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On the Characterizations of f-Biharmonic Legendre Curves in Sasakian Space Forms

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Abstract. We consider f-biharmonic Legendre curves in Sasakian space forms. We find curvature characterizations of these types of curves in four cases.

1. Introduction

Let (M, g) and (N, h) be two Riemannian manifolds and $\phi : (M, g) \to (N, h)$ a smooth map. The *energy functional of* ϕ is defined by

$$E(\phi) = \frac{1}{2} \int_M \left| d\phi \right|^2 v_g,$$

where v_g is the canonical volume form in M. If ϕ is a critical points of the energy functional $E(\phi)$, then it is called *harmonic* [5]. ϕ is called a *biharmonic map* if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M \left| \tau(\phi) \right|^2 v_g,$$

where $\tau(\phi)$ is *the tension field* of ϕ which is defined by $\tau(\phi) = trace \nabla d\phi$. The *Euler-Lagrange equation* of the bienergy functional $E_2(\phi)$ gives the *biharmonic equation*

$$\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = -\Delta^{\phi}\tau(\phi) - traceR^N(d\phi, \tau(\phi))d\phi = 0,$$

where J^{ϕ} is the Jacobi operator of ϕ and $\tau_2(\phi)$ is called the *bitension field of* ϕ [8].

Now, if $\phi : M \to N(c)$ is an isometric immersion from m-dimensional Riemannian manifold M to n-dimensional Riemannian space form $N(c)$ of constant sectional curvature c, then

$$\tau(\phi) = mH$$

and

$$\tau_2(\phi) = -m\Delta^{\phi}H + cm^2 H.$$

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Thus, ϕ is biharmonic if and only if

$$\Delta^{\phi} H = cmH,$$

(see [10]). In a different setting, in [4], B.Y. Chen defined a biharmonic submanifold $M \subset \mathbb{E}^n$ of the Euclidean space as its mean curvature vector field H satisfies $\Delta H = 0$, where Δ is the Laplacian. Replacing $c = 0$ in the above equation, we obtain Chen's definition.

ϕ is called an *f-biharmonic map* if it is a critical point of the *f-bienergy functional*

$$E_{2,f}(\phi) = \frac{1}{2} \int_M f \left|\tau(\phi)\right|^2 v_g.$$

The *Euler-Lagrange equation* of this functional gives the *f-biharmonic equation*

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + (\Delta f)\, \tau\left(\phi\right) + 2\nabla^{\phi}_{grad f} \tau\left(\phi\right) = 0.$$

(see [9]). It is clear that any harmonic map is biharmonic and any biharmonic map is f-biharmonic. If the map is non-harmonic biharmonic map, then it is called *proper biharmonic*. Likewise, if the map is non-biharmonic f-biharmonic map, then it is called *proper f-biharmonic* [11].

f-biharmonic maps were introduced in [9]. Ye-Lin Ou studied f-biharmonic curves in real space forms in [11]. D. Fetcu and C. Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [6] and [7]. We studied biharmonic Legendre curves in generalized Sasakian space forms and $\mathcal{S}$-space forms in [13] and [12], respectively. In the present paper, we consider f-biharmonic Legendre curves in Sasakian space forms. We obtain curvature equations for this kind of curves.

The paper is organized as follows. In Section 2, we give a brief introduction about Sasakian space forms. In Section 3, we obtain our main results. We also give two examples of proper f-biharmonic Legendre curves in $\mathbb{R}^7(-3)$.

2. Sasakian Space Forms

Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a contact metric manifold. If the Nijenhuis tensor of φ equals $-2d\eta \otimes \xi$, then $(M, \varphi, \xi, \eta, g)$ is called *Sasakian manifold* [2]. For a Sasakian manifold, it is well-known that:

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1}$$

$$\nabla_X \xi = -\varphi X. \tag{2}$$

(see [3]).

A *plane section in* T_pM is a φ-section if there exists a vector $X \in T_pM$ orthogonal to ξ such that $\{X, \varphi X\}$ span the section. The sectional curvature of a *φ-section* is called *φ-sectional curvature*. For a Sasakian manifold of constant φ-sectional curvature (i.e. Sasakian space form), the *curvature tensor R of M* is given by

$$\begin{array}{c} R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \\ \frac{c-1}{4}\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z \\ +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{array} \tag{3}$$

for all $X, Y, Z \in TM$ [3].

