

On Decidability Results of the Holomorph of a Finite Cyclic Group

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Abstract: As a next step of the result in paper [1, Theorem 3.1], we study double coset separability, residually finitely and solvability of the power problem of holomorph of a finite cyclic group of order 2^t ($t \in \mathbb{Z}^+$) in this paper.

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1. Introduction

Let H be a subgroup of a group G . Then G is said to be H -separable if, for each $g \in G - H$, there exists a normal subgroup N of finite index in G , denoted by $N \triangleleft_f G$, such that $g \notin NH$. In particular, if $H = \{1\}$ then G is *residually finite*. If G is H -separable for all finitely generated subgroups H of G , then G is called subgroup separable. We note that, especially, if G is $\{x\}^G$ -separable for all $x \in G$, where $\{x\}^G = \{g^{-1}xg : g \in G\}$, then G is called *conjugacy separable*. Since

- ◆ finitely presented residually finite groups have solvable word problem,
- ◆ finitely presented conjugacy separable groups have solvable conjugacy problem and
- ◆ finitely presented subgroup separable groups have solvable generalized word problem,

these kind of separability properties are directly related to decision problems in group theory (see, for instance, [9] and [10]).

Free groups and surface groups can be given as examples of having subgroup and conjugacy separability properties ([4, 12, 13]). Moreover it is known that Fuchsian groups are conjugacy separable ([3]) and double coset separable ([11]).

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In addition, we could not find any references in the literature about these above properties (except subgroup separability) studied on holomorph of cyclic groups. In [1], the authors defined subgroup separability on a special split extension which is actually on holomorph. As a next step of this paper, here, we present some further properties studied in that paper, which is related to separability, on the group G .

We recall that the *holomorph of a group* is the semidirect product of a group with its automorphism group with respect to the obvious action. The automorphism group of a non-trivial finite cyclic group of order r is known to be cyclic if and only if the number r is of the kind $r = 4$, $r = p^t$, $r = 2p^t$, where p is an odd prime. So, in these cases, the holomorph is a split metacyclic group ([5]).

Let $t \geq 2$ and N be the cyclic group of order $r = 2^t$. As usual, we identify the automorphism group of N with the group $\mathbb{Z}_{2^t}^*$ which is the units of \mathbb{Z}_{2^t} . Now let us consider the holomorph

$$G = \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^*$$

of N . For the case $t = 2$, G becomes the dihedral group and the cyclic group \mathbb{Z}_4^* (of order 2) is generated by the class of -1 . We note that this case will not be considered in this paper since decidability results investigated here can be seen easily.

Hence at the rest of the paper, we will assume $t \geq 3$. Now, by [5], the group $\mathbb{Z}_{2^t}^*$ decomposes as a direct product of a copy of \mathbb{Z}_2 generated by the class of -1 , and a copy of $\mathbb{Z}_{2^{t-2}}$ generated by the class of 5. Let us write $s = 2^{t-1}$. Thus, group $G = \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^*$ has the following presentation

$$(1) \quad \wp_G = \langle x, y, z ; y^r, x^s, z^2, xyx^{-1} = y^5, zyz^{-1} = y^{-1}, [x, z] \rangle,$$

where the normal cyclic subgroup N is generated by y and the cyclic subgroups of order s and 2 are generated by x and z , respectively. In fact we have two subgroups, say G_1 and G_2 , of G with presentations

$$(2) \quad \wp_{G_1} = \langle x, y ; y^r, x^s, xyx^{-1} = y^5 \rangle \quad \text{and} \quad \wp_{G_2} = \langle y, z ; y^r, z^2, zyz^{-1} = y^{-1} \rangle$$

respectively ([5]), where $G_2 = N \rtimes \langle z ; z^2 \rangle$.

By considering (2), Ateş and Çevik (in [1]) proved the following result.

Theorem 1.1. Let $G = \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^*$ with presentation (1). Then G is G_1 and G_2 -separable.

In this paper, mostly by using Theorem 1 and, considering double coset separability and residually finitely as decidability properties similarly power problem, we will try to prove decidability of these special problems in Section 2, 3 and 4, respectively.

