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METALLIC SHAPED HYPERSURFACES IN LORENTZIAN SPACE FORMS

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ABSTRACT. We show that metallic shaped hypersurfaces in Lorentzian space forms are isoparametric and obtain their full classification.

1. INTRODUCTION

Let p and q be two positive integers. Consider the quadratic equation

$$x^2 - px - q = 0.$$

The positive solution of this equation is

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$$

and called a *member of the metallic means family* (briefly MMF) [5]. These numbers are called (p, q)-*metallic numbers* [5]. For special values of p and q de Spinadel defined in [6] the following metallic means:

i) For $p = q = 1$ we obtain $\sigma_{\mathrm{G}} = \frac{1+\sqrt{5}}{2}$, which is the *golden mean*,
ii) For $p = 2$ and $q = 1$ we obtain $\sigma_{\mathrm{Ag}} = 1 + \sqrt{2}$, which is the *silver mean*,
iii) For $p = 3$ and $q = 1$ we obtain $\sigma_{\mathrm{Br}} = \frac{3+\sqrt{13}}{2}$, which is the *bronze mean*,
iv) For $p = 1$ and $q = 2$ we obtain $\sigma_{\mathrm{Cu}} = 2$, which is the *copper mean*,
v) For $p = 1$ and $q = 3$ we obtain $\sigma_{\mathrm{Ni}} = \frac{1+\sqrt{13}}{2}$, which is the *nickel mean*.

Hence the metallic means family is a generalization of the golden mean. It is well-known that the golden mean is used widely in mathematics, natural sciences, music, art, etc. The MMF have been used in describing fractal geometry, quasiperiodic dynamics (for more details see [9] and the references therein). Furthermore, El Naschie [7] obtained the relationships between the Hausdorff dimension of higher order Cantor sets and the golden mean or silver mean.

In [9], Hreţcanu and Crasmareanu defined the metallic structure on a manifold M as a $(1, 1)$-tensor field J on M satisfying the equation

$$J^2 = pJ + qI,$$

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where I is the Kronecker tensor field of M and p, q are positive integers. If $p = q = 1$, one obtains a golden structure on a manifold M, which was defined and studied in [4] and [8].

In [11], the present authors defined the metallic shaped hypersurfaces and they obtained the full classification of the metallic shaped hypersurfaces in real space forms. A hypersurface M is called a *metallic shaped hypersurface* if the shape operator A of M is a metallic structure, i.e., $A^2 = pA + qI$, where I is the identity on the tangent bundle of M and p, q are positive integers. If $p = q = 1$, one obtains a golden shaped hypersurface, defined by Crasmareanu, Hreţcanu and Munteanu in [3]. The full classification of golden shaped hypersurfaces in real space forms was given in [3].

In [13], Yang and Fu studied the golden shaped hypersurfaces in Lorentzian space forms and gave the full classification of this type of hypersurfaces. In the present study, as a generalization of [13], we consider the metallic shaped hypersurfaces in Lorentzian space forms and obtain the classification of this type of hypersurfaces. Using Mathematica [12], we draw some pictures (see Figures 1 and 2).

2. Main results

Let M be a hypersurface of the Lorentzian space form $M_1^{n+1}(c)$ and for a certain normal vector field N, let $A = A_N$ be the shape operator. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the principal curvatures of M. If M has constant principal curvatures and its shape operator is diagonalized then it is called an *isoparametric hypersurface* [2].

Definition 2.1 ([11])**.** M is called a metallic shaped hypersurface if the shape operator A is a metallic structure. Hence A satisfies

$$A^2 = pA + qI,$$

where I is the identity on the tangent bundle of M, p and q are positive integers. If $p = q = 1$, then we obtain a golden shaped hypersurface (see [3] and [13]). If $p = 2$ and $q = 1$, then the hypersurface is called silver shaped; if $p = 3$ and $q = 1$, then it is called bronze shaped; if $p = 1$ and $q = 2$, then it is called copper shaped; if $p = 1$ and $q = 3$, then it is called nickel shaped.

