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Research Article

Near best approximation property of interpolation and Poisson polynomials in weighted variable exponent Smirnov classes

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Abstract

Let G be a bounded Jordan domain in the complex plane $\mathbb{C}$. In this work under some restrictions of G the near best approximation property of complex interpolation and Poisson polynomials based on the Faber polynomials of $\overline{G}$ in the weighted variable exponent Smirnov classes $E_{\omega}^{p(\cdot)}(G)$ are proved.

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1. Introduction

Let $G \subset \mathbb{C}$ be a bounded Jordan domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ and let $G^{-} := Ext\ \Gamma$. Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} := Int\ \mathbb{T}$ and $\mathbb{D}^{-} := Ext\ \mathbb{T}$.

The weighted variable exponent Lebesgue spaces $L_{\omega}^{p(\cdot)}(\Gamma)$ for a given weight ω and Lebesgue measurable variable exponent $p(\cdot) \geq 1$ on Γ, we define as the set of Lebesgue measurable functions f, such that $\int_{\Gamma} |f(z)\omega(z)|^{p(z)} |dz| < \infty$. If $ess\sup_{z\in\Gamma} p(z) < \infty$, then $L_{\omega}^{p(\cdot)}(\Gamma)$ is a Banach space equipped with the norm

$$\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} := \inf\left\{\lambda > 0 : \int\limits_{\Gamma} |f(z)\omega(z)/\lambda|^{p(z)} |dz| \leq 1\right\} < \infty.$$

If $p(\cdot) \equiv p$, it is the classical weighted Lebesgue space $L_{\omega}^{p}(\Gamma)$. In the case of $\omega(\cdot) \equiv 1$, it turns to variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ endowed with the norm $\|\|_{L^{p(\cdot)}(\Gamma)} := \|\|_{L_{1}^{p(\cdot)}(\Gamma)}$, investigated in [20]. For real variable exponents $p(\cdot)$ detailed information about the variable exponent Lebesgue spaces can be found in the monographs [4, 10, 28].

By $E^{p}(G)$ we denote the Smirnov class of analytic functions in G. As is known if $f \in E^{p}(G)$, then there exists a sequence (γ_n) of the rectifiable Jordan curves $\gamma_n \subset G$, $n =$

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$1, 2, \ldots$, tending to Γ in the sense that γ_n eventually surrounds each compact subdomain of G such that

$$\int\limits_{\gamma_n} |f(z)|^p |dz| \leq M < \infty, \; 1 \leq p < \infty.$$

Each function $f \in E^p(G)$ has [8, pp. 419-438] non-tangential boundary values almost everywhere ($a.e$) on Γ and the boundary function belongs to $L^p(\Gamma)$. The set

$$E_\omega^{p(\cdot)}(G) := \left\{ f \in E^1(G) : f \in L_\omega^{p(\cdot)}(\Gamma) \right\}$$

is called the weighted variable exponent Smirnov class of analytic functions in G. If $\omega \equiv 1$, then it turns to variable exponent Smirnov class $E^{p(\cdot)}(G)$, considered by us in [18]. Under the condition $1 \leq ess\sup_{z\in\Gamma} p(z) < \infty$, $E_\omega^{p(\cdot)}(G)$ becomes a Banach space with the norm

$$\|f\|_{E_\omega^{p(\cdot)}(G)} := \|f\|_{L_\omega^{p(\cdot)}(\Gamma)}.$$

Let $\mathcal{E}$ be the segment $[0, 2\pi]$ or a Jordan rectifiable curve Γ and let $p(\cdot) : \mathcal{E} \to \mathbb{R}^+ := (0, \infty)$ be a Lebesgue measurable function on $\mathcal{E}$ such that

$$1 < p_- := ess\inf_{z\in\mathcal{E}} p(z) \leq ess\sup_{z\in\mathcal{E}} p(z) =: p^+ < \infty. \tag{1.1}$$

Definition 1.1. We say that $p(\cdot) \in \mathcal{P}_0(\mathcal{E})$, if $p(\cdot)$ satisfies the conditions (1.1) and the inequality

$$|p(z_1) - p(z_2)| \ln(|\mathcal{E}| / |z_1 - z_2|) \leq c(p), \quad \forall z_1, z_2 \in \mathcal{E}, \; z_1 \neq z_2$$

with a positive constant $c(p)$, where $|\mathcal{E}|$ is the Lebesgue measure of $\mathcal{E}$.

Let g be a continuous real variable function and let

$$\omega(g, t) := \sup_{|t_1 - t_2| \leq t} |g(t_1) - g(t_2)|, \quad t_1, t_2 \in (0, \infty), \quad t > 0,$$

be its modulus of continuity, defined on $[0, \infty)$.