A submanifold of a Sasakian manifold is called an *integral submanifold* if $\eta(X) = 0$, for every tangent vector X. A 1-dimensional integral submanifold of a Sasakian manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ is called a *Legendre curve of M* [3]. Hence, a curve $\gamma : I \to M = (M^{2m+1}, \varphi, \xi, \eta, g)$ is called a Legendre curve if $\eta(T) = 0$, where T is the tangent vector field of γ.

3. f-Biharmonic Legendre curves in Sasakian Space Forms

Let $\gamma : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exist orthonormal vector fields $E_1, E_2, ..., E_r$ along γ such that

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\;\;... \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{4}$$

then γ is called a *Frenet curve of osculating order* r, where $\kappa_1, ..., \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$.

It is well-known that a Frenet curve of osculating order 1 is a *geodesic*; a Frenet curve of osculating order 2 is called a *circle* if κ_1 is a non-zero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a *helix of order* r if $\kappa_1, ..., \kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is shortly called a *helix*.

An arc-length parametrized curve $\gamma : (a, b) \to (M, g)$ is called an f-biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if the following equation is satisfied [11]:

$$f(\nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T) + 2f' \nabla_T \nabla_T T + f'' \nabla_T T = 0. \tag{5}$$

Now let $M = (M^{2m+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form and $\gamma : I \to M$ a Legendre Frenet curve of osculating order r. Differentiating

$$\eta(T) = 0 \tag{6}$$

and using (4), we get that

$$\eta(E_2) = 0. \tag{7}$$

Using (3), (4) and (7), it can be seen that

$$\nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= -3\kappa_1 \kappa_1' E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2\right) E_2 \\ &\quad + \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4, \end{aligned}$$

$$R(T, \nabla_T T)T = -\kappa_1 \frac{(c+3)}{4} E_2 - 3\kappa_1 \frac{(c-1)}{4} g(\varphi T, E_2)\varphi T,$$

(see [7]). If we denote the left-hand side of (5) with $f.\tau_3$, we find

$$\begin{aligned} \tau_3 &= \nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T + 2\frac{f'}{f} \nabla_T \nabla_T T + \frac{f''}{f} \nabla_T T \\ &= \left(-3\kappa_1 \kappa_1' - 2\kappa_1^2 \frac{f'}{f}\right) E_1 \\ &\quad + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + \kappa_1 \frac{(c+3)}{4} + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f}\right) E_2 \\ &\quad + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' + 2\kappa_1 \kappa_2 \frac{f'}{f}) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 \\ &\quad + 3\kappa_1 \frac{(c-1)}{4} g(\varphi T, E_2)\varphi T. \end{aligned} \tag{8}$$

Let $k = \min\{r, 4\}$. From (8), the curve γ is f-biharmonic if and only if $\tau_3 = 0$, that is,

(1) $c = 1$ or $\varphi T \perp E_2$ or $\varphi T \in span\{E_2, ..., E_k\}$; and

(2) $g(\tau_3, E_i) = 0$, for all $i = \overline{1, k}$.

So we can state the following theorem:

Theorem 3.1. *Let γ be a non-geodesic Legendre Frenet curve of osculating order r in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $k = \min\{r, 4\}$. Then γ is f- biharmonic if and only if*

(1) *$c = 1$ or $\varphi T \perp E_2$ or $\varphi T \in span\{E_2, ..., E_k\}$; and*

(2) *the first k of the following equations are satisfied (replacing $\kappa_k = 0$):*

$$\begin{aligned} 3\kappa_1' + 2\kappa_1 \tfrac{f'}{f} &= 0, \\ \kappa_1^2 + \kappa_2^2 &= \tfrac{c+3}{4} + \tfrac{3(c-1)}{4}\left[g(\varphi T, E_2)\right]^2 + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + \tfrac{3(c-1)}{4} g(\varphi T, E_2) g(\varphi T, E_3) + 2\kappa_2 \tfrac{f'}{f} + 2\kappa_2 \tfrac{\kappa_1'}{\kappa_1} &= 0, \\ \kappa_2\kappa_3 + \tfrac{3(c-1)}{4} g(\varphi T, E_2) g(\varphi T, E_4) &= 0. \end{aligned}$$

From Theorem 3.1, it can be easily seen that a curve γ with constant geodesic curvature κ_1 is f-biharmonic if and only if it is biharmonic. Since Fetcu and Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [7], we study curves with non-constant geodesic curvature κ_1 in this paper. If γ is a non-biharmonic f-biharmonic curve, then we call it *proper f-biharmonic*.

