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On Statistical Submanifolds in Manifolds of Quasi-Constant Curvature

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Abstract

We mention some properties of statistical submanifolds in statistical manifolds of quasi-constant curvature. We obtain Chen first inequality and a Chen inequality for the $\delta(2,2)$-invariant for these manifolds.

Keywords: *Statistical manifold of quasi-constant curvature, submanifold, Chen inequality.*
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1. Introduction

An important topic submanifold theory is to find out relations between the sectional curvature tensor, the scalar curvature tensor and the mean curvature tensor of a submanifold. First relevant results in this field were obtained by B.-Y. Chen in 1993 [6]. He set up some inequalities between the extrinsic (the squared mean curvature) and intrinsic (the scalar curvature) invariants of a submanifold in a real space form, well-known as Chen first inequalities. Similar problems for submanifolds in Sasakian space form, Kenmotsu space form, Riemannian manifold of quasi-constant curvature etc., has been studied by many geometers, see [20], [7], [8], [13], [14], [15]. All of results related to Chen inequalities were given in [9] and its references.

A differential geometric approach for a statistical model of discrete probability distribution was introduced in [1] . Firstly, Amari was used the notion of a statistical manifold with applications in Information Geometry. The geometry of these manifolds involves deals with conjugate connections and, consequently, is closed related to affine differential geometry. A *statistical manifold* is a Riemannian manifold $(\overline{N}, \overline{g})$ endowed with a pair of torsion-free affine connections $\overline{D}$ and $\overline{D}^*$ satisfying

$$U\overline{g}\left(V,E\right)=\overline{g}\left(\overline{D}_{U}V,E\right)+\overline{g}\left(V,\overline{D}_{U}^{*}E\right), \tag{1.1}$$

for any U, V and $E \in T\overline{N}$. The connections $\overline{D}$ and $\overline{D}^*$are called *conjugate (dual) connections* (see [1] and [22]).

Any torsion-free affine connection $\overline{D}$ always has a dual connection given by

$$\overline{D}+\overline{D}^{*}=2\overline{D}^{0}, \tag{1.2}$$

where $\overline{D}^0$ is Levi-Civita connection of $\overline{N}$ [1]. So, many geometers have been established inequalities for statistical submanifolds of various statistical manifolds, for more details [2], [16], [3], [10], [17], [4], [5].

Motivated by the studies of the above papers, we obtain improved Chen inequality and a Chen inequality for the invariant $\delta(2,2)$ for statistical submanifolds in statistical manifolds of quasi-constant curvature.

2. Preliminaries

In [3], authors give an example of a statistical manifold of quasi-constant curvature and studied the properties of statistical submanifolds of these manifolds.

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The curvature tensor $\overline{R}$ of $\overline{D}$ is defined by

$$\begin{aligned}\overline{R}(U,V)E &= a\{\overline{g}(V,E)U-\overline{g}(U,E)V\} \\ &+b[T(V)T(E)U-\overline{g}(U,E)T(V)P \\ &+\overline{g}(V,E)T(U)P-T(U)T(E)V],\end{aligned} \tag{2.1}$$

where a,b are scalar functions, T is a 1-form given by

$$\overline{g}(U,P)=T(U) \tag{2.2}$$

and P is a unit vector field. The vector field P can be written

$$P=P^{T}+P^{\perp},$$

where P^T and $P^{\perp}$ are the tangent and normal components of P, respectively. If a statistical manifold $\overline{N}$ with its statistical structure $(\overline{D},\overline{g})$ has the curvature tensor $\overline{R}$ in the form (2.1), then it is called a *statistical manifold of quasi-constant curvature [3]. If* $b=0$, then the statistical manifold $\overline{N}$ turns into a statistical manifold of constant curvature [2].

Let $(\overline{N},\overline{g})$ be a statistical manifold given by torsion-free affine connections $\overline{D}$ and $\overline{D}^*$. Denote by $\overline{R}$ and $\overline{R}^*$ the curvature tensor fields of $\overline{D}$ and $\overline{D}^*$, respectively. Then $\overline{R}$ and $\overline{R}^*$ satisfy

$$\overline{g}\left(\overline{R}^*(U,V)E,F\right)=-\overline{g}\left(E,\overline{R}(U,V)F\right), \tag{2.3}$$

(see [12]). From (2.3), if $(\overline{D},\overline{g})$ is a statistical structure of quasi-constant curvature, then $\left(\overline{D}^*,\overline{g}\right)$ is also a statistical structure of quasi-constant curvature. So (2.1) is valid for $\left(\overline{D}^*,\overline{g}\right)$.

