THE DIRICHLET PROBLEM OF A CONFORMABLE ADVECTION-DIFFUSION EQUATION

by

Derya AVCI[*]*, Beyza Billur ISKENDER EROGLU, and Necati OZDEMIR*

Department of Mathematics, Faculty of Science and Arts,
Balikesir University, Balikesir, Turkey

The fractional advection-diffusion equations are obtained from a fractional power law for the matter flux. Diffusion processes in special types of porous media which has fractal geometry can be modelled accurately by using these equations. However, the existing non-local fractional derivatives seem complicated and also lose some basic properties satisfied by usual derivatives. For these reasons, local fractional calculus has recently been emerged to simplify the complexities of fractional models defined by non-local fractional operators. In this work, the conformable, a local, well-behaved and limit-based definition, is used to obtain a local generalized form of advection-diffusion equation. In addition, this study is devoted to give a local generalized description to the combination of diffusive flux governed by Fick's law and the advection flux associated with the velocity field. As a result, the constitutive conformable advection-diffusion equation can be easily achieved. A Dirichlet problem for conformable advection-diffusion equation is derived by applying fractional Laplace transform with respect to time, t, and finite sin-Fourier transform with respect to spatial co-ordinate, x. Two illustrative examples are presented to show the behaviours of this new local generalized model. The dependence of the solution on the fractional order of conformable derivative and the changing values of problem parameters are validated using graphics held by MATLAB codes.

Key words: *conformable derivative, fractional Laplace transform,*
fractional advection-diffusion

Introduction

The standard advection-diffusion equation (ADE) can be interpreted as a diffusion equation with an additional velocity field [1]:

$$\frac{\partial c}{\partial t} = a \, \Delta c - v \, \text{grad} \, c \qquad (1)$$

where c represents variable velocity, and a, (for $v > 0$) is constant coefficients dependent on the relevant physical problem.

Matter transports in a porous media, Brownian motion in financial models, groundwater flow in the nature are just a few applications of eq. (1). Some internal effects in advection-diffusion processes have led them to replacing by fractional order models [2, 3]. The non-local generalization of matter flux defined [4, 5]:

———————
* Corresponding author, e-mail: dkaradeniz@balikesir.edu.tr

$$\boldsymbol{j} = \mathrm{D}_{\mathrm{RL}}^{1-\alpha}\left(-a\,\mathrm{grad}\,c + v\,c\right), \quad 0 < \alpha \le 1, \tag{2}$$

where $\mathrm{D}_{\mathrm{RL}}^{1-\alpha}$ represents the well-known Riemann-Liouville fractional derivative:

$$\mathrm{D}_{\mathrm{RL}}^{1-\alpha} f(t) = \mathrm{D}\left[I^{\alpha} f(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau\right], \quad 0 < \alpha \le 1$$

By the combination of mass balance equation, eq. (2) results the fractional ADE in terms of Caputo fractional derivative:

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\,\Delta c - v\,\mathrm{grad}\,c, \quad 0 < \alpha \le 1 \tag{3}$$

In 1-D spatial co-ordinate, eq. (3) reduces to the following form:

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x} \tag{4}$$

Many real-world problems describing by time fractional ADE have been solved by interested researchers. For example, it was considered to describe the transport dynamics in complex systems which are governed by anomalous diffusion [6]. Moreover, the fractional ADE was effectively used to model the transport of passive tracers by fluid flow in a porous media [7]. The fundamental solutions to the Cauchy and source problems and the theory of diffusive stresses based on fractional ADE were studied [8-10]. Several implicit or explicit numerical schemes were proposed to solve fractional-diffusion/advection-diffusion equations [11-15].

In the previously mentioned studies, the non-local generalizations of ADE were considered. In this manuscript, we are motivated to investigate a local generalized form of AD By using basic mathematical tools in conformable theory, we first propose a local description for the matter flux:

$$\boldsymbol{j} = \left[-A(t)\,\mathrm{grad}\,c(x,t) + V(t)\,c(x,t)\right], \quad 0 < \alpha \le 1 \tag{5}$$

Here, the function $c = c(x,t)$ is the matter concentration a position x of the domain at time, t. In the present study, we consider that the matter diffusion parameter $A(t) = at^{\alpha-1}$ and the velocity of the flow $V(t) = vt^{\alpha-1}$, *i. e.* both parameters are temporally dependent, where a and v are arbitrary positive constants. By successive computations by using conservation of matter, we get the following conformable ADE:

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\,\Delta c - v\,\mathrm{grad}\,c \tag{6}$$

where $\partial^{\alpha}/\partial t^{\alpha}$ denotes the conformable partial derivative of order $0 < \alpha \le 1$. We formulate a Dirichlet problem for a line segment. Note that the non-local form of similar problem with Caputo fractional derivative was formulated by Povstenko and Klekot [16].

