



# Commutator Subgroups of Generalized Hecke and Extended Generalized Hecke Groups

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## Abstract

Let  $p$  and  $q$  be integers such that  $2 \leq p \leq q$ ,  $p + q > 4$  and let  $H_{p,q}$  be the generalized Hecke group associated to  $p$  and  $q$ . The generalized Hecke group  $H_{p,q}$  is generated by  $X(z) = -(z - \lambda_p)^{-1}$  and  $Y(z) = -(z + \lambda_q)^{-1}$  where  $\lambda_p = 2 \cos \frac{\pi}{p}$  and  $\lambda_q = 2 \cos \frac{\pi}{q}$ . The extended generalized Hecke group  $\overline{H}_{p,q}$  is obtained by adding the reflection  $R(z) = 1/\bar{z}$  to the generators of generalized Hecke group  $H_{p,q}$ . In this paper, we study the commutator subgroups of generalized Hecke groups  $H_{p,q}$  and extended generalized Hecke groups  $\overline{H}_{p,q}$ .

## 1 Introduction

In [5], Hecke introduced the groups  $H(\lambda)$  generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where  $\lambda$  is a fixed positive real number. Let  $S = TU$ , i.e.,

$$S(z) = -\frac{1}{z + \lambda}.$$

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Hecke showed that  $H(\lambda)$  is discrete if and only if  $\lambda = \lambda_q = 2 \cos(\frac{\pi}{q})$ ,  $q \geq 3$  integer, or  $\lambda \geq 2$ . We consider the former case  $q \geq 3$  integer and we denote it by  $H_q = H(\lambda_q)$ . The Hecke group  $H_q$  is isomorphic to the free product of two finite cyclic groups of orders 2 and  $q$ ,

$$H_q = \langle T, S : T^2 = S^q = I \rangle \simeq C_2 * C_q.$$

The first few Hecke groups  $H_q$  are  $H_3 = \Gamma = PSL(2, \mathbb{Z})$  (the modular group),  $H_4 = H(\sqrt{2})$ ,  $H_5 = H(\frac{1+\sqrt{5}}{2})$ , and  $H_6 = H(\sqrt{3})$ . It is clear from the above that  $H_q \subset PSL(2, \mathbb{Z}[\lambda_q])$  unlike in the modular group case (the case  $q = 3$ ), the inclusion is strict and the index  $|PSL(2, \mathbb{Z}[\lambda_q]) : H_q|$  is infinite as  $H_q$  is discrete whereas  $PSL(2, \mathbb{Z}[\lambda_q])$  is not for  $q \geq 4$ .

Lehner studied in [11] a more general class  $H_{p,q}$  of Hecke groups  $H_q$ , by taking

$$X = \frac{-1}{z - \lambda_p} \text{ and } V = z + \lambda_p + \lambda_q,$$

where  $2 \leq p \leq q$ ,  $p + q > 4$ . Here if we take  $Y = XV = -\frac{1}{z + \lambda_q}$ , then we have the presentation,

$$H_{p,q} = \langle X, Y : X^p = Y^q = I \rangle \simeq C_p * C_q.$$

In particular  $H_{p,q}$  has the signature  $(0; p, q, \infty)$ . We call these groups as *generalized Hecke groups*  $H_{p,q}$ . We know from [11] that  $H_{2,q} = H_q$ ,  $|H_q : H_{q,q}| = 2$ , and there is no group  $H_{2,2}$ . Furthermore all Hecke groups  $H_q$  are included in generalized Hecke groups  $H_{p,q}$ . Generalized Hecke groups  $H_{p,q}$  have been also studied by Calta and Schmidt in [2] and [3].