It is known that if M is a Lorentzian hypersurface in $M_1^{n+1}(c)$, then the normal vector is spacelike. In [10], M. A. Magid showed that the shape operator is one of the following forms:

$$A = \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{bmatrix}, \tag{1}$$

$$A = \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ 1 & a_0 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-2} \end{bmatrix}, \tag{2}$$

$$A = \begin{bmatrix} a_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_0 & 1 & 0 & \dots & 0 \\ -1 & 0 & a_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-3} \end{bmatrix}, \tag{3}$$

$$A = \begin{bmatrix} a_0 & b_0 & 0 & \dots & 0 \\ -b_0 & a_0 & 0 & \dots & 0 \\ 0 & 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-2} \end{bmatrix}. \tag{4}$$

First we give the following proposition:

Proposition 2.1. *Let M be a metallic shaped hypersurface in a Lorentzian space form $M_1^{n+1}(c)$. Then the shape operator can be diagonalized and the principal curvatures are $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$, which means that the hypersurface is isoparametric.*

Proof. Assume that M is a spacelike hypersurface in the Lorentzian space form $M_1^{n+1}(c)$. Hence the normal vector is timelike and it is known that the shape operator can be diagonalized by choosing the orthogonal frame field on M. Since M is a metallic shaped hypersurface, the relation $A^2 = pA + qI$ gives us that the principal curvatures of the spacelike metallic shaped hypersurface are $\sigma_{p,q}$ and $p - \sigma_{p,q}$. Hence the hypersurface is isoparametric.

Now we consider the above four forms of the shape operators given by M. A. Magid in [10].

If the shape operator is of the form (1) and $A^2 = pA + qI$, then the principal curvatures of the hypersurface are $\sigma_{p,q}$ and $p - \sigma_{p,q}$.

If the shape operator is of the form (2), then $a_0^2 = pa_0 + q$, $2a_0 = p$, $a_1^2 = pa_1 + q$, $\dots$, $a_{n-2}^2 = pa_{n-2} + q$. So we get $q = -\frac{p^2}{4} < 0$, which is impossible because of the definition of q.

If the shape operator is of the form (3), then $a_0^2 = pa_0 + q$, $-1 = 0$, which is a contradiction.

If the shape operator is of the form (4), then $a_0^2 - b_0^2 = pa_0 + q$, $2a_0 b_0 = pb_0$, $a_1^2 = pa_1 + q$, $\dots$, $a_{n-2}^2 = pa_{n-2} + q$. So it follows that $b_0 = 0$ and $a_i = \sigma_{p,q}$ or $p - \sigma_{p,q}$. This proves the proposition. □

In [1], Abe, Koike, and Yamaguchi showed that the isoparametric hypersurfaces in $\mathbb{R}_1^{n+1}$ have the following cases:

(R1) $\mathbb{R}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}_1^{n+1} : x_1 = 0\}$, $A = [0]$,

(R2) $\mathbb{H}^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c},\ c < 0\}$, $A = \pm\sqrt{-c}I$,

(R3) $\mathbb{R}^r \times \mathbb{H}^{n-r} = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = \frac{1}{c},\ c < 0\}$, $A = \pm\left(0_r \oplus \sqrt{-c}I_{n-r}\right)$,

(R4) $\mathbb{R}_1^n = \{x \in \mathbb{R}_1^{n+1} : x_{n+1} = 0\}$, $A = [0]$,

(R5) $\mathbb{S}_1^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c},\ c > 0\}$, $A = \pm\sqrt{c}I$,

(R6) $\mathbb{R}^r \times \mathbb{S}_1^{n-r} = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = \frac{1}{c},\ c > 0\}$, $A = \pm\left(0_r \oplus \sqrt{c}I_{n-r}\right)$.

Here (R1)–(R3) are spacelike hypersurfaces and (R4)–(R6) are Lorentzian hypersurfaces in $\mathbb{R}_1^{n+1}$.

For the metallic shaped hypersurfaces in $\mathbb{R}_1^{n+1}$, we give the following theorem:

Theorem 2.1. *The only metallic shaped hypersurfaces of Minkowski space $\mathbb{R}_1^{n+1}$ are:*

1) $\mathbb{H}^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\}$, $A = \sqrt{-c}I$, *where* $c = -\sigma_{p,q}^2$,

2) $\mathbb{H}^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\}$, $A = -\sqrt{-c}I$, *where* $c = -(p - \sigma_{p,q})^2$,

3) $\mathbb{S}_1^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\}$, $A = \sqrt{c}I$, *where* $c = \sigma_{p,q}^2$,

4) $\mathbb{S}_1^n(c) = \{x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\}$, $A = -\sqrt{c}I$, *where* $c = (p - \sigma_{p,q})^2$.