Let (N,g,D) and $\left(\overline{N},\overline{g},\overline{D}\right)$ be two statistical manifolds. An immersion $\pi: N\longrightarrow\overline{N}$ is called a *statistical immersion* [12]. If there is a statistical immersion between two statistical manifolds (N,g,D,D^*) and $\left(\overline{N},\overline{g},\overline{D},\overline{D}^*\right)$, then N is called a *statistical submanifold* of $\overline{N}$.

Let N be a statistical submanifold of a statistical manifold $\overline{N}$. Then, the Gauss formulas are given by

$$\overline{D}_UV=D_UV+h(U,V),$$

$$\overline{D}^*_UV=D^*_UV+h^*(U,V),$$

where the normal valued tensor fields h and h^* are symmetric and bilinear *the imbedding curvature tensors* of N in $\overline{N}$ for $\overline{D}$ and $\overline{D}^*$. So, D and D^* are called the *induced connections* of these connections, respectively. We have the linear transformations A_ξ and A^*_ξ defined by

$$g(A_\xi U,V)=\overline{g}(h(U,V),\xi) \tag{2.4}$$

and

$$g\left(A^*_\xi U,V\right)=\overline{g}(h^*(U,V),\xi) \tag{2.5}$$

for any unit $\xi\in T^{\perp}N$ and $U,V\in TN$ [22].

Let R, R^* denote the curvature tensors of the submanifold (N,g,D,D^*) in TN. Then we have the following Propositions:

Proposition 2.1. *[22] Let N be a statistical submanifold of $\overline{N}$. Then the Gauss equation with respect to the connection D is*

$$\begin{aligned}\overline{g}\left(\overline{R}(U,V)E,F\right)&=g(R(U,V)E,F)\\ &+\overline{g}(h(U,E),h^*(V,F))-\overline{g}(h^*(U,F),h(V,E))\end{aligned} \tag{2.6}$$

respectively, where $U,V,E,F\in TN$.

Proposition 2.2. *[22] Let N be a statistical submanifold of $\overline{N}$. Then the Gauss equation with respect to the connection D^* is*

$$\begin{aligned}\overline{g}\left(\overline{R}^*(U,V)E,F\right)&=g(R^*(U,V)E,F)\\ &+\overline{g}(h^*(U,E),h(V,F))-\overline{g}(h(U,F),h^*(V,E))\end{aligned}$$

respectively, where $U,V,E,F\in TN$.

In [19], the $\overline{K}$-*sectional curvature* of the statistical manifold was introduced as follows:
Let π be a plane in $T\overline{N}$; for an orthonormal basis $\{U, V\}$ of π, the $\overline{K}$-*sectional curvature* is

$$\overline{K}(\pi) = \frac{1}{2}\left[\overline{R}(U,V) + \overline{R}^{*}(U,V) - 2\overline{R}^{0}(U,V)\right], \tag{2.7}$$

where $\overline{R}^{0}$ is the curvature tensor field of $\overline{D}^{0}$ on $T\overline{N}$.

Example 2.1. [3] Let $\left(\overline{N} = I \times N^n(c), D, D^*\right)$ be a dualistic product (for more details see [21]), I one-dimensional statistical manifold, $N^n(c)$ a statistical manifold of constant curvature c with its projection $\pi : \overline{N} = I \times N^n(c) \to N^n(c)$. Denote by dt^2 the metric on I. Thus we have

$$\overline{g} = dt^2 + g_N,$$

where g_N is a metric on $N^n(c)$. The vector field $U \in \chi\left(\overline{N}\right)$ can be written as

$$U = \pi_*(U) + \overline{g}\left(U, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}, \tag{2.8}$$

where $\frac{\partial}{\partial t} \in \chi(I)$.
For $U, V, E, F \in \chi\left(\overline{N}\right)$, using (2.8), we obtain