Definitions and properties

The conventional fractional derivatives such as Riemann-Liouville, Caputo, and Grunwald-Letnikov have found numerous applications in science and engineering by giving the memory and hereditary effects. Nevertheless, the complexity and the lack of some basic properties satisfied by usual derivative have leaded the scientists to improve new local fraction-

al derivatives and integrals. To this end, Khalil *et al.* [17] introduced the so-called conformable derivative with fractional order $0 < \alpha \le 1$ for a given real valued function, f, defined on $[0,\infty)$. New notions and theorems as conformable chain and product rules, integration by parts, mean value theorem, divergence theorem, Taylor power series expansions, conformable fractional Laplace and Fourier transforms, conformable partial, and sequential derivatives have been proposed and used in applications [18-20]. In recent years, the interest in the conformable differential equations has been increasing [21-26].

Some remarkable differences between the non-local and the conformable operators with fractional orders can be given. Mittag-Leffler function which is a generalization of exponential function plays an important role in conventional fractional calculus while the fractional exponential function $f(t) = e^{t^\alpha/\alpha}$ appears in the applications of conformable calculus. Furthermore, the conformable derivative of a constant function is zero while it is not valid for the Riemann-Liouville fractional derivative. Additional information about mathematical aspects of conformable calculus and also the comparison of the non-local and local fractional operators can be found in pioneering book [27].

The basic definitions and relations based on [18] are given:

(1) Definition: The left conformable derivative of a function $f : [a,\infty) \to \mathbb{R}$ with fractional order $0 < \alpha \le 1$ is defined by:

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \to \infty} \frac{f\left[t + \varepsilon(t-a)^{1-\alpha}\right] - f(t)}{\varepsilon} \tag{7}$$

and if $t = a$, $(T_\alpha^a f)(a) = \lim_{t \to a^+}(T_\alpha^a f)(t)$.

Likewise, the right conformable derivative of a function $f : (-\infty, b] \to \mathbb{R}$, is defined by:

$$({}^b T_\alpha f)(t) = -\lim_{\varepsilon \to \infty} \frac{f\left[t + \varepsilon(b-t)^{1-\alpha}\right] - f(t)}{\varepsilon} \tag{8}$$

(2) Definition: For a given function $f : [a,\infty) \to \mathbb{R}$ the left conformable integral for $0 < \alpha \le 1$ is defined by:

$$(I_\alpha^a f)(t) = \int_a^t (x-a)^{\alpha-1} f(x)\,dx \tag{9}$$

Similarly, in the right case for $f : (-\infty, b] \to \mathbb{R}$, we get:

$$({}^b I_\alpha f)(t) = \int_t^b (b-x)^{\alpha-1} f(x)\,dx \tag{10}$$

(3) Notation: If f is differentiable, then:

$$(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t) \quad \text{and} \quad ({}^b T_\alpha f)(t) = -(b-t)^{1-\alpha} f'(t)$$

(4) Lemma: Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $n < \alpha \le n+1$. Then, for all $t > a$ we have:

$$T_\alpha^a I_\alpha^a f(t) = f(t) \tag{11}$$

This property is valid for the right case and also for the classical derivative within the fundamental theorems of calculus.

Avci, D., *et al.*: The Dirichlet Problem of a Conformable Advection-Diffusion Equation
THERMAL SCIENCE: Year 2017, Vol. 21, No. 1A, pp. 9-18

12

(5) Lemma: Let f be conformable differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have:

$$I_\alpha^a T_\alpha^a f(t) = f(t) - f(a) \tag{12}$$

It can be generalized to the higher order cases.

