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Solution of a Class of Optimization Problems Based on Hyperbolic Penalty Dynamic Framework

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In this study, a gradient-based dynamic system is constructed in order to solve a certain class of optimization problems. For this purpose, the hyperbolic penalty function is used. Firstly, the constrained optimization problem is replaced with an equivalent unconstrained optimization problem via the hyperbolic penalty function. Thereafter, the nonlinear dynamic model is defined by using the derivative of the unconstrained optimization problem with respect to decision variables. To solve the resulting differential system, a steepest descent search technique is used. Finally, some numerical examples are presented for illustrating the performance of the nonlinear hyperbolic penalty dynamic system.

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1. Introduction

Numerous commonly encountered problems in modern science and technology involve a type of optimization problem, and optimization is thus an attractive research area for many scientists in various disciplines [1–4]. In the literature, several efficient methods have been discussed for finding the feasible best solution to these problems. A detailed and modern discussion of these methods can be found in [5].

The gradient-based method is one of these approaches, and was first introduced by Arrow and Hurwicz [6]. In this approach, the optimization problem is transformed into a system of ordinary differential equations, which are equipped with optimality conditions, in order to find the optimal solutions of the optimization problem. Similar studies can be found in the literature [7–17].

In this paper, we construct a hyperbolic penalty dynamic system for solving a certain class of inequality constrained optimization problem. The proposed system shows that the steady state solutions $x(t)$ of the dynamic system can be approximated to the optimal solutions x^* of the optimization problem on a continuous path as $t \to \infty$. The steepest descent search technique is used to achieve the intended results.

2. Preliminaries

2.1. Inequality constrained problem

Consider the nonlinear programming problem with inequality constraints:

$$\begin{aligned} &\min\nolimits_{x\in \mathrm{R}^n} f(x), \\ &\text{s.t. } g(x) \geq 0, \end{aligned} \tag{1}$$

where $f : \mathrm{R}^n \to \mathrm{R}$, $g : \mathrm{R}^n \to \mathrm{R}^m$ are C^2 functions and we assume that the feasible set is non-empty.

By the penalty function methods, the constrained optimization problem is transformed into an unconstrained optimization problem. In the literature, we frequently see that the following penalty function can be used for the problem (1),

$$P_{\text{penalty}}(g(x)) = \sum_{i=1}^{m} \min(g_i(x), 0), \tag{2}$$

where $\mu > 0$ $(\mu \to \infty)$ is an auxiliary penalty variable. Hence, the solutions of the inequality constrained problem (1) can be obtained under some conditions from the following unconstrained optimization problem,

$$\begin{aligned} &\min P_{\text{penalty}}(x, \mu) = f(x) + \mu \sum_{i=1}^{m} \min(g_i(x), 0) \\ &\text{s.t. } x \in \mathrm{R}^n, \end{aligned} \tag{3}$$

where $\mu > 0$ $(\mu \to \infty)$ is an auxiliary penalty variable. This result can be expressed briefly in the following theorem.

Theorem 1. [5, pp. 404] Let $\{x_k\}$ be a sequence generated by the penalty method. Then any limit point of the sequence is a solution to the constrained problem.

2.2. Hyperbolic penalty function method

The hyperbolic penalty method was first presented by Xavier in 1982 [18] in order to solve the constrained optimization problem (1). In a similar manner, the hyperbolic penalty methods transform constrained optimization problem (1) into an unconstrained optimization problem as follows,

$$\min f(x) + \sum_{i=1}^{m} P_{\text{hyp}}(g_i(x)). \tag{4}$$

The second term is the hyperbolic penalty function and can be written as,

$$P_{\text{hyp}}(y, \alpha, \tau) = -\left(\frac{1}{2}\tan\alpha\right) y$$

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$$+\sqrt{\left(\frac{1}{2}\tan\alpha\right)^2 y^2+\tau^2}, \tag{5}$$

where $\alpha \in [0, \pi/2)$ and $\tau \geq 0$. For convenience, the hyperbolic penalty function can be modelled by the following form:

$$P_{\text{hyp}}(y,\lambda,\tau) = -\lambda y + \sqrt{\lambda^2 y^2 + \tau^2}, \tag{6}$$

where $\lambda \geq 0$ ($\lambda \to \infty$) and $\tau \geq 0$ ($\tau \to 0$).