$$\begin{aligned}\overline{g}\left(\overline{R}(U,V)E,F\right) &= c\left[\overline{g}(V,E)\overline{g}(U,F) - \overline{g}(U,E)\overline{g}(V,F)\right] \\ &+ c\left[\overline{g}(U,E)\overline{g}\left(V,\frac{\partial}{\partial t}\right)\overline{g}\left(F,\frac{\partial}{\partial t}\right) - \overline{g}(U,F)\overline{g}\left(V,\frac{\partial}{\partial t}\right)\overline{g}\left(E,\frac{\partial}{\partial t}\right)\right. \\ &\left. + \overline{g}(V,F)\overline{g}\left(U,\frac{\partial}{\partial t}\right)\overline{g}\left(E,\frac{\partial}{\partial t}\right) - \overline{g}(V,E)\overline{g}\left(U,\frac{\partial}{\partial t}\right)\overline{g}\left(F,\frac{\partial}{\partial t}\right)\right].\end{aligned}$$

It is known that (I, D, dt^2) and $\left(N^n(c), \widehat{D}, g_N\right)$ are statistical manifolds if and only if $\left(\overline{N} = I \times N^n(c), D, \overline{g}\right)$ is a statistical manifold [11]. So $\overline{N} = I \times N^n(c)$ is a statistical manifold of quasi-constant curvature with constant functions $a = b = c$.

Let $\{u_1, ..., u_n\}$ and $\{u_{n+1}, ...u_{n+m}\}$ be orthonormal tangent and normal frames, respectively, on N. The mean curvature vector fields are given by

$$H = \tfrac{1}{n}\sum_{i=1}^{n} h(u_i,u_i) = \tfrac{1}{n}\sum_{\alpha=1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)u_{n+\alpha} \quad , \quad h_{ij}^{\alpha} = \overline{g}\left(h(u_i,u_j), u_{n+\alpha}\right)$$

and

$$H^* = \tfrac{1}{n}\sum_{i=1}^{n} h^*(u_i,u_i) = \tfrac{1}{n}\sum_{\alpha=1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right)u_{n+\alpha} \quad , \quad h_{ij}^{*\alpha} = \overline{g}\left(h^*(u_i,u_j), u_{n+\alpha}\right).$$

3. Chen first inequality

In this section, we prove an improved Chen inequality statistical submanifolds in statistical manifolds of quasi-constant curvature. So, we give the following algebraic lemma which will be used in the proof of the main theorem.

Lemma 3.1. *[18] Let $m \geq 3$ be an integer and $\{b_1, ..., b_m\}$ m real numbers. Then we have*

$$\sum_{1 \leq i < j \leq m} b_i b_j - b_1 b_2 \leq \frac{m-2}{2(m-1)}\left(\sum_{i=1}^{m} b_i\right)^2.$$

The equality case of the above inequality holds if and only if $b_1 + b_2 = b_3 = ... = b_m$.

Let $\overline{N}^{n+m}$ be an $(n+m)$-dimensional statistical manifold of quasi-constant curvature, N^n an n-dimensional statistical submanifold of $\overline{N}$, $p \in N$ and π a plane section at p. We consider an orthonormal basis $\{u_1, u_2\}$ of π and $\{u_1, ..., u_n\}$, $\{u_{n+1}, ...u_{n+m}\}$ orthonormal basis of T_pN^n and $T_p^{\perp}N^n$, respectively.

Let K^0 be the sectional curvature of the Levi-Civita connection D^0 on N^n, h^0 the second fundamental form of N^n. From (2.7), the sectional curvature $K(\pi)$ of the plane section π is

$$K(\pi) = \frac{1}{2}\left[g\left(R\left(u_1, u_2\right) u_2, u_1\right) + g\left(R^*\left(u_1, u_2\right) u_2, u_1\right) \right.$$
$$\left. -2g\left(R^0\left(u_1, u_2\right) u_2, u_1\right)\right]. \tag{3.1}$$