Definition 1.2. If

$$\sup_{B_j} |B_j \cap \Gamma|^{-1} \left\| \omega \chi_{B_j} \right\|_{L^{p(\cdot)}(\Gamma)} \left\| \omega^{-1} \chi_{B_j} \right\|_{L^{q(\cdot)}(\Gamma)} < \infty, \quad 1/p(\cdot) + 1/q(\cdot) = 1$$

for a given exponent $p(\cdot)$ defined on Γ, where the supremum is taken over all discs B_j with the characteristic functions χ_{B_j}, then we write $\omega \in A_{p(\cdot)}(\Gamma)$.

Definition 1.3. Let Γ be a smooth Jordan curve and let $\theta(s)$ be the angle between the tangent and the positive real axis expressed as a function of arclength s. If $\theta(s)$ has a modulus of continuity $\omega(\theta, t)$, satisfying the Dini-smooth condition:

$$\int_0^\delta [\omega(\theta, t) / t] \, dt < \infty, \; \delta > 0,$$

then we say that Γ is a Dini-smooth curve. The set of Dini-smooth curves we denote by $\mathfrak{D}$.

Let $\Gamma \in \mathfrak{D}$. By φ we denote the conformal mapping of G^- onto $\mathbb{D}^-$, normalized by the conditions: $\varphi(\infty) = \infty$, $\lim_{z\to\infty} \varphi(z)/z > 0$. Let ψ be the inverse mapping of φ. The mappings φ and ψ have continuous extensions to Γ and $\mathbb{T}$, respectively. Their derivatives φ' and ψ' have definite nontangential boundary values $a.e.$ on Γ and $\mathbb{T}$, and the boundary functions are Lebesgue integrable on Γ and $\mathbb{T}$, respectively [8, p. 419-438].

For $f \in L_\omega^{p(\cdot)}(\Gamma)$, $p \in \mathcal{P}_0(\Gamma)$ and $\omega \in A_{p(\cdot)}(\Gamma)$ we set $f_0 := f \circ \psi$, $p_0 := p \circ \psi$, and $\omega_0 := \omega \circ \psi$. If $\Gamma \in \mathfrak{D}$, then by [33]

$$f \in L_\omega^{p(\cdot)}(\Gamma) \Leftrightarrow f_0 \in L_{\omega_0}^{p_0(\cdot)}(\mathbb{T}), \; p \in \mathcal{P}_0(\Gamma) \Leftrightarrow p_0 \in \mathcal{P}_0(\mathbb{T}) \tag{1.2}$$
$$\text{and } \omega \in A_{p(\cdot)}(\Gamma) \Leftrightarrow \omega_0 \in A_{p_0(\cdot)}(\mathbb{T}).$$

Let F_k, $k = 0, 1, 2, ..$, be the Faber polynomials of $\overline{G}$, which can be defined by the series representation [32]:

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \quad w \in \mathbb{D}^-, \, z \in G, \tag{1.3}$$

i.e., as the coefficients of Laurent series expansion of $\psi'(w) / [\psi(w) - z]$ in a neighborhood of ∞. Therefore, F_k has the integral representation

$$F_k(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^k \psi'(w)}{\psi(w) - z} dw, \quad z \in G. \tag{1.4}$$

Using Cauchy's integral representation

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)\psi'(w)}{\psi(w) - z} dw, \; z \in G,$$

which holds for every $f \in E_{\omega}^{p(\cdot)}(G) \subset E^1(G)$, and (1.3) we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z) \;, z \in G, \tag{1.5}$$

where $a_k(f)$, $k = 0, 1, 2, ..$, are the Faber coefficients of f, defined by

$$a_k = a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw. \tag{1.6}$$

Let γ be an oriented rectifiable curve. For a given $z \in \gamma$ and $\delta > 0$, by $s_+(z, \delta)$ (respectively by $s_-(z, \delta)$) we denote the subarc of γ, in the positive (respectively negative) orientation of γ, with the starting point z, such that arc length from z to each point less than δ.

If γ is smooth and the equality

$$\lim_{\delta \to 0} \left\{ \int_{s_-(z,\delta)} |d_\varsigma \arg(\varsigma - z)| + \int_{s_+(z,\delta)} |d_\varsigma \arg(\varsigma - z)| \right\} = 0$$

holds uniformly for $z \in \gamma$, then it is said [35] that γ is of vanishing rotation (VR). In [35] L. Zhong and L. Zhu proved that there exists a smooth curve which is not of VR . On the other hand, if $\gamma \in \mathfrak{D}$, then γ is VR (see, [35]).

Definition 1.4 ([6]). Let γ be a rectifiable Jordan curve with length L and let $z = z(t)$,$t \in [0, L]$, be its parametric representation. If $\beta(t) := \arg z'(t)$ can be defined on $[0, L]$ to become a function of bounded variation, then γ is called of bounded rotation (BR) and $\int_\Gamma |d\beta(t)|$ is called total rotation of γ.

If $\gamma \in BR$, then there are two half tangents at each point of γ.