Extended generalized Hecke groups  $\overline{H}_{p,q}$  can be defined similar to extended Hecke groups  $\overline{H}_q$ , by adding the reflection  $R(z) = 1/\overline{z}$  to the generators of generalized Hecke group  $H_{p,q}$ . Hence extended generalized Hecke groups  $\overline{H}_{p,q}$  have a presentation

$$\overline{H}_{p,q} = \langle X, Y, R : X^p = Y^q = R^2 = I, RX = X^{-1}R, RY = Y^{-1}R \rangle,$$

that is

$$\overline{H}_{p,q} = \langle X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I \rangle \cong D_p *_{\mathbb{Z}_2} D_q.$$

The groups  $H_{p,q}$  is a subgroup of index 2 in  $\overline{H}_{p,q}$ .

Now we focus on the commutator subgroups of  $H_{p,q}$  and  $\overline{H}_{p,q}$ . The commutator subgroups of  $H_q$ ,  $\overline{H}_q$ ,  $H_q^m$  ( $m$ -th power subgroup of  $H_q$ ) and  $H'_{p,q}$  ( $p$  and  $q$  are relatively prime) have been studied by many authors see [1], [16], [18], [19], [20] and [22].

In this paper, we study the commutator subgroups of  $H_{p,q}$  and  $\overline{H}_{p,q}$ . We give the generators, the group structures and the signatures of the commutator subgroups of  $H_{p,q}$  and  $\overline{H}_{p,q}$ . Here we use the Reidemeister-Schreier method, the permutation method (see, [21]) and the extended Riemann-Hurwitz condition (see, [10]) to get our results.

**Remark 1.** *i) The Hecke groups  $H_q$ , the extended Hecke groups  $\overline{H}_q$  and their normal subgroups have been studied extensively for many aspects in the literature. For examples, see [4], [8], [9], [12] and [14]. Also, there are many relationships between the groups  $H_q$  (or  $\overline{H}_q$ ) and the automorphism groups of Riemann (or Klein) surfaces or of regular maps  $\{p, q\}$ . Naturally, many results related with  $H_q$  and  $\overline{H}_q$  can be generalized to the groups  $H_{p,q}$  and  $\overline{H}_{p,q}$ .*

*ii) Generalized Hecke groups  $H(p_1, p_2, \dots, p_n)$  and extended generalized Hecke groups  $H^*(p_1, p_2, \dots, p_n)$  have been introduced first by Huang in [6]. Our studied groups are the special cases ( $n = 2$ ) of these groups given in [6] and [7].*

## 2 Commutator Subgroups of Generalized Hecke Groups

### $H_{p,q}$

In this section, we study the commutator subgroup  $H'_{p,q}$  of generalized Hecke groups  $H_{p,q}$ . We use standard notation, in particular,  $G^{(n)}$  denotes the  $n^{\text{th}}$  derived group of a group  $G$ .

**Theorem 1.** *Let  $p$  and  $q$  be integers such that  $2 \leq p \leq q$ ,  $p + q > 4$ .*

*i)  $|H_{p,q} : H'_{p,q}| = pq$ .*

*ii) The commutator subgroup  $H'_{p,q}$  of  $H_{p,q}$  is a free group of rank  $(p-1)(q-1)$  with basis  $[X, Y], [X, Y^2], \dots, [X, Y^{q-1}], [X^2, Y], [X^2, Y^2], \dots, [X^2, Y^{q-1}], \dots, [X^{p-1}, Y], [X^{p-1}, Y^2], \dots, [X^{p-1}, Y^{q-1}]$ . And the signature of  $H'_{p,q}$  is  $(\frac{pq-p-q-(p,q)+2}{2}; \infty^{(p,q)})$  where  $(p, q)$  is the greatest common divisor of  $p$  and  $q$ .*

*iii) For  $n \geq 2$ ,  $|H_{p,q} : H_{p,q}^{(n)}| = \infty$ .*

*Proof.* *i)* Firstly, we set up the quotient group  $H_{p,q}/H'_{p,q}$ . The quotient group  $H_{p,q}/H'_{p,q}$  is the group obtained by adding the relation  $XY = YX$  to the relations of  $H_{p,q}$ . Thus we have

$$H_{p,q}/H'_{p,q} = \langle X, Y : X^p = Y^q = I, XY = YX \rangle \simeq C_p \times C_q.$$