Using (2.1), (2.3) and (2.6), we obtain

$$g\left(R\left(u_1, u_2\right) u_2, u_1\right) = a + b\left\{T\left(u_2\right)^2 + T\left(u_1\right)^2\right\} + \sum_{\alpha=1}^{m}\left(h_{11}^{*\alpha}h_{22}^{\alpha} - h_{12}^{*\alpha}h_{12}^{\alpha}\right)$$

and

$$g(R^*\left(u_1, u_2\right) u_2, u_1) = -g\left(R\left(u_1, u_2\right) u_1, u_2\right) = a + b\left\{T\left(u_2\right)^2 + T\left(u_1\right)^2\right\}$$
$$+\sum_{\alpha=1}^{m}\left(h_{11}^{\alpha}h_{22}^{*\alpha} - h_{12}^{\alpha}h_{12}^{*\alpha}\right).$$

If the last equalities are used in (3.1) then

$$K(\pi) = a + b\left\{T\left(u_2\right)^2 + T\left(u_1\right)^2\right\} + \frac{1}{2}\sum_{\alpha=1}^{m}\left(h_{11}^{*\alpha}h_{22}^{\alpha} + h_{11}^{\alpha}h_{22}^{*\alpha} - 2h_{12}^{\alpha}h_{12}^{*\alpha}\right) - K_0(\pi).$$

The last equality can be written as

$$K(\pi) = a + b\left\{T\left(u_2\right)^2 + T\left(u_1\right)^2\right\} + 2\sum_{\alpha=1}^{m}\left[h_{11}^{0\alpha}h_{22}^{0\alpha} - \left(h_{12}^{0\alpha}\right)^2\right] - \frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{*\alpha}h_{22}^{*\alpha} - \left(h_{12}^{*\alpha}\right)^2\right]$$
$$-\frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{\alpha}h_{22}^{\alpha} - \left(h_{12}^{\alpha}\right)^2\right] - K_0(\pi).$$

From the Gauss equation with respect to Levi-Civita connection, we obtain

$$K(\pi) = a + b\left\{T\left(u_2\right)^2 + T\left(u_1\right)^2\right\} + K_0(\pi) - 2\overline{K}_0(\pi)$$
$$-\frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{*\alpha}h_{22}^{*\alpha} - \left(h_{12}^{*\alpha}\right)^2\right] - \frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{\alpha}h_{22}^{\alpha} - \left(h_{12}^{\alpha}\right)^2\right] \tag{3.2}$$

where $\overline{K}_0$ the sectional curvature of the Levi-Civita connection $\overline{D}^0$ on $\overline{N}^{n+m}$.

Moreover, let τ be the scalar curvature of N^n. Then, using (2.7) and (2.3), we get

$$\tau = \frac{1}{2}\sum_{1\le i<j\le n}\left[g\left(R\left(u_i, u_j\right) u_j, u_i\right) + g\left(R^*\left(u_i, u_j\right) u_j, u_i\right) - 2g\left(R^0\left(u_i, u_j\right) u_j, u_i\right)\right]$$
$$= \frac{1}{2}\sum_{1\le i<j\le n}\left[g\left(R\left(u_i, u_j\right) u_j, u_i\right) - g\left(R\left(u_i, u_j\right) u_i, u_j\right)\right] - \tau_0, \tag{3.3}$$

where τ_0 is the scalar curvature of the Levi-Civita connection D^0 on N^n. By the use of (2.6) and (2.1), we obtain

$$\sum_{1\le i<j\le n} g\left(R\left(u_i, u_j\right) u_j, u_i\right) = a\left(\frac{n^2-n}{2}\right) + b(n-1)\left\|P^T\right\|^2 + \sum_{\alpha=1}^{m}\sum_{1\le i<j\le n}\left(h_{ii}^{*\alpha}h_{jj}^{\alpha} - h_{ij}^{*\alpha}h_{ij}^{\alpha}\right).$$

By similar calculations, we get

$$\sum_{1\leq i<j\leq n} g\left(R\left(u_i,u_j\right)u_i,u_j\right) = -a\left(\frac{n^2-n}{2}\right) - b\left(n-1\right)\left\|P^T\right\|^2 + \sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left(h_{ij}^{*\alpha}h_{ij}^{\alpha} - h_{ii}^{\alpha}h_{jj}^{*\alpha}\right).$$

By using the last two equality in (3.3), we obtain

$$\tau = a\left(\frac{n^2-n}{2}\right) + b\left(n-1\right)\left\|P^T\right\|^2 + \frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{\alpha} + h_{ii}^{\alpha}h_{jj}^{*\alpha} - 2h_{ij}^{*\alpha}h_{ij}^{\alpha}\right\} - \tau_0.$$

