

Solving NLP Problems with Dynamic System Approach Based on Smoothed Penalty Function

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Abstract. In this work, a dynamical system approach for solving nonlinear programming (NLP) problem based on a smoothed penalty function is investigated. The proposed approach shows that an equilibrium point of the dynamic system is stable and converge to optimal solutions of the corresponding nonlinear programming problem. Furthermore, relationships between optimal solutions for smooth and nonsmooth penalty problem are discussed. Finally, two practical examples are illustrated the applicability of the proposed dynamic system approach with Euler scheme.

Key words: Nonlinear programming, penalty function, dynamic system, lyapunov stability, smoothing method.

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1.Introduction

Consider the nonlinear programming (NLP) problem

$$(1) \quad \begin{array}{ll} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, 2, \dots, m, \end{array}$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$, $f : R^n \rightarrow R$ and $g = (g_1, g_2, \dots, g_m)^T : R^n \rightarrow R^m$ ($m \leq n$) are continuously differentiable functions. Let $X = \{x : g_i(x) \geq 0, i = 1, 2, \dots, m\}$ be a feasible set for the problem (1). We shall study algorithms for solving the problem (1) based on penalty functions. To obtain a solution

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of (1), the penalty function method solves a sequence of unconstrained optimization problems. A well known penalty function for this problem is given by

$$(2) \quad F(x, p) = f(x) - p \sum_{i=1}^m p(g_i(x))$$

where $p(t) = \min(0, t)$ and $p > 0$ is a auxiliary penalty variable. The penalty function $F(x, p)$ is nonsmooth. The corresponding unconstrained optimization problem of (1) is defined as follows;

$$(3) \quad \min F(x, p) \quad s.t. \quad x \in R^n.$$

Further information can be found in Luenberger [9].

Smoothing approximation for this kind of nonsmooth penalty functions have been seen in literature. In [12], Pinar and Zenios studied a quadratic smoothing approximation to nonsmooth exact penalty function for convex optimization problem. They also used a smoothing method for solving optimization problems with neural network structure in [17]. Moreover, similar studies can be found in the references Chen-Mangasarian [3], Meng et al. [10], Yang et al. [16] and the references therein.

In this work, we will use parallel smoothing techniques as in [10], [12] and [16] for solving the optimization problem (1) with dynamical system approach. This approach shows that the stable equilibrium point of the dynamic system is also the optimal solution of the corresponding optimization problem, respectively. We also put forward relations between the optimal solution of nonsmooth penalty problem and smooth penalty problem.

The rest of work is organized as follows. In Section 2, we will present a smoothing model for nonsmooth penalty function (2) and construct a dynamic system based on smooth penalty function. Moreover, we will prove that the trajectory of dynamical system can converge to an optimal solution of the smooth penalty problem and discuss the stability of equilibrium point. A Lyapunov function is set up during the procedure. In addition, we will investigate relations between smooth and nonsmooth penalty function with dynamic system. In Section 3, two illustrative examples will be given to show the effectiveness of proposed system. Conclusions are given in Section 4.

2. A Dynamical System with Smoothed Penalty Function

We define a function $p_\xi(t)$ for smoothing $p(t) = \min(0, t)$ as follows,

$$p_\xi(t) = \begin{cases} t - 3\xi t^{\frac{1}{3}} - \frac{1}{4}\xi & \text{if } t \leq \xi, \\ \frac{1}{8} \frac{t^3}{\xi^2} & \text{if } \xi \leq t \leq 0, \\ 0 & \text{if } t \geq 0. \end{cases}$$

Then, we see that $\lim_{\xi \rightarrow 0} p_\xi(t) = p(t)$. Now consider the smoothed penalty function for (1) given by

$$(4) \quad F(x, p, \xi) = f(x) - p \sum_{i=1}^m p_\xi(g_i(x)),$$

where $p > 0$. Hence, the unconstrained optimization problem of (1) is defined as

$$(5) \quad \min F(x, p, \xi) \quad s.t. \quad x \in R^n.$$

Thus $\lim_{\xi \rightarrow 0} F(x, p, \xi) = F(x, p)$ for any given p .