The class of bounded rotation curves is sufficiently wide. For example, a curve which is made up of finitely many convex arcs (corners are permitted) is bounded rotation [7]. Every VR curve and also a piecewise VR curve considered in [35] is BR curve. As mentioned above, a BR curve may have cusps or corners. Moreover, there exists a BR curve which is not a VR curve, for instance a rectangle in the plane.

Let Π_n be the class of algebraic polynomials of degree not exceeding n and let

$$E_n(f)_{G,p(\cdot),\omega} := \inf \left\{ \|f - P_n\|_{L_{\omega}^{p(\cdot)}(\Gamma)} : \; P_n \in \Pi_n \right\} \;, \; n = 1, 2, ...$$

be the best approximation numbers of $f \in E_{\omega}^{p(\cdot)}(G)$ in Π_n.

Supposing that all of the zeros z_k, $k = 1, 2, ..., n$, of the nth Faber polynomial F_n are in G, we construct the $(n-1)\,th$ Lagrange interpolating polynomial $L_{n-1}(f, z)$ for $f \in E_{\omega}^{p(\cdot)}(G)$, with the interpolation nodes z_k, $k = 1, 2, ..., n$, having the integral representation

$$L_{n-1}(f, z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\omega_n^*(t) - \omega_n^*(z)}{\omega_n^*(t)} \frac{f(t)}{t - z} dt,$$

where $\omega_n^*(z) := \Pi_{k=1}^n (z - z_k)$.

We also define the Poisson polynomial for a given function $f \in E_{\omega}^{p(\cdot)}(G)$ as

$$V_n(f, z) := \sum_{k=0}^{n} a_k F_k(z) + \sum_{k=n+1}^{2n-1} (2 - k/n)\, a_k F_k(z) \ \ , z \in G,$$

where a_k, $k = 0, 1, ...$, are the Faber coefficients of f, defined by (1.6).

The Faber polynomials have different applications in complex analysis, especially in approximation theory they can be used for construction of approximation polynomials in the classical Smirnov classes. Detailed information about these polynomials and theirs applications in approximation theory are given in the monographs [6, 31, 32]. Relatively new results obtained in the different generalizations of Smirnov classes can be found in the works: [13–17, 22–24]. They are important also in the case of variable exponent Smirnov classes (see, for example: [18, 19, 33]).

In this work we prove the near best approximation property of complex interpolation polynomials, constructed on the zeros of Faber polynomials, and also of Poisson polynomials in the weighted variable exponent Smirnov classes $E_{\omega}^{p(\cdot)}(G)$.

Let

$$M_{\Gamma}(f)(z) := \sup_{r>0} \frac{1}{|\Gamma(z, r)|} \int\limits_{\Gamma(z,r)} |f(\zeta)|\, d\zeta$$

be the Hardy-Littlewood maximal function of $f \in L_{\omega}^{p(\cdot)}(\Gamma)$, where

$$\Gamma(z, r) := \{t \in \Gamma : |t - z| < r\}$$

with the Lebesgue measure $|\Gamma(z, r)|$ for $z \in \Gamma$ and $r > 0$.

Definition 1.5. If the Hardy-Littlewood maximal operator $M_{\Gamma} : f \to M_{\Gamma}(f)$ is bounded in $L_{\omega}^{p(\cdot)}(\Gamma)$ then we write $\omega \in \mathfrak{A}_{p(\cdot)}(\Gamma)$.

Our new results are following:

Theorem 1.6. *Let Γ be a BR curve without cusps. If $p(\cdot) \in \mathcal{P}_0(\Gamma)$ and $\omega^{-p_*} \in \mathfrak{A}_{\frac{p(\cdot)}{p(\cdot)-p_*}}(\Gamma)$ for some $p_* \in (1, p_-)$, then there exists a positive constant $c(p, \Gamma)$ such that for every $f \in E_{\omega}^{p(\cdot)}(G)$ and $n = 1, 2, ...$ the inequality*

$$\|f - L_{n-1}(f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \leq c(p, \Gamma)\, E_{n-1}(f)_{G, p(\cdot), \omega}$$

holds.

Since the Hardy-Littlewood maximal operator $M_{\Gamma} : f \to M_{\Gamma}(f)$ is bounded in the non-weighted Lebesgue space $L^{p(\cdot)}(\Gamma)$, in the case of $\omega \equiv 1$ we have:

Corollary 1.7. *Let Γ be a BR curve without cusps. If $p(\cdot) \in \mathcal{P}_0(\Gamma)$, then there exists a positive constant $c(p, \Gamma)$ such that for every $f \in E^{p(\cdot)}(G)$ and $n = 1, 2, ...$ the inequality*

$$\|f - L_{n-1}(f)\|_{L^{p(\cdot)}(\Gamma)} \leq c(p, \Gamma)\, E_{n-1}(f)_{G, p(\cdot)}$$

holds.