From the above equation, we find

$$\tau = a\left(\frac{n^2-n}{2}\right) + b\left(n-1\right)\left\|P^T\right\|^2 + 2\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{0\alpha}h_{jj}^{0\alpha} - \left(h_{ij}^{0\alpha}\right)^2\right\}$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right\} - \frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right\} - \tau_0.$$

By the Gauss equation for the Levi-Civita connection, we get

$$\tau = a\left(\frac{n^2-n}{2}\right) + b\left(n-1\right)\left\|P^T\right\|^2 + \tau_0 - 2\overline{\tau}_0$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right\} - \frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right\} \tag{3.4}$$

where $\overline{\tau}_0$ the scalar curvature of the Levi-Civita connection $\overline{D}^0$ on $\overline{N}^{n+m}$.
By subtracting (3.2) from (3.4), we get

$$(\tau-\tau_0) - \left(K\left(\pi\right) - K_0\left(\pi\right)\right) = a\left(\frac{n^2-n-2}{2}\right) + b\left[\left(n-1\right)\left\|P^T\right\|^2 - T\left(u_2\right)^2 - T\left(u_1\right)^2\right]$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{*\alpha} - \left(h_{ij}^{*\alpha}\right)^2\right\} - \frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right\} - \frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{*\alpha}h_{22}^{*\alpha} - \left(h_{12}^{*\alpha}\right)^2\right]$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{\alpha}h_{22}^{\alpha} - \left(h_{12}^{\alpha}\right)^2\right] - 2\overline{\tau}_0 + 2\overline{K}_0\left(\pi\right).$$

From the above equality, we obtain

$$(\tau-\tau_0) - \left(K\left(\pi\right) - K_0\left(\pi\right)\right) \geq a\frac{\left(n-2\right)\left(n+1\right)}{2} + b\left[\left(n-1\right)\left\|P^T\right\|^2 - T\left(u_2\right)^2 - T\left(u_1\right)^2\right]$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{*\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha}\right\}$$

$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\leq i<j\leq n}\left\{h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{11}^{\alpha}h_{22}^{\alpha}\right\} - 2\left(\overline{\tau}_0 - \overline{K}_0\left(\pi\right)\right). \tag{3.5}$$

Applying now Lemma 3.1, we have

$$\sum_{1\leq i<j\leq n}\left\{h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{11}^{\alpha}h_{22}^{\alpha}\right\} \leq \frac{\left(n-2\right)}{2\left(n-1\right)}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2 = \frac{n^2\left(n-2\right)}{2\left(n-1\right)}\left(H^{\alpha}\right)^2$$

and

$$\sum_{1\leq i<j\leq n}\left\{h_{ii}^{*\alpha}h_{jj}^{*\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha}\right\} \leq \frac{\left(n-2\right)}{2\left(n-1\right)}\left(\sum_{i=1}^{n}h_{ii}^{*\alpha}\right)^2 = \frac{n^2\left(n-2\right)}{2\left(n-1\right)}\left(H^{*\alpha}\right)^2.$$

Then using the last two inequality in (3.5), we can state the following main theorem:

Theorem 3.1. *Let $\overline{N}$ be an $(n+m)$-dimensional statistical manifold of quasi-constant curvature and N an n-dimensional statistical submanifold of $\overline{N}$. Then we have*

$$\tau_0 - K_0(\pi) \leq \tau - K(\pi) - a\frac{(n-2)(n+1)}{2} - b\left[(n-1)\left\|P^T\right\|^2 - T(u_2)^2 - T(u_1)^2\right]$$
$$+\frac{n^2(n-2)}{4(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + 2\left(\overline{\tau}_0 - \overline{K}_0(\pi)\right).$$

Moreover, the equality case holds in the above inequality if and only if for any $1 \leq \alpha \leq m$ we have

$$h_{11}^{\alpha} + h_{22}^{\alpha} = h_{33}^{\alpha} = ... = h_{nn}^{\alpha},$$
$$h_{11}^{*\alpha} + h_{22}^{*\alpha} = h_{33}^{*\alpha} = ... = h_{nn}^{*\alpha},$$
$$h_{ij}^{\alpha} = h_{ij}^{*\alpha} = 0,\ \ i \neq j,\ \ (i,j) \notin \{(1,2),(2,1)\}.$$