The geometric idea behind the hyperbolic penalty function can be described briefly as follows. Initially, when the constraint is violated, the parameter λ increases; this term goes to infinity in order to apply a penalty to the violated constraint with the help of the hyperbolic penalty term (6) until the violation is over. In this manner, it works as a classical exterior penalty function. On the other hand, when the constraint is feasible, the parameter τ decreases sequentially to zero. In this way, the hyperbolic penalty function behaves like an interior penalty and pushes the iteration point to the boundary of the feasible region. The graphical representation of the hyperbolic penalty function can be found in [19].

3. Hyperbolic penalty dynamic system

In this section, we construct the hyperbolic penalty dynamic system to solve the inequality constrained optimization problem (1). The hyperbolic penalty function (6) is used for the optimization problem (1), and is converted to the unconstrained optimization problem (4). The numerical solution of problem (4) was investigated by the following vector differential equation,

$$\begin{aligned} &\tfrac{\mathrm{d}x}{\mathrm{d}t} = -\nabla f(x) - \sum_{i=1}^{m} \nabla P_{\text{hyp}}(g_i(x)), \\ &\tfrac{\mathrm{d}\lambda}{\mathrm{d}t} = r\lambda,\ r>0, \\ &\tfrac{\mathrm{d}\tau}{\mathrm{d}t} = -q\tau,\ 0<p<1, \end{aligned} \tag{7}$$

where $\nabla f(x)$ and $\nabla P_{\text{hyp}}(g_i(x))$ are the gradient vectors of the objective function and the hyperbolic penalty function with respect to $x \in \mathrm{R}^n$, respectively. For simplicity, the hyperbolic penalty dynamic system (7) can be written as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \begin{pmatrix} -\nabla f(x) - \sum\limits_{i=1}^{m} \nabla P_{\text{hyp}}(g_i(x)) \\ r\lambda \\ -q\tau \end{pmatrix}, \tag{8}$$

where $z^T = (x^T, \lambda, \tau)$.

Definition 1. *A point x_e is referred to an equilibrium point of (8) if it satisfies the right-hand side of Eq. (8).*

The Euler discretization scheme is used to find the stable equilibrium point of the hyperbolic penalty dynamic system (8). The following iteration formulas can be obtained to approximate the solution of (8),

$$\begin{aligned} &x_i^{k+1}(t) = x_i^k(t) + \Delta_t \\ &\quad \times \left(-\nabla x_i f(x) - \textstyle\sum_{s=1}^{m} \nabla x_i P_{\text{hyp}}(g_s(x))\right), \\ &\lambda^{k+1} = \lambda^k + \Delta_t (r\lambda), \\ &\tau^{k+1} = \tau^k - \Delta_t (q\tau), \end{aligned} \tag{9}$$

where ∇x_i, $i = 1...n$ is a gradient of a given function with respect to x_i, and Δ_t is the step size for every interval.

4. Numerical examples and application

In order to demonstrate the effectiveness of the proposed hyperbolic penalty dynamic system, we test several examples using our system (8). We also compare the numerical performance of the proposed dynamic system with various initial points, both feasible and unfeasible.