The following theorem expresses the near best approximation property of Poisson polynomials in $E_{\omega}^{p(\cdot)}(G)$:

Theorem 1.8. *Let $\Gamma \in \mathfrak{D}$. If $p(\cdot) \in \mathcal{P}_0(\Gamma)$ and $\omega \in A_{p(\cdot)}(\Gamma)$, then there exists a positive constant $c(p,\Gamma)$ such that for every $f \in E_\omega^{p(\cdot)}(G)$ and $n = 1, 2, ...$ the inequality*

$$\|f - V_n(f)\|_{L_\omega^{p(\cdot)}(\Gamma)} \leq c(p,\Gamma)\, E_{n-1}(f)_{G,p(\cdot),\omega}$$

holds.

In the classical Smirnov spaces these problems were investigated by many authors. In particular, in [27] X. C. Shen and L. Zhong obtain a series of interpolation nodes in $\overline{G}$ and show that in the case of $\Gamma \in C(2,\alpha)$, $0 < \alpha < 1$, the interpolating and best approximating polynomials have the same order of convergence in $E^p(G)$, *i.e.*, the interpolating polynomials have near best approximation property in $E^p(G)$. Later L. Y. Zhu [36], choosing the interpolation nodes as the zeros of Faber polynomials of $\overline{G}$, obtained similar results under the condition of $\Gamma \in C(1,\alpha)$. Note that in these works Γ does not admit corners. When Γ is a piecewise VR curve without cusps, in [35] L. Zhoung and L. Zhu showed that the interpolating polynomials, based on the zeros of Faber polynomials of $\overline{G}$, converge in the Smirnov classes $E^p(G)$, $1 < p < \infty$. In the reflexive Smirnov-Orlicz classes and weighted symmetric Smirnov classes similar problems, when Γ is a BR curve without cusps, investigated in [1] and [3], respectively.

Relating to the Poisson polynomials it is worth noting that the near best approximation property of these polynomials in the uniform norm and in the weighted Smirnov-Orlicz classes was proved in [30] and [2], respectively.

Note that the quantities $\|f - L_{n-1}(f)\|_{L_\omega^{p(\cdot)}(\Gamma)}$ and $\|f - V_n(f)\|_{L_\omega^{p(\cdot)}(\Gamma)}$, estimated in Theorems 1.6 and 1.8 can be also estimated by the *modulus of smoothness* $\Omega_r(f,\delta)_{G,p(\cdot),\omega}$, defined below.

Definition 1.9. For $g \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\omega \in A_{p(\cdot)}(\mathbb{T})$, we set

$$\Delta_t^r g(w) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} g\left(we^{ist}\right), \quad r = 1, 2, 3, ..., \quad t > 0$$

and

$$\Omega_r(g,\delta)_{\mathbb{T},p(\cdot),\omega} := \sup_{0<|h|\leq\delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r g(w)\, dt \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}.$$

For a given function $f \in L_\omega^{p(\cdot)}(\Gamma)$ we define the Cauchy type integral

$$f_0^+(w) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D}, \quad f_0 := f \circ \psi$$

which is analytic in $\mathbb{D}$.

Motivating from (1.2) we can define the modulus of smoothness for $f \in E_\omega^{p(\cdot)}(G)$ as $\Omega_r(f,\delta)_{G,p(\cdot),\omega} : = \Omega_r\left(f_0^+,\delta\right)_{\mathbb{T},p_0(\cdot),\omega_0}$, $\delta > 0$. The following theorem in the case of $r = 1$ was proved in [33]. For $r > 1$ it can be proved using [34] by similar way.

Theorem 1.10. *Let $\Gamma \in \mathfrak{D}$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$, $r = 1, 2, \ldots$, and $\omega \in A_{p(\cdot)}(\Gamma)$. If $f \in E_\omega^{p(\cdot)}(G)$, then there is a positive constants $c(p,\Gamma)$ such that the inequality*

$$E_n(f)_{G,p(\cdot),\omega} \leq c(p,\Gamma)\, \Omega_r(f, 1/n)_{G,p(\cdot),\omega}, \quad n = 1, 2, 3, ...,$$

holds.

Now combining Theorems 1.6 and 1.8 respectively with Theorem 1.10, we have

Theorem 1.11. *Let* $\Gamma \in \mathfrak{D}$. *If* $p(\cdot) \in \mathcal{P}_0(\Gamma)$ *and* $\omega \in A_{p(\cdot)}(\Gamma)$, *then there exists a positive constant* $c(p,\Gamma)$ *such that for every* $f \in E_\omega^{p(\cdot)}(G)$ *and* $n = 1, 2, \ldots$ *the inequality*

$$\|f - L_{n-1}(f)\|_{L_\omega^{p(\cdot)}(\Gamma)} \leq c(p,\Gamma)\,\Omega_r(f, 1/n)_{G,p(\cdot),\omega}$$

holds.