*ii)* Now we can determine the generators of  $H'_{p,q}$  by the Reidemeister-Schreier method. To do this, we choose the set  $\Sigma = \{I, X, X^2, \dots, X^{p-1}, Y, Y^2, \dots,$

$Y^{q-1}, XY, XY^2, \dots, XY^{q-1}, X^2Y, X^2Y^2, \dots, X^2Y^{q-1}, \dots, X^{p-1}Y, X^{p-1}Y^2, \dots, X^{p-1}Y^{q-1}$  as a Schreier transversal. All possible products are

$$\begin{aligned} I.X.(X)^{-1} &= I, \\ X.X.(X^2)^{-1} &= I, \\ &\vdots \\ X^{p-1}.X.(I)^{-1} &= I, \end{aligned}$$

$$\begin{aligned} Y.X.(XY)^{-1} &= YXY^{-1}X^{-1} = [Y, X], \\ Y^2.X.(XY^2)^{-1} &= Y^2X(XY^2)^{-1} = [Y^2, X], \\ &\vdots \\ Y^{q-1}.X.(XY^{q-1})^{-1} &= Y^{q-1}XY^{-(q-1)}X^{-1} = [Y^{q-1}, X], \\ XY.X.(X^2Y)^{-1} &= XYXY^{-1}X^{-2} = [X, Y].[Y, X^2], \\ XY^2.X.(X^2Y^2)^{-1} &= XY^2XY^{-2}X^{-2} = [X, Y^2].[Y^2, X^2], \\ &\vdots \\ XY^{q-1}.X.(X^2Y^{q-1})^{-1} &= XY^{q-1}XY^{-(q-1)}X^{-2} = [X, Y^{q-1}].[Y^{q-1}, X^2], \\ X^2Y.X.(X^3Y)^{-1} &= X^2YXY^{-1}X^{-3} = [X^2, Y].[Y, X^3], \\ X^2Y^2.X.(X^3Y^2)^{-1} &= X^2Y^2XY^{-2}X^{-3} = [X^2, Y^2].[Y^2, X^3], \\ &\vdots \\ X^2Y^{q-1}.X.(X^3Y^{q-1})^{-1} &= X^2Y^{q-1}XY^{-(q-1)}X^{-3} = [X^2, Y^{q-1}].[Y^{q-1}, X^3], \\ &\vdots \\ X^{p-1}Y.X.(Y)^{-1} &= X^{p-1}YXY^{-1} = [X^{-1}, Y], \\ X^{p-1}Y^2.X.(Y^2)^{-1} &= X^{p-1}Y^2XY^{-2} = [X^{-1}, Y^2], \\ &\vdots \\ X^{p-1}Y^{q-1}.X.(Y^{q-1})^{-1} &= X^{p-1}Y^{q-1}XY^{-(q-1)} = [X^{-1}, Y^{q-1}]. \end{aligned}$$

The other products are equal to the identity. Thus, we find the generators of  $H'_{p,q}$  as  $[X, Y], [X, Y^2], \dots, [X, Y^{q-1}], [X^2, Y], [X^2, Y^2], \dots, [X^2, Y^{q-1}], \dots, [X^{p-1}, Y], [X^{p-1}, Y^2], \dots, [X^{p-1}, Y^{q-1}]$ . The signature of  $H'_{p,q}$  can be obtained by permutation method and Riemann-Hurwitz formula.