In order to solve the unconstrained optimization problem (5), a new dynamic system can be described by the following ordinary differential equations (ODEs)

$$(6) \quad \begin{aligned} \frac{dx}{dt} &= -\nabla_x F(x, p, \xi), \\ x(t_0) &= x(0). \end{aligned}$$

The method based on ODEs for solving optimization problems have been proposed by Arrow and Hurwicz [1], Fiacco and McCormick [6], and Evtushenko [5]. Furthermore, Brown and Biggs [2] and Schropp [14] are shown that ODEs based methods for constrained optimization can be perform better than some conventional methods. Recently, Jin [7], Özdemir [11] and Wang [15] have prepared a new differential equation approach for solving NLP problems.

Stability of nonlinear dynamic systems plays an important role in systems theory and engineering. There are several approaches in the literature. Most of them are based on Lyapunov theory. In the following, we will give some basic definitions of Lyapunov stability theory.

Definition 2.1 *A point x^* is called an equilibrium point of (6) if it satisfies the right hand side of the equations (6), [4].*

Definition 2.2 *Let Ω be an neighbourhood of x^* . A continuously differentiable function V is said to be Lyapunov function at the state x^* for dynamic system (6) if satisfy the following conditions:*

- a) $V(x)$ is equal to zero for the equilibrium point x^* ,
- b) $V(x)$ is positive definite over Ω some neighbourhood of x^* ,
- c) $\frac{dV}{dt}$ is semi-negative definite over Ω some neighbourhood of x^* , [8].

Theorem 2.3 *An equilibrium point x^* is stable if there exists a Lyapunov function in some neighbourhood Ω , [8].*

Definition 2.4 A point x^* is said to be stable in the sense of Lyapunov if for any $x(t_0) = x(0)$ and any scalar $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$, then $\|x(t) - x^*\| < \epsilon$ for $t \geq t_0$, [4].

Now we investigate the relationship between optimal solutions of the nonsmooth penalty problem (3) and the smooth penalty problem (5) with an equilibrium point of the dynamical system (6). The following theorems will give a relation between the stable equilibrium point of the dynamic system (6) and the minimizer of the problem (5), respectively.

Theorem 2.5 If x^* is an stable equilibrium point of the dynamic system (6) under the parameter p and ξ , then x^* is the local minimizer of the problem (5).

Proof. Let $x(t)$ be a solution of the dynamical system (6). Then from the assumption, stability of the equilibrium point x^* , we have for any $x(t_0) = x(0)$ and any scalar $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$ then, $\|x(t) - x^*\| < \epsilon$ for $t \geq t_0$. Therefore

$$\begin{aligned} F(x^*, p, \xi) - F(x(0), p, \xi) &= \int_{x(0)}^{x^*} \frac{dF(x, p, \xi)}{dt} dt \\ &= \int_0^\infty \sum_{k=1}^n \frac{\partial F}{\partial x_k} \frac{dx_k}{dt} dt \\ &= - \int_0^\infty \sum_{k=1}^n \left(\frac{dx_k}{dt} \right)^2 dt \\ &\leq 0. \end{aligned}$$

Hence, x^* is a local minimizer to the problem (5).

Theorem 2.6 If x^* is a local minimizer of the problem (5) under the parameter p and ξ , then x^* is a stable equilibrium point of the dynamic system (6).

Proof. Let we define a suitable Lyapunov function for investigate the stability of the dynamic system (6) as follow;

$$V(x) = F(x, p, \xi) - F(x^*, p, \xi).$$

Note that since x^* is a local minimizer of the problem (5), $V(x)$ is positive definite on some neighbourhood Ω about the point x^* . Hereby, we say that the first two conditions in Definition 2.2 are provided for candidate Lyapunov

function. Furthermore, by the time derivative of $V(x)$, we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x} \frac{dx}{dt} \\ &= \left[\frac{\partial F(x, p, \xi)}{\partial x} \right] \left[-\frac{\partial F(x, p, \xi)}{\partial x} \right]^T \\ &= -\left(\frac{\partial F(x, p, \xi)}{\partial x} \right)^2 \leq 0.\end{aligned}$$

Eventually, x^* is stable for the dynamic system (6).

We now begin to introduce the connection between the nonsmooth (2) and smooth penalty function (4) with the following lemma.