Proof. We know from Proposition 2.1 that a metallic shaped isoparametric hypersurface of a Lorentzian space form has non-zero constant principal curvatures $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$. Because of this reason the cases (R1), (R3), (R4) and (R6) are impossible.

First we consider the case (R2). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $\sqrt{-c} = \sigma_{p,q}$, $c < 0$. So the spacelike metallic shaped hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\},\ A = \sqrt{-c}I,$$

where

$$c = -\sigma_{p,q}^2 = -\frac{p^2 + p\sqrt{p^2 + 4q} + 2q}{2}.$$

If the eigenvalue of the shape operator A is $p - \sigma_{p,q}$ then $-\sqrt{-c} = p - \sigma_{p,q}$, $c < 0$. Hence the spacelike metallic shaped hypersurface is

$$\mathbb{H}^n(c) = \left\{ x \in \mathbb{R}_1^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\},\ A = -\sqrt{-c}I,$$

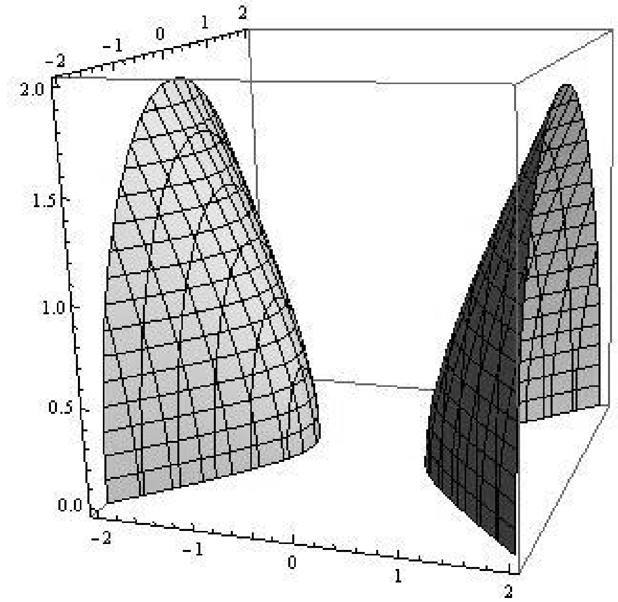

FIGURE 1. The golden shaped hypersurface $H^2\left(-\frac{3+\sqrt{5}}{2}\right) \subset \mathbb{R}^3_1$.

where

$$c = -(p-\sigma_{p,q})^2 = \frac{-p^2+p\sqrt{p^2+4q}-2q}{2}.$$

Now we consider the case (R5). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $\sqrt{c} = \sigma_{p,q}$, $c > 0$. So the Lorentzian metallic shaped hypersurface is

$$\mathbb{S}^n_1(c) = \left\{ x \in \mathbb{R}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\},\ A = \sqrt{c}I,$$

where

$$c = \sigma_{p,q}^2 = \frac{p^2+p\sqrt{p^2+4q}+2q}{2}.$$

If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $-\sqrt{c} = p-\sigma_{p,q}$, $c > 0$. Hence the Lorentzian metallic shaped hypersurface is

$$\mathbb{S}^n_1(c) = \left\{ x \in \mathbb{R}^{n+1}_1 : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c} \right\},\ A = -\sqrt{c}I,$$

where

$$c = (p-\sigma_{p,q})^2 = \frac{p^2-p\sqrt{p^2+4q}+2q}{2}.$$

This proves the theorem. □

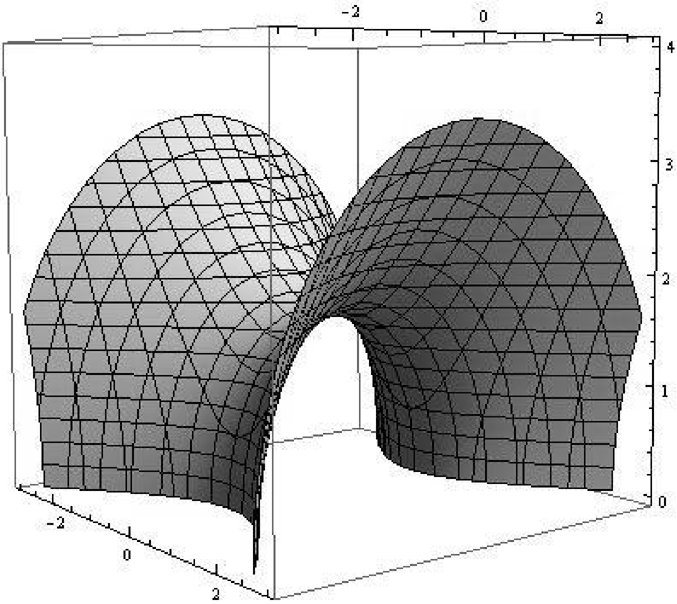

FIGURE 2. The golden shaped hypersurface $S_1^2\left(\frac{3-\sqrt{5}}{2}\right) \subset \mathbb{R}_1^3$.