*iii)* If we take relations and abelianizing, we find that the resulting quotient is infinite. Thus, it follows that  $H''_{p,q}$  has infinite index in  $H'_{p,q}$ . Further since this has infinite index it follows that in each group in the derived series from this point on have infinite index.  $\square$

**Example 1.** If  $p = 3$  and  $q = 4$ , then  $|H_{3,4} : H'_{3,4}| = 12$ . We choose  $\Sigma = \{I, X, X^2, Y, Y^2, Y^3, XY, XY^2, XY^3, X^2Y, X^2Y^2, X^2Y^3\}$  as a Schreier transversal for  $H'_{3,4}$ . Using the Reidemeister-Schreier method, we get the generators of  $H'_{3,4}$  as  $[X, Y], [X, Y^2], [X, Y^3], [X^2, Y], [X^2, Y^2], [X^2, Y^3]$ . Also the signature of  $H'_{3,4}$  is  $(3; \infty)$ .

**Corollary 1.** *If  $p = 2$ , then the generators of  $H'_{2,q}$  are  $[X, Y]$ ,  $[X, Y^2]$ , ...,  $[X, Y^{q-1}]$ . Also, the signatures of  $H'_{2,q}$  is either  $(\frac{q-1}{2}; \infty)$  if  $q$  is odd, or  $(\frac{q-2}{2}; \infty^{(2)})$  if  $q$  is even. These results coincide with the ones given in [1] and [20], for Hecke groups  $H_q$ .*

### 3 Commutator Subgroups of Extended Generalized Hecke Groups $\overline{H}_{p,q}$

In this section, we study the first commutator subgroups  $\overline{H}'_{p,q}$  of extended generalized Hecke groups  $\overline{H}_{p,q}$ .

**Theorem 2.** *Let  $p$  and  $q$  be integers such that  $2 \leq p \leq q$ ,  $p + q > 4$ .*

*i) If  $p$  and  $q$  are both odd numbers, then  $\overline{H}'_{p,q} = \langle X, Y : X^p = Y^q = I \rangle \simeq C_p * C_q$ .*

*ii) If  $p$  is an odd number and  $q$  is an even number, then  $\overline{H}'_{p,q} = \langle X, YXY^{-1}, Y^2 : X^p = (YXY^{-1})^p = (Y^2)^{\frac{q}{2}} = I \rangle \simeq C_p * C_p * C_{q/2}$ .*

*iii) If  $p$  is an even number and  $q$  is an odd number, then  $\overline{H}'_{p,q} = \langle X^2, Y, XYX^{-1} : (X^2)^{\frac{p}{2}} = Y^q = (XYX^{-1})^q = I \rangle \simeq C_{p/2} * C_q * C_q$ .*

*iv) If  $p$  and  $q$  are both even numbers, then  $\overline{H}'_{p,q} = \langle X^2, YX^2Y^{-1}, Y^2, XY^2X^{-1}, XYXY^{-1} : (X^2)^{p/2} = (YX^2Y^{-1})^{p/2} = (Y^2)^{q/2} = (XY^2X^{-1})^{q/2} = I \rangle \simeq C_{p/2} * C_{p/2} * C_{q/2} * C_{q/2} * \mathbb{Z}$ .*

*Proof.* The quotient group  $\overline{H}_{p,q}/\overline{H}'_{p,q}$  is

$$\begin{aligned} \overline{H}_{p,q}/\overline{H}'_{p,q} = \langle X, Y, R : X^p = Y^q = R^2 = I, XR = RX^{-1}, YR = RY^{-1}, \\ XY = YX, XR = RX, YR = RY \rangle. \end{aligned} \quad (3.1)$$

*i) If  $p$  and  $q$  are both odd numbers, then from (3.1), we get*

$$\overline{H}_{p,q}/\overline{H}'_{p,q} = \langle R : R^2 = I \rangle \simeq C_2.$$

Now we can determine the Schreier transversal as  $\Sigma = \{I, R\}$ . According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{aligned} I.X.(I)^{-1} = X, \quad I.Y.(I)^{-1} = Y, \quad I.R.(R)^{-1} = I, \\ R.X.(R)^{-1} = RXR, \quad R.Y.(R)^{-1} = RYR, \quad R.R.(I)^{-1} = I. \end{aligned}$$