Now we give the interpretations of Theorem 3.1.

Case I. $c = 1$.

In this case γ is proper f-biharmonic if and only if

$$\begin{aligned} 3\kappa_1' + 2\kappa_1 \tfrac{f'}{f} &= 0, \\ \kappa_1^2 + \kappa_2^2 &= 1 + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + 2\kappa_2 \tfrac{f'}{f} + 2\kappa_2 \tfrac{\kappa_1'}{\kappa_1} &= 0, \\ \kappa_2\kappa_3 &= 0. \end{aligned} \tag{9}$$

Hence, we can state the following theorem:

Theorem 3.2. *Let γ be a Legendre Frenet curve in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$, $c = 1$ and $m > 1$. Then γ is proper f-biharmonic if and only if either*

(*i*) *γ is of osculating order $r = 2$ with $f = c_1\kappa_1^{-3/2}$ and κ_1 satisfies*

$$t \pm \frac{1}{2}\arctan\left(\frac{2 + c_3\kappa_1}{2\sqrt{-\kappa_1^2 - c_3\kappa_1 - 1}}\right) + c_4 = 0, \tag{10}$$

where $c_1 > 0$, $c_3 < -2$ and c_4 are arbitrary constants, t is the arc-length parameter and

$$\frac{1}{2}(-\sqrt{c_3^2 - 4} - c_3) < \kappa_1(t) < \frac{1}{2}(\sqrt{c_3^2 - 4} - c_3); \text{ or} \tag{11}$$

(*ii*) *γ is of osculating order $r = 3$ with $f = c_1\kappa_1^{-3/2}$, $\frac{\kappa_2}{\kappa_1} = c_2$ and κ_1 satisfies*

$$t \pm \frac{1}{2}\arctan\left(\frac{2 + c_3\kappa_1}{2\sqrt{-(1 + c_2^2)\kappa_1^2 - c_3\kappa_1 - 1}}\right) + c_4 = 0, \tag{12}$$

where $c_1 > 0$, $c_2 > 0$, $c_3 < -2\sqrt{(1 + c_2^2)}$ and c_4 are arbitrary constants, t is the arc-length parameter and

$$\frac{1}{2(1 + c_2^2)}(-\sqrt{c_3^2 - 4(1 + c_2^2)} - c_3) < \kappa_1(t) < \frac{1}{2(1 + c_2^2)}(\sqrt{c_3^2 - 4(1 + c_2^2)} - c_3). \tag{13}$$

Proof. From the first equation of (9), it is easy to see that $f = c_1\kappa_1^{-3/2}$ for an arbitrary constant $c_1 > 0$. So, we find

$$\frac{f'}{f} = \frac{-3}{2}\frac{\kappa_1'}{\kappa_1}, \frac{f''}{f} = \frac{15}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2 - \frac{3}{2}\frac{\kappa_1''}{\kappa_1}. \tag{14}$$

If $\kappa_2 = 0$, then γ is of osculating order $r = 2$ and the first two of equations (9) must be satisfied. Hence the second equation and (14) give us the ODE

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2(\kappa_1^2 - 1). \tag{15}$$

Let $\kappa_1 = \kappa_1(t)$, where t denotes the arc-length parameter. If we solve (15), we find (10). Since (10) must be well-defined, $-\kappa_1^2 - c_3\kappa_1 - 1 > 0$. Since $\kappa_1 > 0$, we have $c_3 < -2$ and (11).

