of void dimensions based on fractal mathematics [1, 2]. The lacunarity analysis originally developed for binary data (binary or presence/absence) can be easily applied to designs with continuous distributed variables [2]. A distance (in scale) is calculated as the ratio of the first and second moments of the counts in all possible boxes of this spacing width. The first moment is the sum of the mean values of all possible blocks in dimension $Z^{(1)}$, and the second moment is the sum of the mean squares in all possible blocks in dimension $Z^{(2)}$. The ratio of the first and second moments is defined as the fractal geometry lacunarity [2, 3].

In this study, the relations between box sizes and the lacunarity values of a geometric pattern are examined both mathematically and numerically. For this purpose, in Section 2, a basic depiction of the lacunarity concept is introduced. In section 3, a third-order non-linear differential equation is firstly proposed for expressing the relationship between the value of the lacunarity and the box size. In addition, the analytical solution for the proposed differential equation is also demonstrated. In Section 4, the aliasing between the analytical solution of the proposed differential equation and the numerical simulations is shown by handling the results of the lacunarity analysis for natural manganese dendrites using the non-linear regression. Finally, we summarize and interpret the findings in Section 5.

2. Lacunarity description

In imaging techniques, a geometric pattern is defined in the form of a matrix in an M-dimensional square lattice. Accordingly, the matrix elements are determined as either a white pixel (filled) or a black pixel valued zero (empty). In the first step, the unit matrix $r \times r$ is calculated by scaling the matrix for each r value, by increasing the value from $r = 1$ to $r \leq L$, until it reaches the value of the upper left corner. As the box is moved to the right, one pixel is displaced, and the white pixels are counted again. These operations are repeated until the matrix is moved over the entire image and the frequency distribution is generated. Accordingly, the number of r-sized boxes containing S occupied sites is denoted by $n[S,r]$, and the total number of r-sized boxes is denoted by $N(r)$. If the size of the image is M, the following relation can be defined:

$$N[r] = (M - r + 1)^2.$$

The number of full sites S is transformed into a probability distribution by dividing the frequency distribution $N(r)$ by $n[S,r]$, i.e., the number of filled sites with box size r. Hence, the probability distribution is:

$$Q(S,\ r) = n[S,r]/N[r].$$

This value can be defined as the probability distribution $Q(S,r)$ of a morphological structure, i.e., the coating ratio of the image. Thus, the first order $Z^{(1)}$ and the second order $Z^{(2)}$ statistical moments are calculated. The first and second moments are given by:

$$\begin{aligned} Z^{(1)} &= \sum S^{*} Q(S,r), \\ Z^{(2)} &= \sum S^{2*} Q(S,r). \end{aligned}$$

The lacunarity value (Λ), which is calculated as the ratio of the second moment to the statistical first moment, is proportional to the box size r and can be defined as follows:

$$\Lambda[r] = Z^{(2)}/[Z^{(1)}]^2.$$

The statistical first moment is

$$Z^{(1)} = M[r],$$

and the second moment is

$$Z^{(2)} = S_s^2[r] + M[r]^2,$$

where $M[r]$ is the mean and $S_s^2[r]$ is the statistical variance of the number of sites per box

$$\Lambda[r] = S_s^2[r]/M[r]^2 + 1. \tag{2.1}$$

Equation (2.1) implies that the lacunarity is not simply dependent on the size of the gliding box r. In a random map, the white squares are occupied by the corresponding environment, and each part of the map is not only bound to $Q(S,r)$ but also to the distribution of gaps (related white squares). Thus, the lacunarity value differs, depending on the statistical distribution of the two different patterned map spaces with the occupancy rates of the occupied sites, i.e., full coverage of the full sites [6, 8, 11, 13].

3. Lacunarity analysis and discussion

A manganese dendrite image was used on the surface of natural magnesite ore for the lacunarity analysis. For this purpose, the manganese dendrite shown in Figure 3.1 was defined as a matrix of $M = 100$ pixels in the computer environment and converted to binary format (BMP) using the image processing method via imageJ [14].

The algorithm used a floating box size $r_{min.} = 1$ to $r_{max.} = 100$, the first and second moment values and statistical values were calculated for lacunarity with MATLAB software and graph diagrams were drawn using Origin 7.0. For the sample used, the lacunarity values varying according to the probability distribution were calculated as $\Lambda(100) = 1.000$ for $r_{max.} = 100$ pixel and $\Lambda(1) = 3.662$ for $r_{min.} = 1$ pixel. In addition, the first and second moments and lacunarity values of the box size $1 \leq r \leq 100$ are shown in Table 3.1, and the relationship of the lacunarity value to the box size is shown graphically in Figure 3.2. The values in Table 3.1 are presented that vary from $r = 1$ to $r = 10$ pixels, then change from $r = 10$ to $r = 100$ pixels in interval 10 pixels.