Theorem 1.12. *Let* $\Gamma \in \mathfrak{D}$. *If* $f \in E_\omega^{p(\cdot)}(G)$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$ *and* $\omega \in A_{p(\cdot)}(\Gamma)$, *then there is a positive constant* $c(p,\Gamma)$ *such that the inequality*

$$\|f - V_n(f,z)\|_{L_\omega^{p(\cdot)}(\Gamma)} \leq c(p,\Gamma)\,\Omega_r(f, 1/n)_{G,p(\cdot),\omega}$$

holds.

Throughout this paper by $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot)$, and $c(\cdot,\cdot)$, $c_1(\cdot,\cdot)$, $c_2(\cdot,\cdot)$,..., we denote the constants depending in general only on the parameters given in the corresponding brackets.

2. Auxiliary results

Let Γ be a *BR* curve without cusps. Then (see, [26])

$$F_n(z) = \frac{1}{\pi}\int_\Gamma [\varphi(\varsigma)]^n\, d_\varsigma \arg(\varsigma - z),\ z \in \Gamma,$$

where the jump of $\arg(\varsigma - z)$ at $\varsigma = z$ is equal to exterior angle $\alpha_z\pi$ in z. Hence we have

$$0 \leq \max_{z\in\Gamma} |\alpha_z - 1| < 1. \tag{2.1}$$

Lemma 2.1 ([1], Lemma 4)**.** *Let* Γ *be a BR curve without cusps. Then for arbitrary* $\varepsilon > 0$, *there exists a positive integer* n_0 *such that the inequality*

$$|F_n(z) - [\varphi(z)]^n| < |\alpha_z - 1| + \varepsilon,\quad z \in \Gamma,$$

holds for every $n > n_0$.

Let

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma\setminus\{\zeta\in\Gamma:\ |\zeta - z|<\varepsilon\}} \frac{f(\zeta)}{\zeta - z}d\zeta,\quad z \in \Gamma$$

be the Cauchy singular integral of $f \in L_\omega^{p(\cdot)}(\Gamma)$. By Privalov's theorem the Cauchy type integrals

$$f^+(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,\quad z \in G,\qquad f^-(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,\quad z \in G^-,$$

have the nontangential inside and outside limits f^+ and f^-, respectively *a.e.* on Γ. Furthermore, the formulas

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z)\quad\text{and}\quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z)$$

are valid *a.e.* on Γ, which imply that

$$f(z) = f^+(z) - f^-(z) \tag{2.2}$$

a.e. on Γ.

Lemma 2.2 ([33], Lemma 5)**.** *Let* $\Gamma \in \mathfrak{D}$ *and* $p(\cdot) \in \mathcal{P}_0(\Gamma)$. *If* $f \in L_\omega^{p(\cdot)}(\Gamma)$ *and* $\omega \in A_{p(\cdot)}(\Gamma)$, *then* $f^+ \in E_\omega^{p(\cdot)}(G)$.

For $z \in \Gamma$ and $\varepsilon > 0$ let $\Gamma(z,\varepsilon) := \{t \in \Gamma : |t-z| < \varepsilon\}$ with the Lebesgue measure $|\Gamma(z,\varepsilon)|$. We recall that a rectifiable Jordan curve Γ is called a Carleson curve if $\sup_{\varepsilon>0}\sup_{z\in\Gamma}|\Gamma(z,\varepsilon)|/\varepsilon < \infty$. The class of Carleson curves is sufficiently wide. In particular, every BR curve and Dini-smooth curve is a Carleson curve. It is also known that if Γ is a Carleson curve and ω belongs to classical Muckenhoupt class $A_p(\Gamma)$, then Cauchy's singular operator $S_\Gamma : f \to S_\Gamma(f)$ is a bounded operator in the weighted space $L^p_\omega(\Gamma)$, $1 < p < \infty$ (see [11, p.89] and [9]). In the case of $\Gamma \in \mathfrak{D}$, similar fact in $L^{p(\cdot)}_\omega(\Gamma)$ was cited in [33] based on *Theorems 2.4* and *2.7* from [5].

Lemma 2.3 ([25], Theorem 4.21)**.** *Let Γ be a Carleson curve and $p \in \mathcal{P}_0(\Gamma)$. If $\omega^{-p_*} \in \mathfrak{A}_{\frac{p(\cdot)}{p(\cdot)-p_*}}(\Gamma)$ for some $p_* \in (1, p_-)$, then the Cauchy singular operator S_Γ is bounded in $L^{p(\cdot)}_\omega(\Gamma)$, i.e., there exists a constant $c(p,\Gamma)$ such that for every $f \in L^{p(\cdot)}_\omega(\Gamma)$ the inequality $\|S_\Gamma(f)\|_{L^{p(\cdot)}_\omega(\Gamma)} \le c(p,\Gamma)\|f\|_{L^{p(\cdot)}_\omega(\Gamma)}$ holds.*

Let $S_n(f) = \sum\limits_{k=-n}^{n} c_k e^{ikt}$, $n \in \mathbb{N}$, be the nth partial sums of Fourier series of $f \in L^1(\mathbb{T})$, with the Fourier coefficients

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}\right)e^{-ikt}dt,$$

and let

$$\sigma_n(f)\left(e^{it}\right) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f)$$

be its Fejér means. Then the inequality

$$\|\sigma_n(f)\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \le c(p)\|f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \tag{2.3}$$

holds [21, Lemma 4] for every $f \in L^{p(\cdot)}_\omega(\mathbb{T})$, $p \in \mathcal{P}_0(\mathbb{T})$ and $\omega \in A_{p(\cdot)}(\mathbb{T})$, which in the non-weighted case was also indicated in [29, Corollary 1].