If $\kappa_2 = constant \neq 0$, we find f is a constant. Hence γ is not proper f-biharmonic in this case. Let $\kappa_2 \neq constant$. From the fourth equation of (9), we have $\kappa_3 = 0$. So, γ is of osculating order $r = 3$. The third equation of (9) gives us $\frac{\kappa_2}{\kappa_1} = c_2$, where $c_2 > 0$ is a constant. Replacing in the second equation of (9), we have the ODE

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1 + c_2^2)\kappa_1^2 - 1]$$

which has the general solution (12) under the condition $c_3 < -2\sqrt{(1 + c_2^2)}$. (13) must be also satisfied. □

Remark 3.3. *If $m = 1$, then M is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write $\kappa_1 > 0$ and $\kappa_2 = 1$. The first and the third equations of (9) give us f is a constant. Hence γ cannot be proper f-biharmonic.*

Case II. $c \neq 1$, $\varphi T \perp E_2$.
In this case, $g(\varphi T, E_2) = 0$. From Theorem 3.1, we obtain

$$\begin{aligned} 3\kappa_1' + 2\kappa_1\tfrac{f'}{f} &= 0, \\ \kappa_1^2 + \kappa_2^2 &= \tfrac{c+3}{4} + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + 2\kappa_2\tfrac{f'}{f} + 2\kappa_2\tfrac{\kappa_1'}{\kappa_1} &= 0, \\ \kappa_2\kappa_3 &= 0. \end{aligned} \tag{16}$$

Firstly, we need the following proposition from [7]:

Proposition 3.4. *[7] Let γ be a Legendre Frenet curve of osculating order 3 in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $\varphi T \perp E_2$. Then $\{T = E_1, E_2, E_3, \varphi T, \nabla_T\varphi T, \xi\}$ is linearly independent at any point of γ. Therefore $m \geq 3$.*

Now we can state the following Theorem:

Theorem 3.5. *Let γ be a Legendre Frenet curve in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi T \perp E_2$. Then γ is proper biharmonic if and only if*

(1) *γ is of osculating order $r = 2$ with $f = c_1\kappa_1^{-3/2}$, $m \geq 2$, $\{T = E_1, E_2, \varphi T, \nabla_T\varphi T, \xi\}$ is linearly independent and*
(*a*) *if $c > -3$, then κ_1 satisfies*

$$t \pm \frac{1}{\sqrt{c+3}}\arctan\left(\frac{c + 3 + 2c_3\kappa_1}{\sqrt{c+3}\sqrt{-4\kappa_1^2 - 4c_3\kappa_1 - c - 3}}\right) + c_4 = 0,$$

(*b*) *if $c = -3$, then κ_1 satisfies*

$$t \pm \frac{\sqrt{-\kappa_1(\kappa_1 + c_3)}}{c_3\kappa_1} + c_4 = 0,$$

(*c*) *if* $c < -3$, *then* κ_1 *satisfies*

$$t \pm \frac{1}{\sqrt{-c-3}} \ln\left(\frac{c+3+2c_3\kappa_1 - \sqrt{-c-3}\sqrt{-4\kappa_1^2 - 4c_3\kappa_1 - c - 3}}{(c+3)\kappa_1} \right) + c_4 = 0; \text{ or}$$

(2) γ *is of osculating order* $r = 3$ *with* $f = c_1\kappa_1^{-3/2}$, $\frac{\kappa_2}{\kappa_1} = c_2 = constant > 0$, $m \geq 3$, $\{T = E_1, E_2, E_3, \varphi T, \nabla_T \varphi T, \xi\}$ *is linearly independent and*

(*a*) *if* $c > -3$, *then* κ_1 *satisfies*

$$t \pm \frac{1}{\sqrt{c+3}} \arctan\left(\frac{c+3+2c_3\kappa_1}{\sqrt{c+3}\sqrt{-4(1+c_2^2)\kappa_1^2 - 4c_3\kappa_1 - c - 3}} \right) + c_4 = 0,$$

(*b*) *if* $c = -3$, *then* κ_1 *satisfies*

$$t \pm \frac{\sqrt{-\kappa_1\left[(1+c_2^2)\kappa_1 + c_3\right]}}{c_3\kappa_1} + c_4 = 0,$$

(*c*) *if* $c < -3$, *then* κ_1 *satisfies*

$$t \pm \frac{1}{\sqrt{-c-3}} \ln\left(\frac{c+3+2c_3\kappa_1 - \sqrt{-c-3}\sqrt{-4(1+c_2^2)\kappa_1^2 - 4c_3\kappa_1 - c - 3}}{(c+3)\kappa_1} \right) + c_4 = 0,$$

where $c_1 > 0$, $c_2 > 0$, c_3 *and* c_4 *are convenient arbitrary constants, t is the arc-length parameter and* $\kappa_1(t)$ *is in convenient open interval.*

Proof. The proof is similar to the proof of Theorem 3.2. □

Case III. $c \neq 1$, $\varphi T \parallel E_2$.