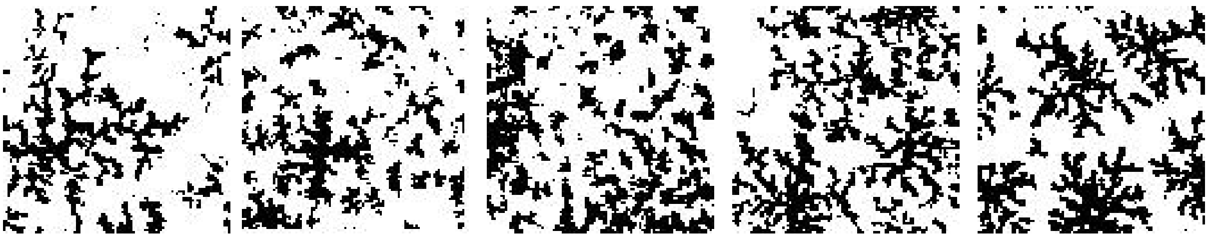

Figure 3.1: Binary format image of manganese dendrite patterns with different probability distributions selected from the magnesite ore surface with dimension $M = 100$ pixels.

Box size (r)	First moment $Z^{(1)}(r)$	Second moment $Z^{(2)}(r)$	Lacunarity $(\Lambda(r))$
1	0.2731	0.2731	3.662
2	1.087746	3.440363	2.907697
3	2.434923	14.99823	2.529703
4	4.306728	42.21086	2.275775
5	6.699219	93.59939	2.08557
6	9.626593	179.1739	1.933434
7	13.09767	310.2645	1.808606
8	17.13285	500.6209	1.705493
9	21.73358	765.5755	1.620786
10	26.90581	1122.259	1.550247
20	111.8363	15770.17	1.260871
30	254.8863	77447.11	1.192099
40	456.0922	234575.4	1.127658
50	711.9516	538348.3	1.062093
60	1025.158	1071731	1.019774
70	1390.106	1939315	1.003581
80	1793.254	3217359	1.000497
90	2186.372	4782181	1.00041
100	2731	7458361	1.0000

Table 3.1: First and second moments and lacunarity values representing distribution of box size r and statistical distribution values, for lattice size $M = 100$ pixels. The probability distribution value is computed as 0.273 for the image b.

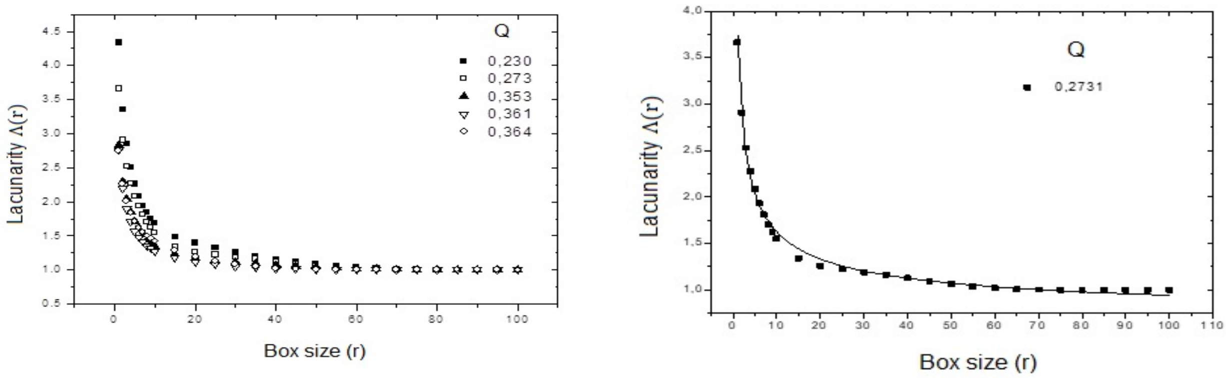

Figure 3.2: Change of lacunarity value $\Lambda(r)$ according to box size value r and non-linear regression implications.

When the $\Lambda(r)$ values in Table 3.1 and Figure 3.1 are examined, the box size of the lacunarity is seen to be similar to the power-law function $r = r_{min}, r_{min+1}, \ldots, r_{max}$. Accordingly, a mathematical model can be defined for this relationship. Thus, the mathematical model function

$$\Lambda(r) = \frac{\beta}{r^{\alpha}} + \gamma, \tag{3.1}$$

can be suggested for the relationship. This is the $\Lambda(r)$, r best interpretation of the lacunarity value between model function $\Lambda(r)$, $r = [r_{min}, r_{max}]$ and $r = r_{min} + r_{min} + 1, \ldots, r_{max}$, which can describe the geometric behaviour of manganese dendrites on the magnesite ore surface. A non-linear regression method can be used to determine solution constants for the function. The constant model parameters $\alpha = 0.311$, $\beta = 3.122$ and $\gamma = 0.494$ are calculated with regression coefficient $R^2 = 0.983$ for a pattern with probability distribution $Q(1,100) = 0.273$. The values of the other examples are also summarized in Table 3.2. Here, the parameters α, β and γ are independent and arbitrary variables for each sample of the system. The calculated results can be shown by the general fixed parameters α^*, β^* and γ^* of the model function, which best show the probability distribution and the regional morphological phase transitions in the surfaces of the images used.