Throughout this paper, G will denote the group $\mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^$.*

2. Double Coset Separability of G

Definition 2.1 Let A_1, \dots, A_n be subgroups of an arbitrary group \overline{G} . We say that \overline{G} is $\{A_1, \dots, A_n\}$ -double coset separable, shortly $\{A_1, \dots, A_n\}$ - d -separable, for any $x \in \overline{G}$ and any subgroups U, V of A_1, \dots , or A_n , if \overline{G} is UxV -separable.

Thus we have the following one of the main results of this paper.

Theorem 2.2. Let G has presentation (1). Assume that G is G_1 and G_2 -separable for the subgroups G_1, G_2 of G with presentations given in (2). Then G is $\{G_1, G_2\}$ - d -separable.

Proof. Actually, Definition 2.1 will be enough to prove this theorem. To do that, for subgroups U, V of G_1 or G_2 , and for a fixed $g \in G$, we need to show that G is UgV -separable. Now let us take U is the trivial subgroup of G_1 (or G_2) and V is a subgroup of G_2 (or G_1) with presentation $\wp_V = \langle y ; y^r \rangle$. In fact the set UgV has the following $r(s+1)$ elements:

$$\{1, y, y^2, \dots, y^{r-1}, x, xy, xy^2, \dots, xy^{r-1}, x^2, x^2y, x^2y^2, \dots, x^2y^{r-1}, \dots, x^{s-1}, x^{s-1}y, x^{s-1}y^2, \dots, x^{s-1}y^{r-1}, z, zy, zy^2, \dots, zy^{r-1}\}.$$

We note that, to obtain the set $G - UgV$, the element g in G must be the identity or must contain just one generator of G . Therefore the set $G - UgV$ which has total $(r-1)(s-1)$ elements can be given as

$$\{yxz, y^2xz, \dots, y^{r-1}xz, y^2x^2z, \dots, y^{r-1}x^2z, \dots, yx^{s-1}z, y^2x^{s-1}z, \dots, y^{r-1}x^{s-1}z\}.$$

In the rest of the proof, by the definition of subgroup separability, we must find a normal subgroup with finite index in G . Clearly N (presented by $\langle y ; y^r \rangle$) is the greatest choice for such a normal subgroup. So the set $N(UgV)$ consists of just suitable powers of y, yx and yz . Therefore, for all $g' \in G - UgV$, we obtain $g' \notin N(UgV)$.

Hence the result.

3. Residually Finitely of G

Definition 3.1. Let \overline{G} be a group and H be a subgroup of \overline{G} . We say that H is finitely compatible in \overline{G} , shortly, \overline{G} is H -finite if, for every $D \triangleleft_f H$, there exists $N_D \triangleleft_f \overline{G}$ such that $N_D \cap H = D$.

Hence

Lemma 3.2. G is G_2 -finite.

Proof. Since $G_2 = N \rtimes \langle z ; z^2 \rangle$ (where N should be thought as in the proof of Theorem 2.2) and $G_1 \triangleleft_f G$ (because $|G/G_1| = 2$) such that $G_1 \cap G_2 = N$, G is G_2 -finite.

We should note that the notation RF will denote the class of *residually finite* groups. By abuse of notation, we also use RF as *residually finite property*. We also note that the following proposition is another way to show that a group is RF without using definition of this property.

Proposition 3.3 [6] Let $H \leq \overline{G}$ and assume that H is RF . If \overline{G} is H -finite and H -separable, then \overline{G} is RF .

Hence we can deduce that

Theorem 3.4 G is RF .

Proof. Since G is G_2 -separable, by Theorem 1.1, and G_2 -finite, by Lemma 3.2, G is RF , as required.

The above result is another way to show that the holomorph of a finite cyclic group is RF . Because the automorphism group of a finitely presented RF group is RF ([8]) and a split extension of a finitely generated RF group by a RF group is RF ([9]).

We recall that a group is *Hopfian* if every epimorphism from this group to itself is an isomorphism. So our group G is Hopfian since every RF group is Hopfian by [9]. We also give the following consequence about the RF groups related to our group G .

Corollary 3.5. The word problem of G is solvable.