Example 1. Consider the following nonlinear programming problem [20, Problem No: 12],

$$\begin{aligned} &\text{minimize} \quad f(x) = 0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2, \\ &\text{subject to} \quad g(x) = 25 - 4x_1^2 - x_2^2 \geq 0. \end{aligned} \tag{10}$$

By using the hyperbolic penalty function (6), problem (10) can be transformed into the unconstrained optimization problem as follows,

$$\begin{aligned} F(x,\lambda,\tau) &= \left(0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2\right) \\ &+ P_{\text{hyp}}(g(x), \lambda, \tau). \end{aligned}$$

The corresponding dynamic system from (7) is constructed as

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t} = -\nabla x_i F(x,\lambda,\tau),\ i = 1...n, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} = r\lambda,\ \frac{\mathrm{d}\tau}{\mathrm{d}t} = -q\tau, \\ x_1(0) = 0,\ x_2(0) = 0. \end{cases} \tag{11}$$

where $x_1(0)$ and $x_2(0)$ are the feasible initial conditions.

By utilizing the Euler discretization scheme (9) with $\lambda = 10$, $\tau = 0.1$, $r = 100$, $q = 0.01$ and step size $\Delta_t = 0.0001$, the trajectory of the system (11) approaches the expected optimal solution $x^* = (2, 3)$ of the optimization problem (10) (see Figs. 1 and 2).

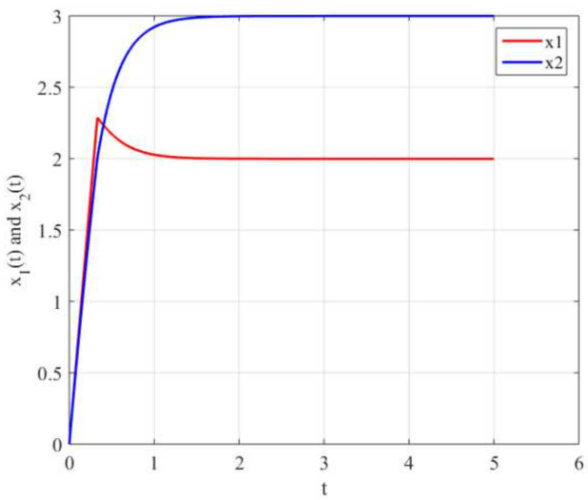

Fig. 1. Transient behavior x_i ($i = 1, 2$) of the dynamic system (10).

Example 2. Consider the following nonlinear programming problem with inequality constraints [21, Problem No: 337],

$$\begin{aligned} &\text{minimize} \quad f(x) = 9x_1^2 + x_2^2 + 9x_3^2, \\ &\text{subject to} \quad \begin{aligned} &g_1(x) = x_1x_2 - 1 \geq 0, \\ &x_2 \geq 1, x_3 \leq 1. \end{aligned} \end{aligned} \tag{12}$$

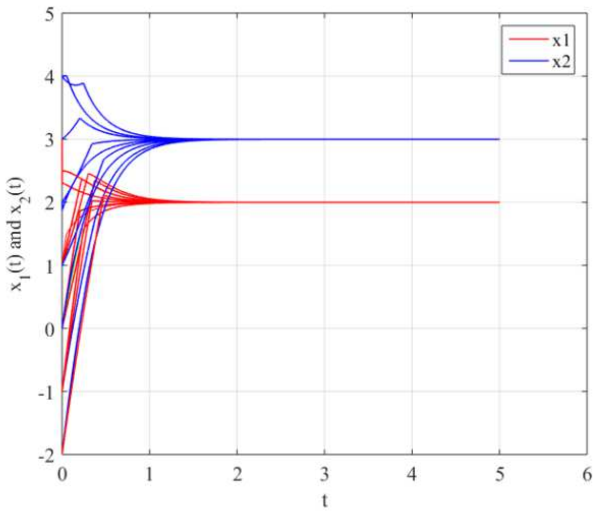

Fig. 2. Transient behavior x_i $(i = 1, 2)$ of the dynamic system (10) with ten random initial points.