Since  $RXR = X^{-1}$  and  $RYR = Y^{-1}$ , the generators of  $\overline{H}'_{p,q}$  are  $X$  and  $Y$ . Thus we have  $\overline{H}'_{p,q} = \langle X, Y : X^p = Y^q = I \rangle \simeq H_{p,q}$ . The signature of  $\overline{H}'_{p,q}$  is  $(0; p, q, \infty)$ .

ii) If  $p$  is an odd number and  $q$  is an even number, then from (3.1), we obtain

$$\overline{H}_{p,q}/\overline{H}'_{p,q} = \langle Y, R : Y^2 = R^2 = I, YR = RY \rangle \simeq C_2 \times C_2.$$

Now we can determine the Schreier transversal as  $\Sigma = \{I, Y, R, YR\}$ . According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll} I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I, & I.R.(R)^{-1} = I, \\ Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(I)^{-1} = Y^2, & Y.R.(YR)^{-1} = I, \\ R.X.(R)^{-1} = RXR, & R.Y.(YR)^{-1} = RYRY^{-1}, & R.R.(I)^{-1} = I, \\ YR.X.(YR)^{-1} = YRXY^{-1}, & YR.Y.(R)^{-1} = YRYR, & YR.R.(Y)^{-1} = I. \end{array}$$

After required calculations, we get the generators of  $\overline{H}'_{p,q}$  as  $X, YXY^{-1}$  and  $Y^2$ . Thus we obtain  $\overline{H}'_{p,q} = \langle X, YXY^{-1}, Y^2 : X^p = (YXY^{-1})^p = (Y^2)^{\frac{q}{2}} = I \rangle \simeq C_p * C_p * C_{q/2}$ . The signature of  $\overline{H}'_{p,q}$  is  $(0; p, p, (q/2), \infty)$ .

iii) The proof is similar to ii). Also, the signature of  $\overline{H}'_{p,q}$  is  $(0; (p/2), q, q, \infty)$ .  
 iv) If  $p$  and  $q$  are both even numbers, then from (3.1), we find

$$\begin{aligned} \overline{H}_{p,q}/\overline{H}'_{p,q} &= \langle X, Y, R : X^2 = Y^2 = R^2 = (XY)^2 = (XR)^2 = (YR)^2 = I \rangle \\ &\simeq C_2 \times C_2 \times C_2. \end{aligned}$$

Now we can determine the Schreier transversal as  $\Sigma = \{I, X, Y, R, XY, XR, YR, XYR\}$ . According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{ll} I.X.(X)^{-1} = I, & I.Y.(Y)^{-1} = I, \\ X.X.(I)^{-1} = X^2, & X.Y.(XY)^{-1} = I, \\ Y.X.(XY)^{-1} = YXY^{-1}X^{-1}, & Y.Y.(I)^{-1} = Y^2, \\ R.X.(XR)^{-1} = RXRX^{-1}, & R.Y.(YR)^{-1} = RYRY^{-1}, \\ XY.X.(Y)^{-1} = XYXY^{-1}, & XY.Y.(X)^{-1} = XY^2X^{-1}, \\ XR.X.(R)^{-1} = XRXR, & XR.Y.(XYR)^{-1} = XRYRY^{-1}X^{-1}, \\ YR.X.(XYR)^{-1} = YRXY^{-1}X^{-1}, & YR.Y.(R)^{-1} = YRYR, \\ XYR.X.(YR)^{-1} = XYRXY^{-1}, & XYR.Y.(XR)^{-1} = XYRYRX^{-1}, \end{array}$$