In this case, $\varphi T = \pm E_2, g(\varphi T, E_2) = \pm 1$, $g(\varphi T, E_3) = g(\pm E_2, E_3) = 0$ and $g(\varphi T, E_4) = g(\pm E_2, E_4) = 0$. From Theorem 3.1, γ is biharmonic if and only if

$$\begin{aligned} 3\kappa_1' + 2\kappa_1\tfrac{f'}{f} &= 0, \\ \kappa_1^2 + \kappa_2^2 &= c + \tfrac{\kappa_1''}{\kappa_1} + \tfrac{f''}{f} + 2\tfrac{\kappa_1'}{\kappa_1}\tfrac{f'}{f}, \\ \kappa_2' + 2\kappa_2\tfrac{f'}{f} + 2\kappa_2\tfrac{\kappa_1'}{\kappa_1} &= 0, \\ \kappa_2\kappa_3 &= 0. \end{aligned} \tag{17}$$

Since $\varphi T \parallel E_2$, it is easily proved that $\kappa_2 = 1$. Then, the first and the third equations of (17) give us f is a constant. Thus, we give the following Theorem:

Theorem 3.6. *There does not exist any proper* f*-biharmonic Legendre curve in a Sasakian space form* $(M^{2m+1}, \varphi, \xi, \eta, g)$ *with* $c \neq 1$ *and* $\varphi T \parallel E_2$.

Case IV. $c \neq 1$ and $g(\varphi T, E_2)$ is not constant 0, 1 or −1.

Now, let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form and $\gamma : I \to M$ a Legendre curve of osculating order r, where $4 \leq r \leq 2m+1$ and $m \geq 2$. If γ is f-biharmonic, then $\varphi T \in span\{E_2, E_3, E_4\}$. Let $\theta(t)$ denote the angle function between φT and E_2, that is, $g(\varphi T, E_2) = \cos\theta(t)$. Differentiating $g(\varphi T, E_2)$ along γ and using (1) and (4), we find

$$\begin{aligned} -\theta'(t)\sin\theta(t) &= \nabla_T g(\varphi T, E_2) = g(\nabla_T\varphi T, E_2) + g(\varphi T, \nabla_T E_2) \\ &= g(\xi + \kappa_1\varphi E_2, E_2) + g(\varphi T, -\kappa_1 T + \kappa_2 E_3) \\ &= \kappa_2 g(\varphi T, E_3). \end{aligned} \tag{18}$$

If we write $\varphi T = g(\varphi T, E_2)E_2 + g(\varphi T, E_3)E_3 + g(\varphi T, E_4)E_4$, Theorem 3.1 gives us

$$3\kappa_1' + 2\kappa_1 \frac{f'}{f} = 0, \tag{19}$$

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2\theta + \frac{\kappa_1''}{\kappa_1} + \frac{f''}{f} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f}, \tag{20}$$

$$\kappa_2' + \frac{3(c-1)}{4}\cos\theta g(\varphi T, E_3) + 2\kappa_2\frac{f'}{f} + 2\kappa_2\frac{\kappa_1'}{\kappa_1} = 0, \tag{21}$$

$$\kappa_2\kappa_3 + \frac{3(c-1)}{4}\cos\theta g(\varphi T, E_4) = 0. \tag{22}$$

If we put (14) in (20) and (21) respectively, we obtain

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2\theta - \frac{\kappa_1''}{2\kappa_1} + \frac{3}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2, \tag{23}$$

$$\kappa_2' - \frac{\kappa_1'}{\kappa_1}\kappa_2 + \frac{3(c-1)}{4}\cos\theta g(\varphi T, E_3) = 0. \tag{24}$$

If we multiply (24) with $2\kappa_2$, using (18), we find

$$2\kappa_2\kappa_2' - 2\frac{\kappa_1'}{\kappa_1}\kappa_2^2 + \frac{3(c-1)}{4}(-2\theta'\cos\theta\sin\theta) = 0. \tag{25}$$