Now let $\mathcal{T}_n$ be the class of trigonometric polynomials of degree not exceeding n and let $E_n(f)_{p(\cdot),\omega} := \inf_{T_n\in\mathcal{T}_n}\left\{\|f-T_n\|_{L^{p(\cdot)}_\omega(\mathbb{T})}\right\}$, $n = 1, 2, ...$ be the best approximation number of $f \in L^{p(\cdot)}_\omega(\mathbb{T})$ in $\mathcal{T}_n$.

The following lemma holds:

Lemma 2.4. *Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p \in \mathcal{P}_0(\mathbb{T})$ and $\omega \in A_{p(\cdot)}(\mathbb{T})$. Then there exists a positive constant $c(p)$ such that*

$$\|\sigma_n(f) - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \le c(p)E_n(f)_{p(\cdot),\omega}, n \in \mathbb{N}.$$

Proof of Lemma 2.4. Let T_n, $n = 1, 2, ...$, be the trigonometric polynomials of the best best approximation to $f \in L^{p(\cdot)}(\mathbb{T})$. Then by (2.3) we have

$$\begin{aligned}
\|\sigma_n(f) - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} &\le \|\sigma_n(f) - T_n\|_{L^{p(\cdot)}_\omega(\mathbb{T})} + \|T_n - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\
&\le \|\sigma_n(f) - \sigma_n(T_n)\|_{L^{p(\cdot)}_\omega(\mathbb{T})} + \|T_n - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\
&= \|\sigma_n(f - T_n)\|_{L^{p(\cdot)}_\omega(\mathbb{T})} + \|T_n - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\
&\le c_1(p)\|T_n - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} + \|T_n - f\|_{L^{p(\cdot)}_\omega(\mathbb{T})} \\
&= c(p)E_n(f)_{p(\cdot),\omega}.
\end{aligned}$$

$\square$

For $z \in G$ we consider the operator

$$T(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w)\,\psi'(w)}{\psi(w) - z} dw, \quad f \in E_{\omega}^{p(\cdot)}(\mathbb{D}) .$$

Lemma 2.5 ([33], Theorem 9)**.** *Let $\Gamma \in \mathfrak{D}$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$ and $\omega \in A_{p(\cdot)}(\Gamma)$. The operator $T : E_{\omega_0}^{p_0(\cdot)}(\mathbb{D}) \to E_{\omega}^{p(\cdot)}(G)$ is linear, bounded, one-to-one and onto. Moreover, $T\left(f_0^+\right) = f$ for every $f \in E_{\omega}^{p(\cdot)}(G)$.*

Lemma 2.6 ([33], Lemma 6)**.** *Let $\Gamma \in \mathfrak{D}$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$ and $\omega \in A_{p(\cdot)}(\Gamma)$. If $f \in E_{\omega}^{p(\cdot)}(G)$ then there exist the positive constants $c_i(p, \Gamma)$, $i = 2, 3$, such that the following inequalities hold:*

$$E_n\left(f_0^+\right)_{p_0(\cdot),\omega_0} \le c_2(p, \Gamma) E_n(f)_{G,p(\cdot),\omega} \le c_3(p, \Gamma) E_n\left(f_0^+\right)_{p_0(\cdot),\omega_0} .$$

3. Proof of main results

Proof of Theorem 1.6. Let $\kappa := \max_{z\in\Gamma} |\alpha_z - 1|$. Then by (2.1) we have $0 \le \kappa < 1$. Setting $\varepsilon := (1 - \kappa)/2$ in Lemma 2.1, for sufficiently large n we get

$$|F_n(z) - [\varphi(z)]^n| < (1 + \kappa)/2, \; z \in \Gamma. \tag{3.1}$$

Since $F_n(z) - [\varphi(z)]^n$ is analytic on $C\overline{G} := \overline{\mathbb{C}}\backslash\overline{G}$, by maximum principle

$$|F_n(z) - [\varphi(z)]^n| < (1 + \kappa)/2, \; z \in CG,$$

and therefore,

$$|F_n(z)| \ge |[\varphi(z)]^n| - (1 + \kappa)/2 \ge (1 - \kappa)/2 > 0, \;\; z \in CG.$$

By choosing the interpolation nodes as the zeros of Faber polynomials, we have

$$\begin{aligned} f(z') - L_{n-1}(f, z') &= \frac{F_n(z')}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{F_n(\varsigma)(\varsigma - z')} d\varsigma \\ &= F_n(z') \left[(f/F_n)^+(z')\right], \;\; z' \in G. \end{aligned}$$