We also recall that deciding whether a given element w of a group belongs to a subgroup H is called the *generalized word problem (membership problem)* for H in that group. In our case, by considering Theorem 1.1, we get the following result for the membership problem.

Theorem 3.6. The generalized word problem is solvable for G_1 and G_2 in G .

4. Solvability of Power Problem of G

We say that any group \overline{G} has *solvable power problem* if we have an algorithm such that given u, v in \overline{G} whether there exists $v = u^n$ for some $n \neq 0$. Notice that solvability of power problem implies solvability of the word problem as well as solvability of the *order problem* which requires if there exists an algorithm such that given $u \in \overline{G}$ computes the order of u .

The usefulness of geometric methods in combinatorial group theory is well-established. Therefore we will draw a van Kampen diagram to show the solvability of power problem of G . In this whole section, for a van Kampen diagram Δ of G , $\partial\Delta$ denotes its boundary, and for a path ρ in this diagram, $\phi(\rho)$ denotes its label. We can refer the reader to [7] for a complete description about van Kampen diagrams.

We note that, for brevity, we will draw the van Kampen diagram of G for $t = 3$ (see Figure 1). For $t \geq 4$, the diagrams are all similar except the number of each different parts in drawn diagram. In other words, a diagram drawn, for $t = n$, contains two times of each parts from the diagram drawn for $t = n - 1$ without loss of generality of construction of diagram.

It is seen from Figure 1 that if we traverse $\partial\Delta$ in a clockwise manner starting from 0, then we get

$$\begin{aligned} \phi(\rho) = & y^5 x^{-2} z x^{-1} y^{-1} x z^{-1} x^2 y^5 x^{-2} z x^{-1} y^{-1} x z^{-1} x^2 \\ & y^5 x^{-2} z x^{-1} y^{-1} x z^{-1} x^2 y^5 x^{-2} z x^{-1} y^{-1} x z^{-1} x^2. \end{aligned}$$

where ρ is the boundary path. So using relations $xy = y^5x$, $zy = y^7z$ and $xz = zx$, we obtain the path $\phi(\rho)$ is freely equivalent to the empty word. In fact, by [7], this guarentees the existence of the diagram of G .

We know that a van Kampen diagram is formed by drawing parts of all relations in relation set with a corresponding manner. In other words, there must be a convenient common path of two relations in such a diagram. So Figure 1 is the greatest choice as a diagram for G , where $t = 3$ since every part in this figure has a symmetrical form of itself. That is the real reason why we examine the solvability of the power problem for G by using this diagram.

Now we can give another result as follows.

Theorem 4.1. The power problem is solvable for G .

Proof. We must check whether $v = u^n$ for u, v in G and some $n \neq 0$. To do that we give the following algorithm obtained by ourselves which gives a solution to power problem. We must note that the following algorithm is based on the symmetrical form of diagram (Figure 1) of G .