Let us denote $v(t) = \kappa_2^2(t)$, where t is the arc-length parameter. Then (25) becomes

$$v' - 2\frac{\kappa_1'}{\kappa_1}v = -\frac{3(c-1)}{4}(-2\theta'\cos\theta\sin\theta), \tag{26}$$

which is a linear ODE. If we solve (26), we obtain the following results:

i) If θ is a constant, then

$$\frac{\kappa_2}{\kappa_1} = c_2, \tag{27}$$

where $c_2 > 0$ is an arbitrary constant. From (18), we find $g(\varphi T, E_3) = 0$. Since $\left\|\varphi T\right\| = 1$ and $\varphi T = \cos\theta E_2 + g(\varphi T, E_4)E_4$, we get $g(\varphi T, E_4) = \pm\sin\theta$. By the use of (20) and (27), we find

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+c_2^2)\kappa_1^2 - \frac{c+3+3(c-1)\cos^2\theta}{4}].$$

ii) If $\theta = \theta(t)$ is a non-constant function, then

$$\kappa_2^2 = -\frac{3(c-1)}{4}\cos^2\theta + \lambda(t).\kappa_1^2, \tag{28}$$

where

$$\lambda(t) = -\frac{3(c-1)}{2}\int \frac{\cos^2\theta\kappa_1'}{\kappa_1^3}dt. \tag{29}$$

If we write (28) in (23), we have

$$[1+\lambda(t)].\kappa_1^2 = \frac{c+3+6(c-1)\cos^2\theta}{4} - \frac{\kappa_1''}{2\kappa_1} + \frac{3}{4}\left(\frac{\kappa_1'}{\kappa_1}\right)^2.$$

Now we can state the following Theorem:

Theorem 3.7. *Let $\gamma : I \to M$ be a Legendre curve of osculating order r in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$, where $r \geq 4$, $m \geq 2$, $c \neq 1$, $g(\varphi T, E_2) = \cos\theta(t)$ is not constant 0, 1 or −1. Then γ is proper f-biharmonic if and only if $f = c_1\kappa_1^{-3/2}$ and*

(*i*) *if θ is a constant,*

$$\frac{\kappa_2}{\kappa_1} = c_2,$$

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+c_2^2)\kappa_1^2 - \frac{c+3+3(c-1)\cos^2\theta}{4}],$$

$$\kappa_2\kappa_3 = \pm\frac{3(c-1)\sin 2\theta}{8},$$

(*ii*) *if θ is a non-constant function,*

$$\kappa_2^2 = -\frac{3(c-1)}{4}\cos^2\theta + \lambda(t).\kappa_1^2,$$

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1+\lambda(t))\kappa_1^2 - \frac{c+3+6(c-1)\cos^2\theta}{4}],$$

$$\kappa_2\kappa_3 = \pm\frac{3(c-1)\sin 2\theta \sin w}{8},$$

where c_1 and c_2 are positive constants, $\varphi T = \cos\theta E_2 \pm \sin\theta\cos wE_3 \pm \sin\theta\sin wE_4$, w is the angle function between E_3 and the orthogonal projection of φT onto span $\{E_3, E_4\}$. w is related to θ by $\cos w = \frac{-\theta'}{\kappa_2}$ and $\lambda(t)$ is given by

$$\lambda(t) = -\frac{3(c-1)}{2}\int\frac{\cos^2\theta\kappa_1'}{\kappa_1^3}dt.$$

We can give the following direct corollary of Theorem 3.7:

Corollary 3.8. *Let $\gamma : I \to M$ be a Legendre curve of osculating order r in a Sasakian space form $(M^{2m+1}, \varphi, \xi, \eta, g)$, where $r \geq 4$, $m \geq 2$, $c \neq 1$, $g(\varphi T, E_2) = \cos\theta$ is a constant and $\theta \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Then γ is proper f-biharmonic if and only if $f = c_1\kappa_1^{-3/2}$, $\frac{\kappa_2}{\kappa_1} = c_2 =$ constant > 0,*

$$\kappa_2\kappa_3 = \pm\frac{3(c-1)\sin 2\theta}{8},$$

$$\kappa_4 = \pm\frac{\eta(E_5) + g(\varphi E_2, E_5)\kappa_1}{\sin\theta} \text{ (if } r > 4\text{); and}$$