Taking here the limit $z' \to z \in \Gamma$, along all nontangential paths inside of Γ, and using (2.2) and Lemma 2.3 we get

$$\begin{aligned} \|f - L_{n-1}(f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} &= \left\| F_n\left[S_{\Gamma}(f/F_n) + \frac{1}{2} f/F_n\right]\right\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \\ &\le \|F_n[S_{\Gamma}(f/F_n)]\|_{L_{\omega}^{p(\cdot)}(\Gamma)} + \frac{1}{2}\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \\ &\le \left\{\max_{z\in\Gamma}|F_n(z)|\right\} \|S_{\Gamma}(f/F_n)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} + \frac{1}{2}\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \\ &\le c_4(p, \Gamma)\left\{\max_{z,\varsigma\in\Gamma}|F_n(z)/F_n(\varsigma)|\right\} \|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} + \frac{1}{2}\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} . \end{aligned}$$

By (3.1), $(1 - \kappa)/2 < |F_n(z)| < (3 + \kappa)/2$, $\; z \in \Gamma$, and hence

$$\|f - L_{n-1}(f, \cdot)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \le \left(c_4(p, \Gamma)\frac{3 + \kappa}{1 - \kappa} + \frac{1}{2}\right)\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)}, \; z \in \Gamma.$$

Since

$$\begin{aligned} \|L_{n-1}(f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} &\le \|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} + \|f - L_{n-1}(f, \cdot)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \\ &\le \left(c_4(p, \Gamma)\frac{3 + \kappa}{1 - \kappa} + \frac{3}{2}\right) c_5(p, \Gamma)\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)} \\ &= c_6(p, \Gamma)\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)}, \end{aligned} \tag{3.2}$$

we obtain that for the large values of n, $L_{n-1}(f,z)$ is uniformly bounded in $E_{\omega}^{p(\cdot)}(G)$.

Let P_{n-1} be $(n-1)\,th$ best approximating polynomial to f in $E_{\omega}^{p(\cdot)}(G)$. Since L_{n-1} is a linear interpolating polynomial operator, by (3.2) we have

$$\begin{aligned}
\|f-L_{n-1}(f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)} &\leq \|f-P_{n-1}\|_{L_{\omega}^{p(\cdot)}(\Gamma)}+\|P_{n-1}-L_{n-1}(f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)}\\
&= \|f-P_{n-1}\|_{L_{\omega}^{p(\cdot)}(\Gamma)}+\|L_{n-1}(P_{n-1}-f)\|_{L_{\omega}^{p(\cdot)}(\Gamma)}\\
&\leq (1+\|L_{n-1}\|)\,\|f-P_{n-1}\|_{L_{\omega}^{p(\cdot)}(\Gamma)}\\
&\leq \left(1+c_6(p,\Gamma)\,\|f\|_{L_{\omega}^{p(\cdot)}(\Gamma)}\right)E_{n-1}(f)_{G,p(\cdot),\omega}\\
&= c(p,\Gamma)\,E_{n-1}(f)_{G,p(\cdot),\omega}.
\end{aligned}$$

□

Proof of Theorem 1.8 . By simple calculations we have

$$\begin{aligned}
V_n(f,z) \quad : &= \sum_{k=0}^{n}a_kF_k(z)+\sum_{k=n+1}^{2n-1}\left(2-\frac{k}{n}\right)a_kF_k(z)\\
&= \sum_{k=0}^{2n-1}\left(2-\frac{k}{n}\right)a_kF_k(z)-\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)a_kF_k(z)\\
&= 2\sum_{k=0}^{2n-1}\left(1-\frac{k}{2n}\right)a_kF_k(z)-\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)a_kF_k(z).
\end{aligned}$$

Denoting by $\sigma_{n-1}(f,G):=\frac{1}{n}\sum\limits_{k=0}^{n-1}\sum\limits_{j=0}^{k}a_jF_j(z)$ the Fejér means for f, constructed via the Faber polynomials $F_j(z)$ of $\overline{G}$, and taken the relations

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)a_kF_k(z) &= \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)a_kF_k(z)\\
&= a_0F_0(z)+\sum_{k=1}^{n-1}\frac{n-k}{n}a_kF_k(z)\\
&= \frac{1}{n}\left[na_0F_0(z)+\sum_{k=1}^{n-1}(n-k)\,a_kF_k(z)\right]\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k}a_jF_j(z)=:\sigma_{n-1}(f,G),
\end{aligned}$$

and

$$\sum_{k=0}^{2n-1}\left(1-\frac{k}{2n}\right)a_kF_k(z)=\frac{1}{2n}\sum_{k=0}^{2n-1}\sum_{j=0}^{k}a_jF_j(z)=\sigma_{2n-1}(f,G)$$