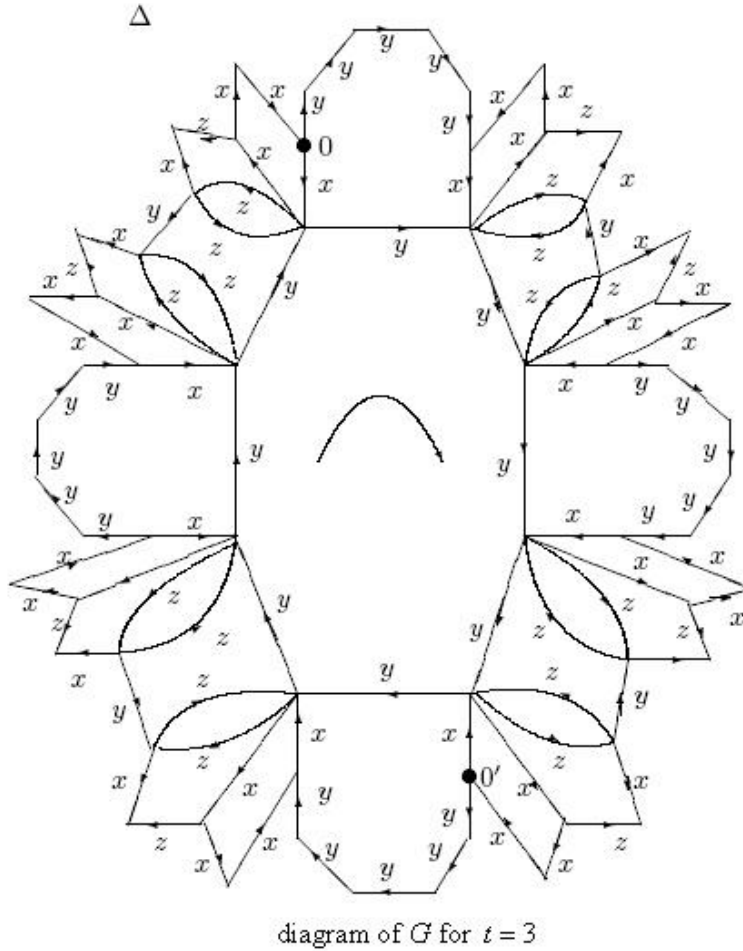


Figure 1

Algorithm:

Step 1: Start from any vertex on diagram.

Step 2: Travel around in a clockwise (or in an anticlockwise) manner. Do not pass through any edge passed before.

Step 3: If you arrive at a symmetrical vertex (by using a path without any cycles) of the beginning vertex, then stop the algorithm and go to Step 5. Otherwise go to next step.

Step 4: Continue to travel around of diagram and go to Step 3.

Step 5: Label the paths travel around.

We note that if algorithm stops at the first symmetrical vertex of the beginning vertex in Step 3, this means $n = 1$ for $v = u^n$. But, as it is seen from Figure 1,

since $r = 2^3 = 8$ and so each part has $8/2 = 4$ copies, the value of n can be at most 4.

To supply the above algorithm, let us take a word

$$v = xyx^3zx^{-2}y^5zxx^{-1}xy^5zx^{-3}y^{-1}xz^{-1}x^2.$$

By using relations $xy = y^5x$, $zy = y^7z$ and $xz = zx$, it is clear that v is equivalent to the word $u = (y^5x^{-2}zx^{-1}y^{-1}xz^{-1}x^2)^2$. We should note that the initial vertex is 0 with clockwise manner as a path of this word and also $n = 2$ has been obtained by traveling the diagram ending at $0'$ which is symmetrical vertex of 0.

In the following two remarks we will leave some problems associated with the subject studied in this paper.

Remark. Since conjugacy separability does not respect extensions ([2]), it worths to study in our group $G = \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^*$ which it will be left as a future project. Moreover, for any group \bar{G} , two elements u and v in that group are said to be Φ -conjugate or *twisted conjugate* (where Φ is an automorphism of \bar{G}), denoted $u \approx_\Phi v$, if there exists $g \in \bar{G}$ such that $(g\Phi)^{-1}ug = v$. Hence \bar{G} is called Φ -conjugacy separable with respect to an automorphism $\Phi : \bar{G} \rightarrow \bar{G}$ if any pair g, h of non- Φ -conjugate elements of \bar{G} are non- Φ -conjugate in some finite quotient of \bar{G} respecting Φ . Clearly this case coincides with the definition of conjugacy separability in the case $\Phi = Id$.

Therefore one can ask the following question:

Question 1: *Is the group G Φ -conjugacy separable?*

In particular, in [2] an example of a group which is not conjugacy separable but contains a subgroup of index 2 that is conjugacy separable is given. Thus conjugacy separability can be still study on the group $\mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^*$. (Clearly this is a special case of Question 1).

Remark. To investigate the Φ -twisted conjugacy problem for a group \bar{G} implies decidability the conjugacy problem for that group. In fact it is said that the Φ -twisted conjugacy problem is solvable in \bar{G} if, for any elements $u, v \in \bar{G}$, we can algorithmically decide if $u \approx_\Phi v$. It is explicit that this coincides with the conjugacy problem in the case $\Phi = Id$. Finally it is said that the twisted conjugacy problem is solvable in \bar{G} if the Φ -twisted conjugacy problem is solvable for any $\Phi \in Aut(\bar{G})$ ([2]).

So the following question arises.

Question 2: *Is twisted conjugacy problem solvable for the group G ?*

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