The other products are equal to the identity. Since  $RXRX^{-1} = X^{-2}$ ,  $XRXR = I$ ,  $YRXY^{-1}X^{-1} = YXY^{-1}X^{-1}$ ,  $XYRXY^{-1} = XYXY^{-1}$ ,  $RYRY^{-1} = Y^{-2}$ ,  $XRYRY^{-1}X^{-1} = I$ ,  $YRYR = I$  and  $XYRYRX^{-1} = XY^2X^{-1}$ , we have the generators of  $\overline{H}'_{p,q}$  as  $X^2, YX^2Y^{-1}, Y^2, XY^2X^{-1}, XYXY^{-1}$ . Then we get  $\overline{H}'_{p,q} = \langle X^2, YX^2Y^{-1}, Y^2, XY^2X^{-1}, XYXY^{-1} : (X^2)^{p/2} = (YX^2Y^{-1})^{p/2} = (Y^2)^{q/2} = (XY^2X^{-1})^{q/2} = I \rangle \simeq C_{p/2} * C_{p/2} * C_{q/2} * C_{q/2} * \mathbb{Z}$ . Also, the signature of  $\overline{H}'_{p,q}$  is  $(0; (p/2)^{(2)}, (q/2)^{(2)}, \infty^{(2)})$ .  $\square$

**Corollary 2.** 1) *The first commutator subgroup of  $\overline{H}_{p,q}$  does not contain any reflection and so  $\overline{H}'_{p,q}$  is a subgroup of  $H_{p,q}$ .*

2) *If  $p = 2$ , then the first commutator subgroups  $\overline{H}'_{2,q}$  coincide with the ones given in [16] for extended Hecke groups  $\overline{H}_q$ .*

3)  $H'_{p,q} \leq \overline{H}'_{p,q} \leq \overline{H}_{p,q}$ .

Now we study the second commutator subgroup  $\overline{H}''_{p,q}$  of  $\overline{H}_{p,q}$ , except the case  $p$  and  $q$  are both even numbers, since in this case  $\overline{H}''_{p,q}$  has infinite index in  $\overline{H}_{p,q}$ .

**Theorem 3.** *Let  $p$  and  $q$  be integers such that  $2 \leq p \leq q$ ,  $p + q > 4$ .*

*i) If  $p$  and  $q$  are both odd numbers, then  $\overline{H}''_{p,q} = H'_{p,q}$ .*

*ii) If  $p$  is an odd number and  $q$  is an even number, then  $\overline{H}''_{p,q}$  is a free group of rank  $(q - 1)p^2 - pq + 1$ .*

*iii) If  $p$  is an even number and  $q$  is an odd number, then  $\overline{H}''_{p,q}$  is a free group of rank  $(p - 1)q^2 - pq + 1$ .*

*Proof.* The case *i)* can be seen from the Theorem 2.1. Since the cases *ii)* and *iii)* are similar, we prove only the case *iii)*. If we take  $a = X^2$ ,  $b = Y$ ,  $c = XYX^{-1}$ , then the quotient group  $\overline{H}'_{p,q}/\overline{H}''_{p,q}$  is the group obtained by adding the relations  $ab = ba$ ,  $ac = ca$  and  $bc = cb$  to the relations of  $\overline{H}'_{p,q}$ . Then

$$\overline{H}'_{p,q}/\overline{H}''_{p,q} \cong C_{p/2} \times C_q \times C_q.$$

Therefore, we obtain  $|\overline{H}'_{p,q} : \overline{H}''_{p,q}| = pq^2/2$ . Now we choose  $\Sigma = \{I, a, a^2, \dots, a^{(p/2)-1}, b, b^2, \dots, b^{q-1}, c, c^2, \dots, c^{q-1}, ab, ab^2, \dots, ab^{q-1}, a^2b, a^2b^2, \dots, a^2b^{q-1}, \dots, a^{(p/2)-1}b, a^{(p/2)-1}b^2, \dots, a^{(p/2)-1}b^{q-1}, ac, ac^2, \dots, ac^{q-1}, a^2c, a^2c^2, \dots, a^2c^{q-1}, \dots, a^{(p/2)-1}c, a^{(p/2)-1}c^2, \dots, a^{(p/2)-1}c^{q-1}, bc, bc^2, \dots, bc^{q-1}, b^2c, b^2c^2, \dots, b^2c^{q-1}, \dots, b^{q-1}c, b^{q-1}c^2, \dots, b^{q-1}c^{q-1}, abc, abc^2, \dots, abc^{q-1}, ab^2c, ab^2c^2, \dots, ab^2c^{q-1}, \dots, ab^{q-1}c, ab^{q-1}c^2, \dots, ab^{q-1}c^{q-1}, \dots, a^{(p/2)-1}bc, a^{(p/2)-1}bc^2, \dots, a^{(p/2)-1}bc^{q-1}, a^{(p/2)-1}b^2c, a^{(p/2)-1}b^2c^2, \dots, a^{(p/2)-1}b^2c^{q-1}, \dots, a^{(p/2)-1}b^{q-1}c, a^{(p/2)-1}b^{q-1}c^2, \dots, a^{(p/2)-1}b^{q-1}c^{q-1}\}$  as a Schreier transversal for  $\overline{H}''_{p,q}$ .