into account, we have

$$V_n(f,z)=2\sigma_{2n-1}(f,G)-\sigma_{n-1}(f,G). \tag{3.3}$$

If $f\in E_{\omega}^{p(\cdot)}(G)$, $p(\cdot)\in\mathcal{P}_0(\Gamma)$ and $\omega\in A_{p(\cdot)}(\Gamma)$, then by (1.2) and Lemma 2.2 we have $f_0^+\in E_{\omega_0}^{p_0(\cdot)}(\mathbb{D})\subset E^1(\mathbb{D})$. Hence the boundary function of f_0^+ belongs to $L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})$. On the other hand, f_0^+ is analytic function on the unit disk $\mathbb{D}$, it has the Taylor series expansion:

$$f_0^+(w)=\sum_{k=0}^{\infty}\beta_k\left(f_0^+\right)w^k,\ \ w\in\mathbb{D}.$$

By Theorem 3.4 in [12, p. 38]

$$c_k\left(f_0^+\right)=\begin{cases}\beta_k\left(f_0^+\right) & k\geq 0\\ 0 & k<0,\end{cases}$$

where $c_k\left(f_0^+\right)$, $k\in\mathbb{Z}$, are the Fourier coefficients of the boundary function of $f_0^+\in L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})\subset L^1(\mathbb{T})$, and then we have

$$f_0^+(w)=\sum_{k=0}^{\infty}c_k\left(f_0^+\right)w^k. \tag{3.4}$$

On the other hand, by (2.2), $f_0=f_0^+-f_0^-$ on $\mathbb{T}$ with $f_0^+\in E_{\omega_0}^{p_0(\cdot)}(\mathbb{D})$ and $f_0^-\in E_{\omega_0}^{p_0(\cdot)}(\mathbb{D}^-)$. Then

$$a_k(f)=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_0^-(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_0^+(w)}{w^{k+1}}dw=\beta_k\left(f_0^+\right),$$

i.e., the Faber coefficients $a_k(f)$, $k=0,1,2,...$, of $f\in E_\omega^{p(\cdot)}(G)$ are the Taylor coefficients of $f_0^+\in E_{\omega_0}^{p_0(\cdot)}(\mathbb{D})$. If $\sum_{k=0}^{\infty}a_k(f)F_k(z)$ is the Faber series expansion of $f\in E_\omega^{p(\cdot)}(G)$, then by (3.4), (1.5) and by definition of the operator $T(f)$, we have that $T\left(\sum_{k=0}^{n}c_k\left(f_0^+\right)w^k\right)=\sum_{k=0}^{n}a_k(f)F_k(z)$ and $T\left(\sigma_n\left(f_0^+\right)\right)=\sigma_n(f,G)$.

Now, using the decreasing property of the sequence $\left\{E_n\left(f_0^+\right)_{p_0(\cdot),\omega_0}\right\}_{n=1}^{\infty}$ and also (3.3), Lemmas 2.5, 2.4 and 2.6 we get

$$\begin{aligned}
\|f-V_n(f,z)\|_{L_\omega^{p(\cdot)}(\Gamma)} &\leq 2\|f-\sigma_{2n-1}(f,G)\|_{L_\omega^{p(\cdot)}(\Gamma)}\\
&\quad+\|f-\sigma_{n-1}(f,G)\|_{L_\omega^{p(\cdot)}(\Gamma)}\\
&= 2\left\|T\left(f_0^+\right)-T\left(\sigma_{2n-1}\left(f_0^+\right)\right)\right\|_{L_\omega^{p(\cdot)}(\Gamma)}\\
&\quad+\left\|T\left(f_0^+\right)-T\left(\sigma_{n-1}\left(f_0^+\right)\right)\right\|_{L_\omega^{p(\cdot)}(\Gamma)}\\
&= 2\left\|T\left(f_0^+-\sigma_{2n-1}\left(f_0^+\right)\right)\right\|_{L_\omega^{p(\cdot)}(\Gamma)}\\
&\quad+\left\|T\left(f_0^+-\sigma_{n-1}\left(f_0^+\right)\right)\right\|_{L_\omega^{p(\cdot)}(\Gamma))}\\
&\leq c_7(p,\Gamma)\left\|f_0^+-\sigma_{2n-1}\left(f_0^+\right)\right\|_{L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})}\\
&\quad+c_8(p,\Gamma)\left\|f_0^+-\sigma_{n-1}\left(f_0^+\right)\right\|_{L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})}\\
&\leq c_9(p,\Gamma)E_{2n-1}\left(f_0^+\right)_{p_0(\cdot),\omega_0}\\
&\quad+c_{10}(p,\Gamma)E_{n-1}\left(f_0^+\right)_{p_0(\cdot),\omega_0}\\
&\leq c_{11}(p,\Gamma)E_{n-1}\left(f_0^+\right)_{p_0(\cdot),\omega_0}\\
&\leq c(p,\Gamma)E_{n-1}(f)_{G,p(\cdot),\omega}.
\end{aligned}$$

□

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