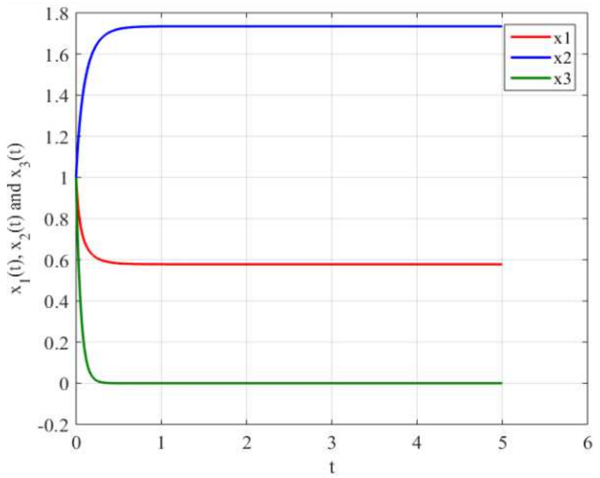

Fig. 3. Transient behavior x_i $(i = 1, 2, 3)$ of the dynamic system (13).

This is a practical problem with an unknown exact solution. The expected solution is $x^* = \left(0.5774,\ 1.732,\ -0.2026 \times 10^{-5}\right)$.

By using the hyperbolic penalty function (6), we have the following unconstrained optimization problem,

$$F\left(x, \lambda, \tau\right) = \left(9x_1^2 + x_2^2 + 9x_3^2\right) + \sum_{s=1}^{3} P_{\text{hyp}}\left(g_s(x), \lambda, \tau\right),$$

where P_{hyp} is the hyperbolic penalty function. The corresponding dynamic system from (7) is defined as

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t} = -\nabla x_i F\left(x, \lambda, \tau\right),\ i = 1 \ldots n, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} = r\lambda,\ \frac{\mathrm{d}\tau}{\mathrm{d}t} = -q\tau, \\ x_1(0) = 1,\ x_2(0) = 1,\ x_2(0) = 1, \end{cases} \tag{13}$$

where $x_1(0)$, $x_2(0)$ and $x_3(0)$ are the feasible initial conditions. Utilizing the Euler discretization scheme (9) with $\lambda = 100$, $\tau = 0.1$, $r = 10$, $q = 0.001$ and step size $\Delta_t = 0.0001$, the trajectory of the system (13) approaches the optimal solution x^* of the optimization problem (12) (see Figs. 3 and 4).

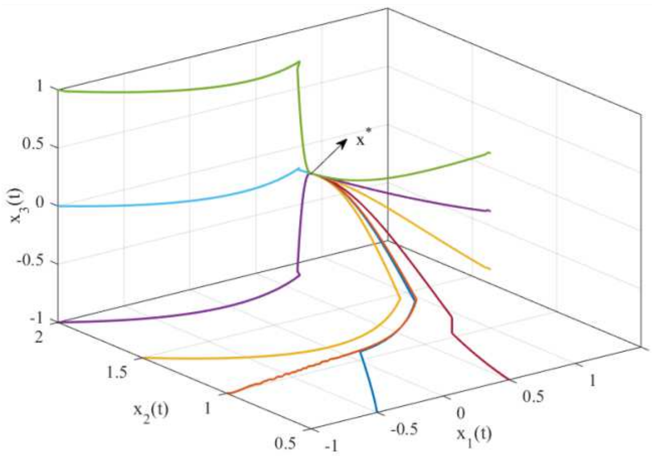

Fig. 4. Transient behavior x_i $(i = 1, 2, 3)$ of the dynamic system (13) with ten different initial points.

5. Conclusions

In this study, we construct a nonlinear dynamic system using the hyperbolic penalty function for a certain class of inequality constrained optimization problems. For this purpose, the steepest descent search technique is adapted to the dynamic system, and the Euler discretization scheme is used to find the steady state solution of the hyperbolic penalty dynamic system. Numerical examples show that the structure of the proposed dynamic system is both effective and reliable in solving a class of inequality constrained optimization problems with various initial points.

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