(6) Definition: Let $a \in \mathbb{R}$, $0 < \alpha \leq 1$, and $f : [a, \infty) \to \mathbb{R}$. Then the fractional Laplace transform of order α of f is defined by:

$$L_\alpha^a \{f(t)\}(s) = F_\alpha^a(s) = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} (t-a)^{\alpha-1} f(t) \, dt \tag{13}$$

(7) Lemma: Let $f : [a, \infty) \to \mathbb{R}$ be a function such that $L_\alpha^a \{f(t)\}(s) = F_\alpha^a(s)$ exists. Then:

$$F_\alpha^a(s) = L\left\{ f\left[a + (\alpha t)^{1/\alpha} \right] \right\}(s) \tag{14}$$

where

$$L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) \, dt$$

(8) Theorem: Let $f : [a, \infty) \to \mathbb{R}$ be a conformable differentiable function and $0 < \alpha \leq 1$. Then:

$$L_\alpha^a \{T_\alpha f(t)\}(s) = sF_\alpha(s) - f(a) \tag{15}$$

Now, we shall give the mathematical formulation of the main problem.

Statement of the problem

To give the motivation of this work, we first clarify the physical background of an ADE as a result of the combination of the Fick's law and advection effect by giving interesting real world problems. For example, [28], some tracer particles of pollutants or immiscible solutes unfortunately damage the hydro-environment balance in the surface water bodies and aquifers because of human activities. Solutes are transported down the stream along the water flow and disperse due to the effects both of diffusion and advection. As a result, the relevant matter concentration in the water changes with respect to position x and time, t. Such an environmental process is modelled by a partial differential equation called ADE. A cellular protein transport or a chemical reaction can also be modelled by these equations.

For normal systems in the nature, the particles diffuse freely via Brownian motion, *i. e.* the mean-square displacement of particles in n-dimensional space is proportional to time defined as $\langle \Delta r^2 \rangle = 2nDt$ where D denotes the diffusion coefficient. In contrast, anomalous mobility may occur in the complex diffusive dynamics and so it needs to consider a power law time scaling as $\langle \Delta r^2 \rangle = 2n\Gamma t^\alpha$ where α represents the magnitude of anomalous transport and the constant Γ denotes the transport factor changing with respect to the matter under diffusion [29, 30]. In physical sense, if $\alpha < 1$ and then mobility is called anomalous subdiffusion. Similarly, $\alpha > 1$ determines the anomalous superdiffusion. Likewise to physical difference in the diffusion models, the mathematical descriptions differ for the anomalous diffusion processes. The standard diffusion equation governed by Fick's law can not describe the non-linear time dependence of the mean-square displacement. Moreover, there is no unique, simple, and standard model to obtain all of the anomalous diffusions in the nature. Therefore, many different models have been introduced to give a real-like description to diffusion problems.

A mathematical common used model for anomalous diffusion is based on a time-dependent diffusion coefficient, $D(t)$. In addition, a more physically correct value to describe this temporal anomalous diffusion in terms of the time-dependent slope of the mean square displacement is defined by $D(t) = \Gamma t^{\alpha-1}$. Note that it depends on the time scale of measurements, *i. e.* there is no characteristic length scale which is expected at all anomalous diffusive processes. In the physical and biological applications, researchers aim to find the optimum α parameter which defines the power law of the temporal diffusion coefficient by curve fitting to experimental data of diffusive particles. In mathematical sense, extended diffusion equations, which naturally arise from the laws of matter fluxes, have been improved to obtain the anomalous diffusion propagator. In the resulting equations, the anomalous exponent α describes the order of derivative.

In the present study, we are motivated to obtain a mathematical model based upon the physical law, which is in terms of temporally dependent coefficients, for a matter flux:

$$\boldsymbol{j} = \left[-at^{\alpha-1}\operatorname{grad} c(x,t) + vt^{\alpha-1}c(x,t)\right], \quad 0 < \alpha \le 1 \tag{16}$$

This relation can be rewritten by using the definition of conformable integral given by eq. (9):

$$\boldsymbol{j} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \tau^{\alpha-1}\left[-a\operatorname{grad} c(x,\tau) + vc(x,\tau)\right]\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}I_t^\alpha\left[-a\operatorname{grad} c(x,t) + vc(x,t)\right]$$

By using the conservation of matter, we get:

$$\frac{\partial c}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}I_t^\alpha\left(a\Delta c - v\operatorname{grad} c\right) \tag{17}$$

Now, we first integrate the eq. (17) with respect to time, t, and then take conformable derivative of both sides. Therefore, we obtain the conformable partial ADE that models an anomalous diffusion process acting by a law of matter flux with the temporally dependent diffusion and advection coefficients. As it can be seen from this constitutive ADE, these time-dependent coefficients are hidden in conformable partial derivative by successive computations. Therefore, we have and formulate a conformable ADE with constant coefficients.