The isoparametric hypersurfaces of de Sitter space $\mathbb{S}_1^{n+1}(1)$ were given by Abe, Koike, and Yamaguchi in [1] as follows:

(S1) $\mathbb{R}^n = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_1 = x_{n+2} + t_0,\ t_0 > 0\},\ A = \pm I,$

(S2) $\mathbb{S}^n(c) = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_1 = \sqrt{\frac{1}{c} - 1},\ 0 < c \leq 1\},\ A = \pm\sqrt{1-c}I,$

(S3) $\mathbb{H}^n(c) = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}},\ c < 0\},\ A = \pm\sqrt{1-c}I,$

(S4) $\mathbb{S}^r(c_1) \times \mathbb{H}^{n-r}(c_2) = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : \sum_{i=1}^{r+2} x_i^2 = \frac{1}{c_1},\ -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\},$
$(\frac{1}{c_1} + \frac{1}{c_2} = 1,\ c_1 > 0,\ c_2 < 0),\ A = \pm\left(\sqrt{1-c_1}I_r \oplus \sqrt{1-c_2}I_{n-r}\right),$

(S5) $\mathbb{S}_1^n(c) = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}},\ c \geq 1\},\ A = \pm\sqrt{c-1}I,$

(S6) $\mathbb{S}^r(c_1) \times \mathbb{S}_1^{n-r}(c_2) = \{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : \sum_{i=2}^{r+2} x_i^2 = \frac{1}{c_1},\ -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\},$
$(\frac{1}{c_1} + \frac{1}{c_2} = 1,\ c_1 > 0,\ c_2 > 0),\ A = \pm\left(\sqrt{c_1-1}I_r \oplus \left(-\sqrt{c_2-1}I_{n-r}\right)\right).$

Here (S1)–(S4) are spacelike hypersurfaces and (S5), (S6) are Lorentzian hypersurfaces of $\mathbb{S}_1^{n+1}(1)$.

Theorem 2.2. *The only metallic shaped hypersurfaces of de Sitter space $\mathbb{S}_1^{n+1}(1)$ are:*

1) $\mathbb{R}^n = \left\{x \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} : x_1 = x_{n+2} + t_0, t_0 > 0\right\},\ A = -I,\ q = p + 1.$

2) $\mathbb{S}^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_1 = \sqrt{\frac{1}{c} - 1}, 0 < c < 1\right\}$, $A = -\sqrt{1-c}I$, *where* $c = 1 - (p - \sigma_{p,q})^2$, $q - 1 < p$.

3) $\mathbb{H}^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}\right\}$, $A = \sqrt{1-c}I$, *where* $c = 1 - \sigma_{p,q}^2$.

4) $\mathbb{H}^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}, c < 0\right\}$, $A = -\sqrt{1-c}I$, *where* $c = 1 - (p - \sigma_{p,q})^2$ *and* $q > 1 + p$.

5) $\mathbb{S}_1^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}\right\}$, $A = \sqrt{c-1}I$, *where* $c = 1 + \sigma_{p,q}^2$.

6) $\mathbb{S}_1^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_{n+2} = \sqrt{1 - \frac{1}{c}}\right\}$, $A = -\sqrt{c-1}I$, *where* $c = 1 + (p - \sigma_{p,q})^2$.

7) $\mathbb{S}^r(c_1) \times \mathbb{S}_1^{n-r}(c_2) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : \sum_{i=2}^{r+2} x_i^2 = \frac{1}{c_1},\ -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\right\}$, $(\frac{1}{c_1} + \frac{1}{c_2} = 1)$, $A = \left(\sqrt{c_1 - 1}I_r \oplus \left(-\sqrt{c_2 - 1}I_{n-r}\right)\right)$, *where* $c_1 = 1 + \sigma_{p,1}^2$ *and* $c_2 = 1 + (p - \sigma_{p,1})^2$.