According to the Reidemeister-Schreier method, we get the generators of  $\overline{H}''_{p,q}$  as follows. There are total  $(p - 1)q^2 - pq + 1$  generators obtained by using the theorem of Nielsen in [13]. These generators are  $[a, b], [a, b^2], \dots, [a, b^{q-1}], [a^2, b], [a^2, b^2], \dots, [a^2, b^{q-1}], \dots, [a^{(p/2)-1}, b], [a^{(p/2)-1}, b^2], \dots, [a^{(p/2)-1}, b^{q-1}], [a, c], [a, c^2], \dots, [a, c^{q-1}], [a^2, c], [a^2, c^2], \dots, [a^2, c^{q-1}], \dots, [a^{(p/2)-1}, c], [a^{(p/2)-1}, c^2], \dots, [a^{(p/2)-1}, c^{q-1}], [b, c], [b, c^2], \dots, [b, c^{q-1}], [b^2, c], [b^2, c^2], \dots, [b^2, c^{q-1}], \dots, [b^{q-1}, c], [b^{q-1}, c^2], \dots, [b^{q-1}, c^{q-1}], [a, bc], [a, b^2c], \dots, [a, b^{q-1}c], [a, bc^2], [a, b^2c^2], \dots, [a, b^{q-1}c^2], \dots, [a, bc^{q-1}], [a, b^2c^{q-1}], \dots, [a, b^{q-1}c^{q-1}], [a^2, bc], [a^2, b^2c], \dots,$

$[a^2, b^{q-1}c], [a^2, bc^2], [a^2, b^2c^2], \dots, [a^2, b^{q-1}c^2], \dots, [a^2, bc^{q-1}], [a^2, b^2c^{q-1}], \dots,$   
 $[a^2, b^{q-1}c^{q-1}], \dots, [a^{(p/2)-1}, bc], [a^{(p/2)-1}, b^2c], \dots, [a^{(p/2)-1}, b^{q-1}c], [a^{(p/2)-1}, bc^2],$   
 $[a^{(p/2)-1}, b^2c^2], \dots, [a^{(p/2)-1}, b^{q-1}c^2], \dots, [a^{(p/2)-1}, bc^{q-1}], [a^{(p/2)-1}, b^2c^{q-1}], \dots,$   
 $[a^{(p/2)-1}, b^{q-1}c^{q-1}], [ab, bc], [ab, b^2c], \dots, [ab, b^{q-1}c], [ab, bc^2], [ab, b^2c^2], \dots, [ab, b^{q-1}c^2],$   
 $\dots, [ab, bc^{q-1}], [ab, b^2c^{q-1}], \dots, [ab, b^{q-1}c^{q-1}], [a^2b, bc], [a^2b, b^2c], \dots, [a^2b, b^{q-1}c],$   
 $[a^2b, bc^2], [a^2b, b^2c^2], \dots, [a^2b, b^{q-1}c^2], \dots, [a^2b, bc^{q-1}], [a^2b, b^2c^{q-1}], \dots, [a^2b, b^{q-1}c^{q-1}],$   
 $\dots, [a^{(p/2)-1}b, bc], [a^{(p/2)-1}b, b^2c], \dots, [a^{(p/2)-1}b, b^{q-1}c], [a^{(p/2)-1}b, bc^2], [a^{(p/2)-1}b, b^2c^2],$   
 $\dots, [a^{(p/2)-1}b, b^{q-1}c^2], \dots, [a^{(p/2)-1}b, bc^{q-1}], [a^{(p/2)-1}b, b^2c^{q-1}], \dots, [a^{(p/2)-1}b, b^{q-1}c^{q-1}].$   
□