Let us consider the obtained conformable advection-diffusion equation:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x} \tag{18}$$

with the initial and boundary conditions:

$$c(x,0) = 0 \tag{19}$$

$$c(0,t) = f(t) \quad \text{and} \quad c(L,t) = 0 \tag{20}$$

where $v \ge 0, a > 0, \ 0 < \alpha \le 1, 0 < x < L, t > 0$, and $\partial^\alpha/\partial t^\alpha$ denotes the conformable partial derivative of order α. We use a well-known method in the classical theory of partial differential equations such that the solution is assumed to be:

$$c(x,t) = e^{\beta x}u(x,t) \tag{21}$$

Then, we find $\beta = v/2a$ by substituting eq. (21) into eq. (18). Therefore, we have:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = a\frac{\partial^2 u}{\partial x^2} - \frac{v^2}{4a}u \tag{22}$$

with the initial and boundary conditions:

$$u(x,0) = 0,$$

$$u(0,t) = f(t) \quad \text{and} \quad u(L,t) = 0$$

Before applying the integral transforms, we remind the finite sin-Fourier transforms with respect to x [31]:

$$F\{u(x)\} = \tilde{u}(\xi_k) = \int_0^L u(x)\sin(\xi_k x)\mathrm{d}x \tag{23}$$

$$F^{-1}\{\tilde{u}(\xi_k)\} = u(x) = \frac{2}{L}\sum_{k=1}^{\infty}\tilde{u}(\xi_k)\sin(\xi_k x) \tag{24}$$

where $\xi_k = k\pi/L$, $k = 1,2,3,\dots$. The application of finite sin-Fourier transform to second order derivative can be easily calculated:

$$F\left\{\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right\} = -\xi_k^2\tilde{u}(\xi_k) + \xi_k\left[u(0) - (-1)^k u(L)\right] \tag{25}$$

Now, applying the finite sin-Fourier transform eq. (23) to eq. (22) and using the property (25), we obtain:

$$\frac{\partial^{\alpha}\tilde{u}(\xi_k,t)}{\partial t^{\alpha}} = \left(-a\xi_k^2 - \frac{v^2}{4a}\right)\tilde{u}(\xi_k,t) + a\xi_k f(t) \tag{26}$$

with the initial condition:

$$\tilde{u}(\xi_k,0) = 0 \tag{27}$$

Next, we apply fractional Laplace transform with respect to time, t, and so eq. (26) reduces to:

$$\tilde{u}_{\alpha}(\xi_k,s) = a\xi_k f(s)\frac{1}{s + a\xi_k^2 + \dfrac{v^2}{4a}} \tag{28}$$

where $f(s) = L_{\alpha}\{f(t)\}(s)$.

(1) Example: For simplicity, we first choose $f(t) = 1$ and then can easily calculate $L_{\alpha}\{1\}(s) = 1/s$.

After arranging eq. (28), we have that:

$$\tilde{u}_{\alpha}(\xi_k,s) = \frac{a\xi_k}{a\xi_k^2 + \dfrac{v^2}{4a}}\left[\frac{1}{s} - \frac{1}{s + a\xi_k^2 + \dfrac{v^2}{4a}}\right] \tag{29}$$

By taking the inverse fractional Laplace transform, we arrive at:

Avci, D., *et al*.: The Dirichlet Problem of a Conformable Advection-Diffusion Equation
THERMAL SCIENCE: Year 2017, Vol. 21, No. 1A, pp. 9-18

15

$$\tilde{u}\left(\xi_k,t\right) = \frac{a\xi_k}{a\xi_k^2 + \dfrac{v^2}{4a}}\left[1 - e^{-\left(a\xi_k^2 + \frac{v^2}{4a}\right)\frac{t^\alpha}{\alpha}}\right]$$

(30)

where

$$L_\alpha\left\{e^{\lambda\frac{t^\alpha}{\alpha}}\right\} = \frac{1}{s - \lambda}$$

Next by taking the inverse finite sin-Fourier transform, we see that:

$$u\left(x,t\right) = \frac{2a}{L}\sum_{k=1}^{\infty}\frac{1}{a\xi_k^2 + \dfrac{v^2}{4a}}\xi_k\sin(\xi_k x)\left[1 - e^{-\left(a\xi_k^2 + \frac{v^2}{4a}\right)\frac{t^\alpha}{\alpha}}\right]$$

(31)