8) $\mathbb{S}^r(c_1) \times \mathbb{S}_1^{n-r}(c_2) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : \sum_{i=2}^{r+2} x_i^2 = \frac{1}{c_1},\ -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\right\}$, $(\frac{1}{c_1} + \frac{1}{c_2} = 1)$, $A = \left(-\sqrt{c_1 - 1}I_r \oplus \left(\sqrt{c_2 - 1}I_{n-r}\right)\right)$, *where* $c_1 = 1 + (p - \sigma_{p,1})^2$ *and* $c_2 = 1 + \sigma_{p,1}^2$.

Proof. By the use of Proposition 2.1, since a metallic shaped isoparametric hypersurface of a Lorentzian space form has non-zero constant principal curvatures $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma_{p,q} = \frac{p-\sqrt{p^2+4q}}{2}$, the case (S4) is not possible.

First we consider the case (S1). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $1 = \sigma_{p,q}$. This gives us $p + q = 1$. Since p and q are positive integers, this is not possible. Now assume that the eigenvalue of the shape operator A is $p - \sigma_{p,q}$. Then $-1 = p - \sigma_{p,q}$. This gives us $p + 1 = q$. Hence the spacelike metallic shaped hypersurface is

$$\mathbb{R}^n = \left\{x \in \mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2} : x_1 = x_{n+2} + t_0, t_0 > 0\right\},\ A = -I,\ q = p + 1.$$

Now we consider the case (S2). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $\sqrt{1-c} = \sigma_{p,q}$. Hence $c = \frac{2-p^2-p\sqrt{p^2+4q}-2q}{2}$. But for (S2), since $0 < c \leq 1$, p and q are positive integers, and this is not possible. If the eigenvalue of the shape operator A is $p - \sigma_{p,q}$, then $-\sqrt{1-c} = p - \sigma_{p,q}$. So we have $q - 1 < p$. Hence the spacelike metallic shaped hypersurface is

$$\mathbb{S}^n(c) = \left\{x \in \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2} : x_1 = \sqrt{\frac{1}{c} - 1}, 0 < c < 1\right\},\ A = -\sqrt{1-c}I,$$

where

$$c = 1 - (p - \sigma_{p,q})^2 = \frac{2 - p^2 + p\sqrt{p^2 + 4q} - 2q}{2}.$$

Now we consider the case (S3). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $\sqrt{1-c}=\sigma_{p,q}$. Hence $c=1-\sigma_{p,q}^2=\frac{2-p^2-p\sqrt{p^2+4q}-2q}{2}$. Since $c<0$ for (S3) the spacelike hypersurface is

$$\mathbb{H}^n(c)=\left\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:x_{n+2}=\sqrt{1-\frac{1}{c}},c<0\right\},\ A=\sqrt{1-c}I,$$

where $c=1-\sigma_{p,q}^2$. If the eigenvalue of the shape operator A is $p-\sigma_{p,q}$ then $-\sqrt{1-c}=p-\sigma_{p,q}$. This gives us $c=1-(p-\sigma_{p,q})^2=\frac{2-p^2+p\sqrt{p^2+4q}-2q}{2}$. Since $c<0$, we obtain $q>1+p$. Then the spacelike hypersurface is

$$\mathbb{H}^n(c)=\left\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:x_{n+2}=\sqrt{1-\frac{1}{c}},c<0\right\},\ A=-\sqrt{1-c}I,$$

where $c=1-(p-\sigma_{p,q})^2$ and $q>1+p$.

Now we consider the case (S5). If the eigenvalue of the shape operator A is $\sigma_{p,q}$ then $\sqrt{c-1}=\sigma_{p,q}$. This gives us $c=1+\sigma_{p,q}^2=\frac{2+p^2+p\sqrt{p^2+4q}+2q}{2}$. For all positive integers p,q we have $c>1$. Hence the Lorentzian hypersurface of $\mathbb{S}_1^{n+1}(1)$ is

$$\mathbb{S}_1^n(c)=\left\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:x_{n+2}=\sqrt{1-\frac{1}{c}}\right\},\ A=\sqrt{c-1}I,$$

where

$$c=1+\sigma_{p,q}^2.$$

If the eigenvalue of the shape operator A is $p-\sigma_{p,q}$ then $-\sqrt{c-1}=p-\sigma_{p,q}$. This gives us $c=1+(p-\sigma_{p,q})^2=\frac{2+p^2-p\sqrt{p^2+4q}+2q}{2}$. For all positive integers p,q, we have $c>1$. Hence the Lorentzian hypersurface of $\mathbb{S}_1^{n+1}(1)$ is