If we consider statistical submanifold in statistical manifold of constant curvature we have the following corollary:

Corollary 3.1. *Let $\overline{N}$ be an $(n+m)$-dimensional statistical manifold of constant curvature and N an n-dimensional statistical submanifold of $\overline{N}$. Then we have*

$$\tau_0 - K_0(\pi) \leq \tau - K(\pi) - a\frac{(n-2)(n+1)}{2} + \frac{n^2(n-2)}{4(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + 2\left(\overline{\tau}_0 - \overline{K}_0(\pi)\right).$$

Moreover, one of the equality holds in the all cases if and only if for any $1 \leq \alpha \leq m$ we have

$$\sigma_{11}^{\alpha} + \sigma_{22}^{\alpha} = \sigma_{33}^{\alpha} = ... = \sigma_{nn}^{\alpha},$$
$$\sigma_{11}^{*\alpha} + \sigma_{22}^{*\alpha} = \sigma_{33}^{*\alpha} = ... = \sigma_{nn}^{*\alpha},$$
$$\sigma_{ij}^{\alpha} = \sigma_{ij}^{*\alpha} = 0,\ \ i \neq j,\ \ (i,j) \notin \{(1,2),(2,1)\}.$$

4. A Chen $\delta(2,2)$ inequality

In this section, we establish Chen inequality for the invariant $\delta(2,2)$ for submanifolds in statistical manifolds of quasi-constant curvature. The following lemma has a major role in the proof of the our main result.

Lemma 4.1. *[18] Let $m \geq 4$ be an integer and $\{b_1, ..., b_m\}$ m real numbers. Then we have*

$$\sum_{1\leq i<j\leq m} b_i b_j - b_1 b_2 - b_3 b_4 \leq \frac{m-3}{2(m-2)}\left(\sum_{i=1}^{m} b_i\right)^2.$$

Equality holds if and only if $b_1 + b_2 = b_3 + b_4 = b_5 = ... = b_m$.

Let $p \in N$, $\pi_1, \pi_2 \subset T_pN$, mutually orthogonal, spanned respectively by $sp\{u_1, u_2\} = \pi_1$, $sp\{u_3, u_4\} = \pi_2$. Consider $\{u_1, ..., u_n\} \subset T_pN$, $\{u_{n+1}, ..., u_{n+m}\} \subset T_p^{\perp}N$. Then from (3.2), for the planes π_1 and π_2 we have

$$K(\pi_1) = a + b\left\{T(u_2)^2 + T(u_1)^2\right\} + K_0(\pi_1) - 2\overline{K}_0(\pi_1)$$
$$-\frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\right] - \frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\right] \tag{4.1}$$

and

$$K(\pi_2) = a + b\left\{T(u_4)^2 + T(u_3)^2\right\} + K_0(\pi_2) - 2\overline{K}_0(\pi_2)$$
$$-\frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{33}^{*\alpha}h_{44}^{*\alpha} - (h_{34}^{*\alpha})^2\right] - \frac{1}{2}\sum_{\alpha=1}^{m}\left[h_{33}^{\alpha}h_{44}^{\alpha} - (h_{34}^{\alpha})^2\right]. \tag{4.2}$$

From (3.4), (4.1) and (4.2),

$$(\tau - \tau_0) - (K(\pi_1) - K_0(\pi_1)) - (K(\pi_2) - K_0(\pi_2)) \geq a\frac{(n^2-n-4)}{2}$$

$$+b\left\{(n-1)\left\|P^T\right\|^2-T(u_2)^2-T(u_1)^2-T(u_4)^2-T(u_3)^2\right\}$$
$$-\frac{1}{2}\sum_{\alpha=1}^{m}\sum_{1\le i<j\le n}\left\{\left[h_{ii}^{\alpha}h_{jj}^{\alpha}-h_{11}^{\alpha}h_{22}^{\alpha}-h_{33}^{\alpha}h_{44}^{\alpha}\right]+\left[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-h_{11}^{*\alpha}h_{22}^{*\alpha}-h_{33}^{*\alpha}h_{44}^{*\alpha}\right]\right\}$$
$$-2\left(\overline{\tau}_0-\overline{K}_0(\pi_1)-\overline{K}_0(\pi_2)\right).$$