Samples		Probability distribution	Model parameters			Regression coefficient
		$Q(S,r)$	α^*	β^*	γ^*	R^2
	a	0.231	0.522±0,011	3.815±0,003	0.304±0,001	0.988
	b	0.273	0.494±0,021	3.122±0,001	0.311±0,003	0.983
Manganese dendrites	c	0.353	0.532±0,010	2.118±0,004	0.392±0,001	0.971
	d	0.361	0.637±0,038	1.963±0,001	0.430±0,002	0.952
	e	0.374	0.411±0,052	2.103±0,002	0.347±0,003	0.978

Table 3.2: The probability distributions and values of the proposed mathematical model parameters for the observed samples.

The three parameters (α, β and γ) of the mathematical model have a single meaning for the lacunarity function of each manganese dendrites image. In particular, the value α represents the convergence of the $\Lambda(r)$ function, β represents the graph depression for lacunarity and γ represents a transition term. Calculations showed that while the β value takes on values over a very large numerical range, the parameters α and γ remain small. A small change is defined as a power-law function with low lacunarity value, while a large pit is a power-law function with high lacunarity value and a growing β value. The power-law function parameters can be correlated with the α and β constants, which can define the occupancy and morphological structure of the images. In particular, a small variation of the α value corresponds to a significant change in the value of β.

4. Model characterization

The relationship between lacunarity and the box-size r can be defined by:

$$r\frac{d\Lambda}{dr}\frac{d\Lambda^3}{dr^3} = r(\frac{d\Lambda^2}{dr^2})^2 - \frac{d\Lambda}{dr}\frac{d\Lambda^2}{dr^2}, \tag{4.1}$$

where $r \leq M$. This third order non-linear differential equation describes the edge size of the pixels that define the space box. To get the analytical solution of the non-linear differential equation (4.1), we need to do some variable transformations as this $\frac{d\Lambda}{dr} = m$, $\frac{d\Lambda^2}{dr^2} = \frac{dm}{dr}$ and $\frac{d\Lambda^3}{dr^3} = \frac{dm^2}{dr^2}$. According to this variable transformation, equation (4.1) is reduced to the second order differential equation as following:

$$rm\frac{dm^2}{dr^2} - r\left(\frac{dm}{dr}\right)^2 - m\frac{dm}{dr} = 0. \tag{4.2}$$

For this kind of nonlinear differential equation, the variable transform $m = e^u$ is applied to get the solution. By this transformation, if we write $\frac{dm}{dr} = e^u \frac{du}{dr}$ and $\frac{dm^2}{dr^2} = e^u \left(\frac{du}{dr}\right)^2 + e^u \frac{du^2}{dr^2}$ equalities in equation (4.2),

$$r(e^u)^2 \left(\frac{du^2}{dr^2} + \left(\frac{du}{dr}\right)^2\right) - r(e^u)^2 \left(\frac{du}{dr}\right)^2 + (e^u)^2 \frac{du}{dr} = 0,$$

and when the proper arrangement is made,

$$r\frac{du^2}{dr^2} + \frac{du}{dr} = 0, \tag{4.3}$$

second order linear differential equation is obtained.

Again, if we take $\frac{du}{dr} = v$ and $\frac{du^2}{dr^2} = \frac{dv}{dr}$ variable transformation for equation (4.3), the third order non-linear ordinary differential equation (4.1) is reduced to the following first order linear differential equation,

$$r\frac{dv}{dr} + v = 0. \tag{4.4}$$

By separation of variables, the equation (4.4) gives the following form,

$$\frac{dv}{v} + \frac{dr}{r} = 0, \tag{4.5}$$

and the analytical solution of equation (4.5) is obtained as,

$$v = \frac{c_1}{r}.$$

By using the variable transformation $\frac{du}{dr} = v$, $m = e^u$ and $\frac{d\Lambda}{dr} = m$ in this order, the analytical solution of third order non-linear differential equation (4.1) is obtained as follow,

$$\Lambda(r) = \frac{C_2}{C_1+1} r^{C_1+1} + c_3, \tag{4.6}$$

where c_1, c_2 and c_3 are the parameters that describe the system under investigation. The mathematical model (3.1) coincides with the analytical solution of the proposed third order differential equation (4.6) with the assumptions $\beta = \frac{c_2}{c_1+1}$, $\alpha = -(c_1+1)$ and $\gamma = c_3$. Thus, the change between the lacunarity values and box size was modeled.

5. Conclusions

In this study, the relationship between the lacunarity values and the box size is used to determine the heterogeneity of fractal and non-fractal geometric patterns on the deposit surface. For this purpose, the lacunarity value according to the box size for the manganese dendrite sediment patterns formed in the pores and cracks of the natural magnesite ore surface was calculated by using the gliding-box algorithm and the relations of the lacunarity value with the box size were defined. For describing this relation, the nonlinear regression method is used and the numerical power-law function is derived. It is shown that it is possible to determine the heterogeneity of the pattern system with three numerical model parameters. By taking the numerical model function as a reference, a third-order nonlinear differential equation is derived and its analytical solution is performed. The numerical solution function is compatible with the analytical solution function. This study's findings can be utilized to estimate the heterogeneity of similar deposit surfaces.

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Author's contributions

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