(*i*) *if $a > 0$, then κ_1 satisfies*

$$t \pm \frac{1}{2\sqrt{a}}\arctan\left(\frac{1}{2\sqrt{a}}\frac{2a + c_3\kappa_1}{\sqrt{-(1+c_2^2)\kappa_1^2 - c_3\kappa_1 - a}}\right) + c_4 = 0,$$

(*ii*) *if $a = 0$, then κ_1 satisfies*

$$t \pm \frac{\sqrt{-\kappa_1\left[(1+c_2^2)\kappa_1 + c_3\right]}}{c_3\kappa_1} + c_4 = 0,$$

(*iii*) *if* $a < 0$, *then* κ_1 *satisfies*

$$t \pm \frac{1}{2\sqrt{-a}} \ln\left(\frac{2a + c_3\kappa_1 - 2\sqrt{-a}\sqrt{-(1+c_2^2)\kappa_1^2 - c_3\kappa_1 - a}}{2a\kappa_1}\right) + c_4 = 0,$$

where $a = \left[c + 3 + 3(c-1)\cos^2\theta\right]/4$, $\varphi T = \cos\theta E_2 \pm \sin\theta E_4$, $c_1 > 0$, $c_2 > 0$, c_3 *and* c_4 *are convenient arbitrary constants,* t *is the arc-length parameter and* $\kappa_1(t)$ *is in convenient open interval.*

In order to obtain explicit examples, we will first need to recall some notions about the Sasakian space form $\mathbb{R}^{2m+1}(-3)$ [3]:

Let us consider $M = \mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1, ..., x_m, y_1, ..., y_m, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ and the tensor field φ given by

$$\varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

The associated Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{m}((dx_i)^2 + (dy_i)^2)$. Then $(M, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$. The vector fields

$$X_i = 2\frac{\partial}{\partial y_i},\ X_{m+i} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}), i = \overline{1,m},\ \xi = 2\frac{\partial}{\partial z} \tag{30}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\nabla_{X_i} X_j = \nabla_{X_{m+i}} X_{m+j} = 0, \nabla_{X_i} X_{m+j} = \delta_{ij}\xi, \nabla_{X_{m+i}} X_j = -\delta_{ij}\xi,$$
$$\nabla_{X_i}\xi = \nabla_\xi X_i = -X_{m+i}, \nabla_{X_{m+i}}\xi = \nabla_\xi X_{m+i} = X_i,$$

(see [3]).

Now, let us produce examples of proper f-biharmonic Legendre curves in $\mathbb{R}^7(-3)$:

Let $\gamma = (\gamma_1, ..., \gamma_7)$ be a unit speed curve in $\mathbb{R}^7(-3)$. The tangent vector field of γ is

$$T = \frac{1}{2}\left[\gamma_4' X_1 + \gamma_5' X_2 + \gamma_6' X_3 + \gamma_1' X_4 + \gamma_2' X_5 + \gamma_3' X_6 + (\gamma_7' - \gamma_1'\gamma_4 - \gamma_2'\gamma_5 - \gamma_3'\gamma_6)\xi\right].$$

Thus, γ is a unit speed Legendre curve if and only if $\eta(T) = 0$ and $g(T, T) = 1$, that is,

$$\gamma_7' = \gamma_1'\gamma_4 + \gamma_2'\gamma_5 + \gamma_3'\gamma_6$$

and

$$\left(\gamma_1'\right)^2 + ... + \left(\gamma_6'\right)^2 = 4.$$

For a Legendre curve, we can use the Levi-Civita connection and (30) to write

$$\nabla_T T = \frac{1}{2}\left(\gamma_4'' X_1 + \gamma_5'' X_2 + \gamma_6'' X_3 + \gamma_1'' X_4 + \gamma_2'' X_5 + \gamma_3'' X_6\right), \tag{31}$$

$$\varphi T = \frac{1}{2}(-\gamma_1' X_1 - \gamma_2' X_2 - \gamma_3' X_3 + \gamma_4' X_4 + \gamma_5' X_5 + \gamma_6' X_6). \tag{32}$$