$$\mathbb{S}_1^n(c)=\left\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:x_{n+2}=\sqrt{1-\frac{1}{c}}\right\},\ A=-\sqrt{c-1}I,$$

where

$$c=1+(p-\sigma_{p,q})^2.$$

Now we consider the case (S6). If $\sqrt{c_1-1}=\sigma_{p,q}$ and $-\sqrt{c_2-1}=p-\sigma_{p,q}$, then $c_1=1+\sigma_{p,q}^2=\frac{2+p^2+p\sqrt{p^2+4q}+2q}{2}$ and $c_2=1+(p-\sigma_{p,q})^2=\frac{2+p^2-p\sqrt{p^2+4q}+2q}{2}$. Since $\frac{1}{c_1}+\frac{1}{c_2}=1$ we get $q=1$. Hence the Lorentzian hypersurface of $\mathbb{S}_1^{n+1}(1)$ is

$$\mathbb{S}^r(c_1)\times\mathbb{S}_1^{n-r}(c_2)=\left\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:\sum_{i=2}^{r+2}x_i^2=\frac{1}{c_1},\ -x_1^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\},$$
$$(\frac{1}{c_1}+\frac{1}{c_2}=1),\ A=\left(\sqrt{c_1-1}I_r\oplus\left(-\sqrt{c_2-1}I_{n-r}\right)\right),$$

where $c_1=1+\sigma_{p,1}^2$ and $c_2=1+(p-\sigma_{p,1})^2$. If $-\sqrt{c_1-1}=p-\sigma_{p,q}$ and $\sqrt{c_2-1}=\sigma_{p,q}$, then $c_1=1+(p-\sigma_{p,q})^2$ and $c_2=1+\sigma_{p,q}^2$. Since $\frac{1}{c_1}+\frac{1}{c_2}=1$ we get $q=1$. Hence the Lorentzian hypersurface of $\mathbb{S}_1^{n+1}(1)$ is

$$\mathbb{S}^r(c_1)\times\mathbb{S}_1^{n-r}(c_2)=\{x\in\mathbb{S}_1^{n+1}(1)\subset\mathbb{R}_1^{n+2}:\sum_{i=2}^{r+2}x_i^2=\frac{1}{c_1},\ -x_1^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\},$$
$$(\frac{1}{c_1}+\frac{1}{c_2}=1),\ A=\left(-\sqrt{c_1-1}I_r\oplus\left(\sqrt{c_2-1}I_{n-r}\right)\right),$$

where $c_1=1+(p-\sigma_{p,1})^2$ and $c_2=1+\sigma_{p,1}^2$.

This proves the theorem. □

Abe, Koike, and Yamaguchi classified isoparametric hypersurfaces of $\mathbb{H}_1^{n+1}(-1)$ as follows ([1]):

(H1) $\mathbb{H}^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1},\ c\le-1\right\}$, $A=\pm\sqrt{-1-c}I$,

(H2) $\mathbb{H}^r(c_1)\times\mathbb{H}^{n-r}(c_2)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:-x_1^2+\sum_{i=3}^{r+2}x_i^2=\frac{1}{c_1},-x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\}$, $(\frac{1}{c_1}+\frac{1}{c_2}=-1,\ c_1<0,\ c_2<0)$,
$A=\pm\left(\sqrt{-1-c_1}I_r\oplus\left(-\sqrt{-1-c_2}\right)I_{n-r}\right)$.

(H3) $\mathbb{R}_1^n=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=x_{n+2}+t_0,\ t_0>0\right\}$, $A=\pm I$,

(H4) $\mathbb{S}_1^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1},\ c>0\right\}$, $A=\pm\sqrt{1+c}I$,

(H5) $\mathbb{H}_1^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_{n+2}=\sqrt{-1-\frac{1}{c}},\ -1\le c<0\right\}$,
$A=\pm\sqrt{1+c}I$,

(H6) $\mathbb{S}_1^r(c_1)\times\mathbb{H}^{n-r}(c_2)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:-x_1^2+\sum_{i=3}^{r+2}x_i^2=\frac{1}{c_1},\ -x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\}$, $(\frac{1}{c_1}+\frac{1}{c_2}=1,\ c_1>0,\ c_2<0)$,
$A=\pm\left(\sqrt{1+c_1}I_r\oplus\sqrt{1+c_2}I_{n-r}\right)$.