Lemma 2.7 *For all $x \in R^n$ and $\xi < 0$ we have*

$$\frac{1}{4}mp\xi \leq F(x, p, \xi) - F(x, p) \leq 0.$$

Proof. From the definition of $p_\xi(t)$ and $p(t)$, it follows that

$$\frac{1}{4}\xi \leq p(t) - p_\xi(t) \leq 0.$$

Similarly, we can write that

$$\frac{1}{4}\xi \leq p(g_i(x)) - p_\xi(g_i(x)) \leq 0, \quad i = 1, 2, \dots, m.$$

Therefore,

$$\frac{1}{4}m\xi \leq \sum_{i=1}^m p(g_i(x)) - \sum_{i=1}^m p_\xi(g_i(x)) \leq 0.$$

By the equations (2) and (4) give the result.

Theorem 2.8 *Let $\{\xi_j\} \rightarrow 0$ be a sequence of negative numbers and x_j an optimal solution to the problem $\min_{x \in R^n} F(x, p, \xi_j)$. Assume that x^* be an accumulating point of the sequence $\{x_j\}$. Then x^* is an optimal solution to $\min_{x \in R^n} F(x, p)$.*

Proof. Since

$$\frac{1}{4}mp\xi_j \leq F(x, p, \xi_j) - F(x, p) \leq 0$$

for every j , we have

$$(7) \quad \min_{x \in R^n} \left\{ \frac{1}{4}mp\xi_j \right\} \leq x_j - \min_{x \in R^n} F(x, p) \leq 0.$$

Here, if $j \rightarrow \infty$ in (7), we obtain

$$\{x_j\}_{j=1}^{\infty} \rightarrow \min_{x \in R^n} F(x, p)$$

since $\{\xi_j\} \rightarrow 0$. Also from assumption, x^* is a accumulating point of the $\{x_j\}$, there is a subsequence $\{x_{j_k}\}$ of the $\{x_j\}$ such that

$$\{x_{j_k}\} \rightarrow x^*, \quad k \rightarrow \infty.$$

So, by the uniqueness of the limit, we get

$$\min_{x \in R^n} F(x, p) = x^*.$$

Definition 2.9 If $x \in X$ and if $g_i(x) \geq \xi$ for $i = 1, 2, \dots, m$, then x is said to be an ξ -feasible.

Theorem 2.10 Suppose that x^* be an optimal solution to the problem (3) and \bar{x} be an optimal solution to (5). Then

$$\frac{1}{4}mp\xi \leq F(\bar{x}, p, \xi) - F(x^*, p) \leq 0.$$

Proof. Using Lemma 2.7 and optimality conditions for x^* and \bar{x} , it follows that

$$F(\bar{x}, p, \xi) \leq F(x^*, p, \xi) \leq F(x^*, p)$$

which gives

$$(8) \quad F(\bar{x}, p, \xi) - F(x^*, p) \leq 0.$$

Similarly we have

$$\begin{aligned} F(\bar{x}, p) &\geq F(x^*, p) \\ F(\bar{x}, p, \xi) - \frac{1}{4}mp\xi &\geq F(\bar{x}, p) \geq F(x^*, p) \end{aligned}$$

that is

$$(9) \quad F(\bar{x}, p, \xi) - F(x^*, p) \geq \frac{1}{4}mp\xi.$$

The above two inequalities (8) and (9) indicate that the theorem is true.

Theorem 2.8 and Theorem 2.10 say that the optimal solution of smooth penalty problem (5) is also the optimal solution to the nonsmooth penalty problem (3) in a small error.

Theorem 2.11 Let x^* be an optimal solution to (3) and $\bar{x} \in R^n$ an optimal solution to (5). If x^* be a feasible for the problem (1) and \bar{x} be an ξ -feasible for the problem (1), then

$$\frac{1}{2}mp\xi \leq f(\bar{x}) - f(x^*) \leq 0.$$

Proof. Let \bar{x} be an ξ -feasible for the problem (1). Using the Definition 2.9, we have $\sum_{i=1}^m p_\xi(g_i(\bar{x})) \geq \frac{1}{4}m\xi$. Also from the optimality conditions of x^* , we obtain

$$\sum_{i=1}^m p(g_i(x^*)) = 0.$$

From the Theorem 2.10 and the definitions of $F(x, p)$, $F(x, p, \xi)$, we get

$$\frac{1}{4}mp\xi \leq \left(f(\bar{x}) - p \sum_{i=1}^m p_\xi(g_i(\bar{x})) \right) - \left(f(x^*) - p \sum_{i=1}^m p(g_i(x^*)) \right) \leq 0,$$

which gives

$$\frac{1}{2}mp\xi \leq f(\bar{x}) - f(x^*) \leq 0.$$

and we complete the proof.