From Lemma 4.1,

$$\sum_{1\le i<j\le n}\left[h_{ii}^{\alpha}h_{jj}^{\alpha}-h_{11}^{\alpha}h_{22}^{\alpha}-h_{33}^{\alpha}h_{44}^{\alpha}\right]$$
$$\le\frac{n-3}{2(n-2)}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2=\frac{n^2(n-3)}{2(n-2)}(H^{\alpha})^2,$$

and similarly

$$\sum_{1\le i<j\le n}\left[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-h_{11}^{*\alpha}h_{22}^{*\alpha}-h_{33}^{*\alpha}h_{44}^{*\alpha}\right]$$
$$\le\frac{n-3}{2(n-2)}\left(\sum_{i=1}^{n}h_{ii}^{*\alpha}\right)^2=\frac{n^2(n-3)}{2(n-2)}(H^{*\alpha})^2.$$

Using the last two inequlities, we obtain the following inequality:

$$(\tau-\tau_0)-(K(\pi_1)-K_0(\pi_1))-(K(\pi_2)-K_0(\pi_2))\ge a\frac{(n^2-n-4)}{2}$$
$$+b\left\{(n-1)\left\|P^T\right\|^2-T(u_2)^2-T(u_1)^2-T(u_4)^2-T(u_3)^2\right\}$$
$$-\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)-2\left(\overline{\tau}_0-\overline{K}_0(\pi_1)-\overline{K}_0(\pi_2)\right).$$

So we state the following theorem.

Theorem 4.1. *Let $\overline{N}$ be an $(n+m)$-dimensional statistical manifold of quasi-constant curvature and N an n-dimensional statistical submanifold of $\overline{N}$. Then*

$$\tau_0-K_0(\pi_1)-K_0(\pi_2)\le\tau-K(\pi_1)-K(\pi_2)-a\frac{(n^2-n-4)}{2}$$
$$-b\left\{(n-1)\left\|P^T\right\|^2-T(u_2)^2-T(u_1)^2-T(u_4)^2-T(u_3)^2\right\}$$
$$+\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)+2\left(\widetilde{\tau}_0-\widetilde{K}_0(\pi_1)-\widetilde{K}_0(\pi_2)\right).$$

Moreover, the equality holds if and only if for any $1\le\alpha\le m$ we have

$$h_{11}^{\alpha}+h_{22}^{\alpha}=h_{33}^{\alpha}=...=h_{nn}^{\alpha},$$
$$h_{11}^{*\alpha}+h_{22}^{*\alpha}=h_{33}^{*\alpha}=...=h_{nn}^{*\alpha},$$
$$h_{ij}^{\alpha}=h_{ij}^{*\alpha}=0,\ \ i\ne j,\ \ (i,j)\notin\{(1,2),(2,1),(3,4),(4,3)\}.$$

If we consider statistical submanifold in statistical manifold of constant curvature we have the following corollary:

Corollary 4.1. *Let $\overline{N}$ be an $(n+m)$-dimensional statistical manifold of constant curvature and N an n-dimensional statistical submanifold of $\overline{N}$. Then*

$$\tau_0-K_0(\pi_1)-K_0(\pi_2)\le\tau-K(\pi_1)-K(\pi_2)-a\frac{(n^2-n-4)}{2}$$
$$+\frac{n^2(n-3)}{4(n-2)}\left(\|H\|^2+\|H^*\|^2\right)+2\left(\widetilde{\tau}_0-\widetilde{K}_0(\pi_1)-\widetilde{K}_0(\pi_2)\right).$$

Moreover, the equality is attained in the above inequality if and only if for any $1\le\alpha\le m$ we have

$$h_{11}^{\alpha}+h_{22}^{\alpha}=h_{33}^{\alpha}=...=h_{nn}^{\alpha},$$
$$h_{11}^{*\alpha}+h_{22}^{*\alpha}=h_{33}^{*\alpha}=...=h_{nn}^{*\alpha},$$
$$h_{ij}^{\alpha}=h_{ij}^{*\alpha}=0,\ \ i\ne j,\ \ (i,j)\notin\{(1,2),(2,1),(3,4),(4,3)\}.$$

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