Theorem 2.3. *The only metallic shaped hypersurfaces of anti-de Sitter space* $\mathbb{H}_1^{n+1}(1)$ *are:*

1) $\mathbb{H}^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1}\right\}$, $A=\sqrt{-1-c}I$, *where* $c=-(1+\sigma_{p,q}^2)$.

2) $\mathbb{H}^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1}\right\}$, $A=-\sqrt{-1-c}I$, *where* $c=-\left[1+(p-\sigma_{p,q})^2\right]$.

3) $\mathbb{H}^r(c_1)\times\mathbb{H}^{n-r}(c_2)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:-x_1^2+\sum_{i=3}^{r+2}x_i^2=\frac{1}{c_1},-x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\}$, $(\frac{1}{c_1}+\frac{1}{c_2}=-1)$, $A=\left(\sqrt{-1-c_1}I_r\oplus\left(-\sqrt{-1-c_2}\right)I_{n-r}\right)$, *where* $c_1=-(1+\sigma_{p,1}^2)$, $c_2=-\left[1+(p-\sigma_{p,1})^2\right]$.

4) $\mathbb{H}^r(c_1)\times\mathbb{H}^{n-r}(c_2)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:-x_1^2+\sum_{i=3}^{r+2}x_i^2=\frac{1}{c_1},-x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\}$, $(\frac{1}{c_1}+\frac{1}{c_2}=-1)$, $A=\left(-\sqrt{-1-c_1}I_r\oplus\left(\sqrt{-1-c_2}\right)I_{n-r}\right)$, *where* $c_1=-\left[1+(p-\sigma_{p,1})^2\right]$, $c_2=-(1+\sigma_{p,1}^2)$.

5) $\mathbb{R}_1^n=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=x_{n+2}+t_0,\,t_0>0\right\}$, $A=-I$, $q=p+1$.

6) $\mathbb{S}_1^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1}\right\}$, $A=\sqrt{1+c}I$, *where* $c=\sigma_{p,q}^2-1$.

7) $\mathbb{S}_1^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1},\,c>0\right\}$, $A=-\sqrt{1+c}I$, *where* $c=(p-\sigma_{p,q})^2-1$ *and* $q>1+p$.

8) $\mathbb{H}_1^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_{n+2}=\sqrt{-1-\frac{1}{c}},\,-1\le c<0\right\}$, $A=-\sqrt{1+c}I$, *where* $c=(p-\sigma_{p,q})^2-1$ *and* $q<1+p$.

Proof. Here (H1) and (H2) are spacelike hypersurfaces and (H3)–(H6) are Lorentzian hypersurfaces of $\mathbb{H}_1^{n+1}(-1)$.

Since the eigenvalues of the shape operator of the hypersurface in $\mathbb{H}_1^{n+1}(-1)$ are $\sigma_{p,q}>0$ and $p-\sigma_{p,q}<0$, (H6) is not possible.

We consider the case (H1). If the eigenvalue of the shape operator is $\sqrt{-1-c}=\sigma_{p,q}$, then we have $c=-(1+\sigma_{p,q}^2)=-\frac{2+p^2+p\sqrt{p^2+4q}+2q}{2}$. Hence the spacelike hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{H}^n(c)=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1}\right\},\ A=\sqrt{-1-c}I,$$

where $c=-(1+\sigma_{p,q}^2)$. If the eigenvalue of the shape operator is $-\sqrt{-1-c}=p-\sigma_{p,q}$, then we have $c=-\left[1+(p-\sigma_{p,q})^2\right]=-\frac{2+p^2-p\sqrt{p^2+4q}+2q}{2}$. Hence the spacelike hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{H}^n(c)=\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:x_1=\sqrt{\frac{1}{c}+1}\},\ A=-\sqrt{-1-c}I,$$

where $c=-\left[1+(p-\sigma_{p,q})^2\right]$.