As a consequence of the Theorem 2.11, we proved the relationship between the optimal solution to the problem (1) and (5).

3. Illustrative Examples

Example 3.1 An example is taken from Matlab to show the applicability of our approach,

$$\begin{aligned} \text{minimize} \quad & f(x) = \exp(x_1) (4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ \text{subject to} \quad & g_1(x) = x_1 + x_2 - x_1x_2 - 1.5 \geq 0 \\ & g_2(x) = x_1x_2 + 10 \geq 0 \end{aligned}$$

where its optimal solution locates at the point $(-9.5474, 1.0474)$. Its smooth penalty function is written as

$$\begin{aligned} F(x, p, \xi) = & \exp(x_1) (4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ & -p \begin{cases} g_1(x) + \frac{1}{2}\xi (g_1(x))^2 - \frac{1}{4}\xi, & \text{if } t \leq \xi \\ \frac{1}{8} \frac{(g_1(x))^3}{\xi^2}, & \text{if } \xi \leq t \leq 0 \\ 0, & \text{if } t \geq 0 \end{cases} \\ & -p \begin{cases} g_2(x) + \frac{1}{2}\xi (g_2(x))^2 - \frac{1}{4}\xi, & \text{if } t \leq \xi \\ \frac{1}{8} \frac{(g_2(x))^3}{\xi^2}, & \text{if } \xi \leq t \leq 0 \\ 0, & \text{if } t \geq 0. \end{cases} \end{aligned}$$

The corresponding differential equations system from (6) are

$$\begin{aligned}\frac{dx_1}{dt} &= -\exp(x_1)(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) - \exp(x_1)(8x_1 + 4x_2) \\ &\quad + p\nabla_{x_1}p\xi(g_1) + p\nabla_{x_1}p\xi(g_2) \\ \frac{dx_2}{dt} &= -\exp(x_1)(4x_2 + 4x_1 + 2) + p\nabla_{x_2}p\xi(g_1) + p\nabla_{x_2}p\xi(g_2).\end{aligned}$$

The Euler method is used to solve the differential equations system. If we take the initial values for $x_1(0) = 0$, $x_2(0) = 0$, $p = 10$ and $\xi = -0.0001$ with step size $dt = 0.0001$, simulation result shows the equilibrium point of the dynamical system, which is very close to the optimal point of the nonlinear programming problem, at the point $(-9.5240, 1.0479)$.