Also, the signature of  $\overline{H}_{p,q}''$  is  $(\frac{q^2(p-1)-q(p+1)+2}{2}; \infty(q))$ .

**Example 2.** Let  $p = 4$  and  $q = 5$ . Then we have  $|\overline{H}'_{4,5} : \overline{H}''_{4,5}| = 50$ . We choose  $\Sigma = \{I, a, b, b^2, b^3, b^4, c, c^2, c^3, c^4, ab, ab^2, ab^3, ab^4, ac, ac^2, ac^3, ac^4, bc, bc^2, bc^3, bc^4, b^2c, b^2c^2, b^2c^3, b^2c^4, b^3c, b^3c^2, b^3c^3, b^3c^4, b^4c, b^4c^2, b^4c^3, b^4c^4, abc, abc^2, abc^3, abc^4, ab^2c, ab^2c^2, ab^2c^3, ab^2c^4, ab^3c, ab^3c^2, ab^3c^3, ab^3c^4, ab^4c, ab^4c^2, ab^4c^3, ab^4c^4\}$  as a Schreier transversal for  $\overline{H}''_{4,5}$ . According to the Reidemeister-Schreier method, we get total 56 generators of  $\overline{H}''_{4,5}$  as  $[a, b], [a, b^2], [a, b^3], [a, b^4], [a, c], [a, c^2], [a, c^3], [a, c^4], [b, c], [b, c^2], [b, b^3], [b, c^4], [b^2, c], [b^2, c^2], [b^2, c^3], [b^2, c^4], [b^3, c], [b^3, c^2], [b^3, c^3], [b^3, c^4], [b^4, c], [b^4, c^2], [b^4, c^3], [b^4, c^4], [a, bc], [a, b^2c], [a, b^3c], [a, b^4c], [a, bc^2], [a, b^2c^2], [a, b^3c^2], [a, b^4c^2], [a, bc^3], [a, b^2c^3], [a, b^3c^3], [a, b^4c^3], [a, bc^4], [a, b^2c^4], [a, b^3c^4], [a, b^4c^4], [ab, bc], [ab, b^2c], [ab, b^3c], [ab, b^4c], [ab, bc^2], [ab, b^2c^2], [ab, b^3c^2], [ab, b^4c^2], [ab, bc^3], [ab, b^2c^3], [ab, b^3c^3], [ab, b^4c^3], [ab, bc^4], [ab, b^2c^4], [ab, b^3c^4], [ab, b^4c^4]$ . Also, the signature of  $\overline{H}''_{4,5}$  is  $(26; \infty^{(5)})$ .

**Corollary 3.** Let  $p$  and  $q$  be integers such that  $2 \leq p \leq q, p + q > 4$ . In case  $p$  and  $q$  are both even numbers,  $|\overline{H}_{p,q} : \overline{H}_{p,q}^{(n)}| = \infty$  for  $n \geq 2$  and, in other cases of  $p$  and  $q$ ,  $|\overline{H}_{p,q} : \overline{H}_{p,q}^{(n)}| = \infty$  for  $n \geq 3$ .

**Corollary 4.** If  $p = q$  odd number, then the commutator subgroups  $\overline{H}_{q,q}''$  coincides with the ones given in [15] for extended Hecke groups  $\overline{H}_q$ .

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