Now we consider the case (H2). If the eigenvalue of the shape operator is $\sqrt{-1-c_1}=\sigma_{p,q}$ and $-\sqrt{-1-c_2}=p-\sigma_{p,q}$, then we have $c_1=-(1+\sigma_{p,q}^2)=-\frac{2+p^2+p\sqrt{p^2+4q}+2q}{2}$, $c_2=-\left[1+(p-\sigma_{p,q})^2\right]=-\frac{2+p^2-p\sqrt{p^2+4q}+2q}{2}$ and $q=1$. Hence the spacelike hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\begin{aligned}&\mathbb{H}^r(c_1)\times\mathbb{H}^{n-r}(c_2)\\&\quad=\left\{x\in\mathbb{H}_1^{n+1}(-1)\subset\mathbb{R}_2^{n+2}:-x_1^2+\sum_{i=3}^{r+2}x_i^2=\frac{1}{c_1},-x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{c_2}\right\},\\&\qquad(\frac{1}{c_1}+\frac{1}{c_2}=-1),\ A=\left(\sqrt{-1-c_1}I_r\oplus\left(-\sqrt{-1-c_2}\right)I_{n-r}\right),\end{aligned}$$

where $c_1 = -(1+\sigma_{p,1}^2)$ and $c_2 = -\left[1+(p-\sigma_{p,1})^2\right]$. If the eigenvalue of the shape operator is $-\sqrt{-1-c_1} = p-\sigma_{p,q}$ and $\sqrt{-1-c_2} = \sigma_{p,q}$, then we have $c_1 = -\left[1+(p-\sigma_{p,q})^2\right]$, $c_2 = -(1+\sigma_{p,q}^2)$ and $q=1$. Hence the spacelike hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is $\mathbb{H}^r(c_1)\times\mathbb{H}^{n-r}(c_2) = \{x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : -x_1^2 + \sum_{i=3}^{r+2} x_i^2 = \frac{1}{c_1}$, $-x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{c_2}\}$, $(\frac{1}{c_1}+\frac{1}{c_2} = -1)$, $A = \left(-\sqrt{-1-c_1}I_r \oplus \left(\sqrt{-1-c_2}\right) I_{n-r}\right)$, where $c_1 = -\left[1+(p-\sigma_{p,1})^2\right]$ and $c_2 = -(1+\sigma_{p,1}^2)$.

Now we consider the case (H3). If the eigenvalue of the shape operator is $1 = \sigma_{p,q}$, then we have $p+q=1$. Since p and q are positive integers, this is not possible. If the eigenvalue of the shape operator is $-1 = p-\sigma_{p,q}$, then $p+1=q$. Hence the Lorentzian hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{R}_1^n = \left\{x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = x_{n+2}+t_0,\ t_0>0\right\},\ A=-I,\ q=p+1.$$

Now we consider the case (H4). If the eigenvalue of the shape operator is $\sqrt{1+c} = \sigma_{p,q}$, then $c = \sigma_{p,q}^2 - 1 = \frac{p^2+p\sqrt{p^2+4q}+2q-2}{2}$. Hence the Lorentzian hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{S}_1^n(c) = \left\{x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c}+1},\ c>0\right\},\ A = \sqrt{1+c}I,$$

where $c = \sigma_{p,q}^2 - 1$. If the eigenvalue of the shape operator is $-\sqrt{1+c} = p-\sigma_{p,q}$ then $c = (p-\sigma_{p,q})^2 - 1 = \frac{p^2-p\sqrt{p^2+4q}+2q-2}{2}$. Since $c>0$, this gives us $q>1+p$. Hence the Lorentzian hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{S}_1^n(c) = \left\{x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_1 = \sqrt{\frac{1}{c}+1},\ c>0\right\},\ A = -\sqrt{1+c}I,$$

where $c = (p-\sigma_{p,q})^2 - 1$ and $q>1+p$.

Now we consider the case (H5). If the eigenvalue of the shape operator is $\sqrt{1+c} = \sigma_{p,q}$, then $c = \frac{p^2+p\sqrt{p^2+4q}+2q-2}{2}$. Since $-1 \le c < 0$, this is impossible. If the eigenvalue of the shape operator is $-\sqrt{1+c} = p-\sigma_{p,q}$, then $c = (p-\sigma_{p,q})^2 - 1 = \frac{p^2-p\sqrt{p^2+4q}+2q-2}{2}$. Since $-1 \le c < 0$, we obtain $q < 1+p$. Hence the Lorentzian hypersurface of $\mathbb{H}_1^{n+1}(-1)$ is

$$\mathbb{H}_1^n(c) = \left\{x \in \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} : x_{n+2} = \sqrt{-1-\frac{1}{c}},\ -1 \le c < 0\right\},$$
$$A = -\sqrt{1+c}I,$$

where $c = (p-\sigma_{p,q})^2 - 1$ and $q < 1+p$. □

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