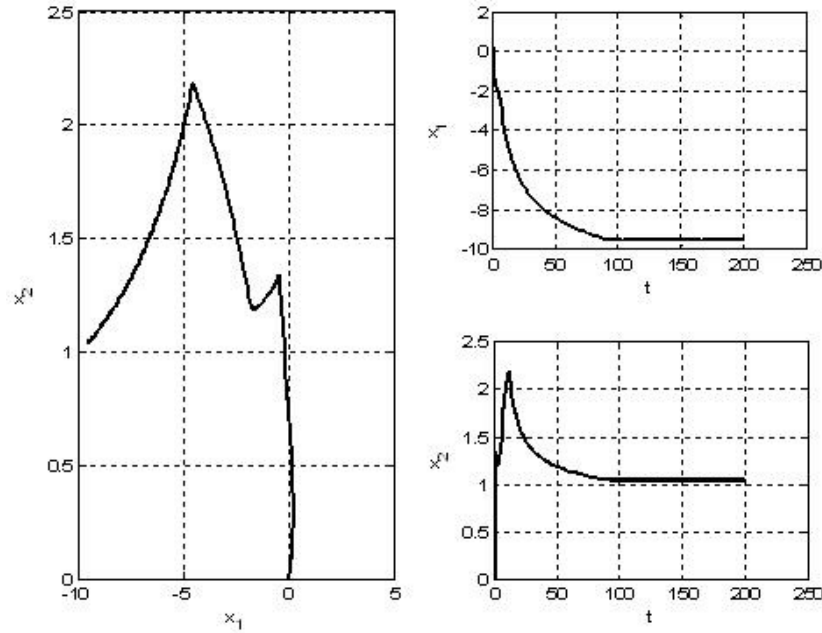


Figure 1. The trajectories of example problem for the given initial values

Example 3.2 Consider the following NLP problem with inequality constrained [13],

$$\begin{aligned}\text{minimize} \quad & f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ \text{subject to} \quad & g_1(x) = x_1^2 + x_2^2 - 0.25 \geq 0.\end{aligned}$$

The optimal solution is $x^* = (1, 1)^T$. Its smooth penalty function is written as

$$F(x, p, \xi) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 - p \begin{cases} g_1(x) + \frac{1}{2}\xi(g_1(x))^2 - \frac{1}{4}\xi, & \text{if } t \leq \xi \\ \frac{1}{8}\frac{(g_1(x))^3}{\xi^2}, & \text{if } \xi \leq t \leq 0 \\ 0, & \text{if } t \geq 0. \end{cases}$$

The corresponding differential equations system are

$$\begin{aligned} \frac{dx_1}{dt} &= 400x_1(x_2 - x_1^2) + 2(1 - x_1) + p\nabla_{x_1}p\xi(g_1) \\ \frac{dx_2}{dt} &= -200(x_2 - x_1^2) + p\nabla_{x_2}p\xi(g_1). \end{aligned}$$

The Euler method is used to solve the differential equations system. Taking step size $dt = 0.001$, when the initial point is taken as $x_1(0) = -2$, $x_2(0) = 2$, $p = 1$ with $\xi = -0.0001$, the trajectories of the dynamical system converge to theoretical optimal solution $x^* = (1, 1)^T$.

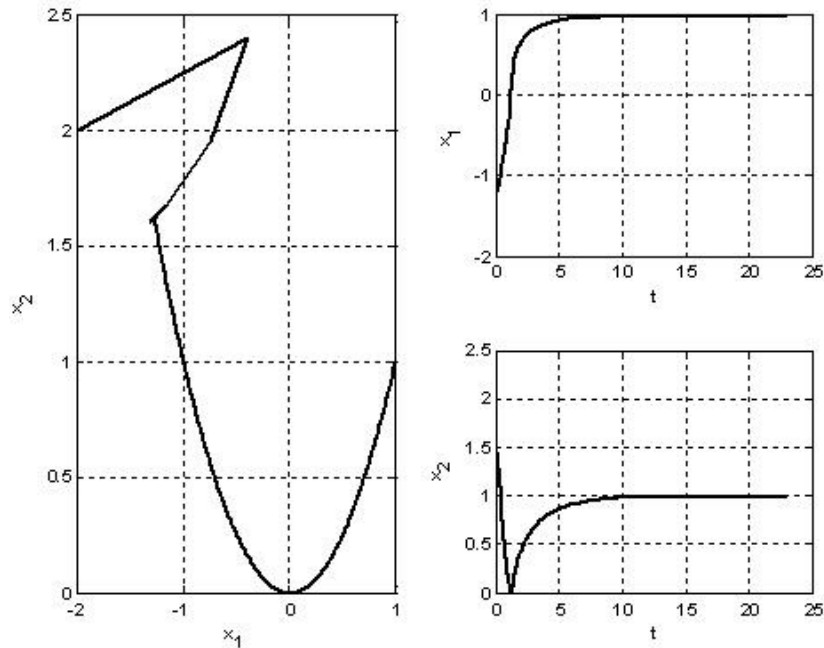


Figure 2. The trajectories of example problem for the given initial values

4. Conclusions

We have proposed a dynamic system approach for solving NLP problems. Based on the smooth penalty function (4), corresponding the dynamic system model (6) has been constructed for this purpose. Through theoretical analysis and example computations, it has been shown that the equilibrium point of the dynamic system coincides with the optimal solution of the NLP problem depends on the chosen parameter and ρ . Moreover, the equilibrium point of the system was proved to be stable in the sense of Lyapunov method. Furthermore, we showed that optimal values for smooth and nonsmooth penalty method are very close.

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