From (31) and (32), $\varphi T \perp E_2$ if and only if

$$\gamma_1''\gamma_4' + \gamma_2''\gamma_5' + \gamma_3''\gamma_6' = \gamma_1'\gamma_4'' + \gamma_2'\gamma_5'' + \gamma_3'\gamma_6''.$$

Finally, we can give the following explicit examples:

Example 3.9. *Let us take* $\gamma(t) = (2\sinh^{-1}(t), \sqrt{1+t^2}, \sqrt{3}\sqrt{1+t^2}, 0, 0, 0, 1)$ *in* $\mathbb{R}^7(-3)$*. Using the above equations and Theorem* 3.5*,* γ *is a proper* f*-biharmonic Legendre curve with osculating order* $r = 2$*,* $\kappa_1 = \frac{1}{1+t^2}$*,* $f = c_1(1+t^2)^{3/2}$ *where* $c_1 > 0$ *is a constant. We can easily check that the conditions of Theorem* 3.5 *(i.e.* $c \neq 1, \varphi T \perp E_2$*) are verified, where* $c_3 = -1$ *and* $c_4 = 0$*.*

Example 3.10. *Let* $\gamma(t) = (a_1, a_2, a_3, \sqrt{2}t, 2\sinh^{-1}(\frac{t}{\sqrt{2}}), \sqrt{2}\sqrt{2+t^2}, a_4)$ *be a curve in* $\mathbb{R}^7(-3)$*, where* $a_i \in \mathbb{R}$*,* $i = \overline{1,4}$*. Then we calculate*

$$T = \frac{\sqrt{2}}{2}X_1 + \frac{1}{\sqrt{2+t^2}}X_2 + \frac{\sqrt{2}t}{2\sqrt{2+t^2}}X_3,$$

$$E_2 = \frac{-t}{\sqrt{2+t^2}}X_2 + \frac{\sqrt{2}}{\sqrt{2+t^2}}X_3,$$

$$E_3 = \frac{\sqrt{2}}{2}X_1 - \frac{1}{\sqrt{2+t^2}}X_2 - \frac{\sqrt{2}t}{2\sqrt{2+t^2}}X_3,$$

$$\kappa_1 = \kappa_2 = \frac{1}{2+t^2}, \; r = 3.$$

From Theorem 3.5*, it follows that* γ *is proper* f*-biharmonic with* $f = c_1(2+t^2)^{3/2}$*, where* $c_1 > 0$*,* $c_2 = 1$*,* $c_3 = -1$ *and* $c_4 = 0$*.*

References

[1] C. Baikoussis, D. E. Blair, On Legendre curves in contact 3-manifolds, Geom. Dedicata 49 (1994) 135–142.
[2] D. E. Blair, Geometry of manifolds with structural group $\mathcal{U}(n) \times O(s)$, J. Differential Geometry 4 (1970) 155–167.
[3] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhauser, Boston, 2002.
[4] B.Y. Chen, A report on submanifolds of finite type, Soochow J. Math. 22 (1996) 117–337.
[5] Jr. J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109–160.
[6] D. Fetcu, Biharmonic Legendre curves in Sasakian space forms, J. Korean Math. Soc. 45 (2008) 393–404.
[7] D. Fetcu, C. Oniciuc, Explicit formulas for biharmonic submanifolds in Sasakian space forms, Pacific J. Math. 240 (2009) 85–107.
[8] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, Chinese Ann. Math. Ser. A 7 (1986) 389–402.
[9] W. J. Lu, On f-biharmonic maps between Riemannian manifolds, arXiv:1305.5478.
[10] S. Montaldo, C. Oniciuc, A short survey on biharmonic maps between Riemannian manifolds, Rev. Un. Mat. Argentina 47 (2006) no. 2 1–22.
[11] Y. L. Ou, On f-biharmonic maps and f-biharmonic submanifolds, arXiv:1306.3549v1.
[12] C. Özgür, Ş. Güvenç, On biharmonic Legendre curves in S-space forms, Turkish J. Math. 38 (2014) no. 3 454–461.
[13] C. Özgür, Ş. Güvenç, On some classes of biharmonic Legendre curves in generalized Sasakian space forms, Collect. Math. 65 (2014) no. 2 203–218.