Consequently, we find the fundamental solution $c(x,t)$ of Dirichlet problem:

$$c\left(x,t\right) = \frac{2a}{L}e^{\frac{v}{2a}x}\sum_{k=1}^{\infty}\frac{1}{a\xi_k^2 + \dfrac{v^2}{4a}}\xi_k\sin(\xi_k x)\left[1 - e^{-\left(a\xi_k^2 + \frac{v^2}{4a}\right)\frac{t^\alpha}{\alpha}}\right]$$

(32)

For a real physical application, this solution represents a density distribution function of diffusive particles, also called the propagator. Figures 1(a) and 1(b) illustrate the character of solution for the fractional and integer order values of α by choosing parameters $a = L = v = 1$. We also use this assumption to obtain fig. (2). In figs. 3(a) and 3(b), we analyze the effects of the parameter v in the case of $a = L = 1$ and the parameter a for $v = L = 1$. To plot this figures, the fractional order of conformable derivative is taken as $\alpha = 0.75$.

(2) **Example:** Now, we consider $f(t) = 1$ and repeat the calculations. Therefore, eq. (30) is replaced by:

$$\tilde{u}_\alpha(\xi_k,s) = a\xi_k\frac{\alpha^{1/\alpha}\Gamma\left(1 + \dfrac{1}{\alpha}\right)}{s^{1+\frac{1}{\alpha}}}\frac{1}{s + a\xi_k^2 + \dfrac{v^2}{4a}}$$

(33)

where

$$L_\alpha\{t\}(s) = \alpha^{1/\alpha}\frac{\Gamma\left(1 + \dfrac{1}{\alpha}\right)}{s^{1+\frac{1}{\alpha}}}$$

Before taking the inverse Laplace transform, we give the convolution theorem for fractional Laplace transform can be seen by using the Lemma (eq. 14) and classical convolution theorem for Laplace transform [26].

(3) **Theorem:** Let $f,g:[0,\infty)\to\mathbb{R}$ be functions such that $L_\alpha\{f(t)\} = F_\alpha(s)$ and $L_\alpha\{g(t)\} = G_\alpha(s)$ exist for $0 < \alpha \le 1$. Then:

$$L_\alpha\left\{(f * g)(\mathrm{t})\right\} = F_\alpha(s)G_\alpha(s)$$

(34)

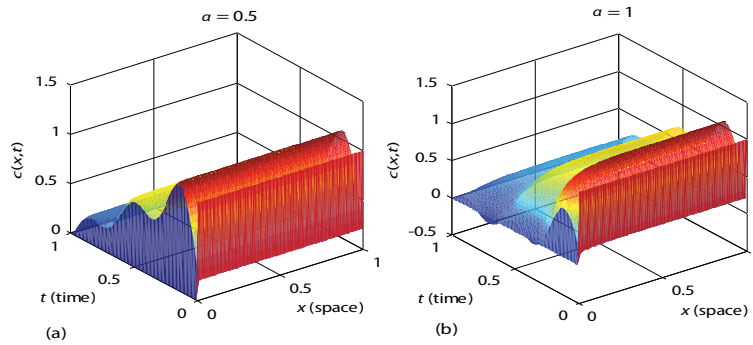where $(f * g)$ is the convolution of f and g functions defined by:

Figure 1. Dependence of the surface solution with constant initial condition $f(t) = 1$ on the variation of fractional order α
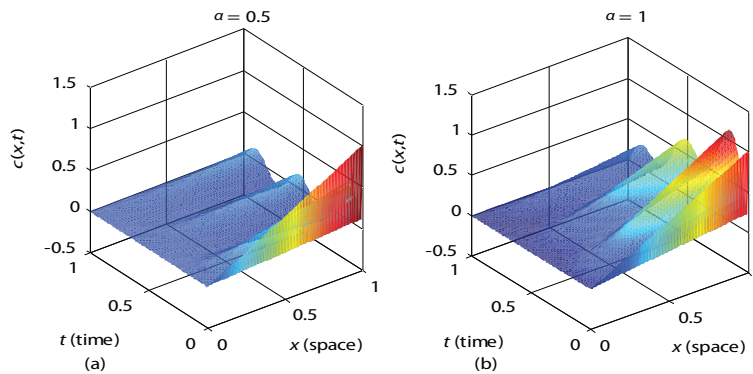(*for color image see journal web site*)

Figure 2. Dependence of the surface solution with constant initial condition $f(t) = 1$ on the variation of α
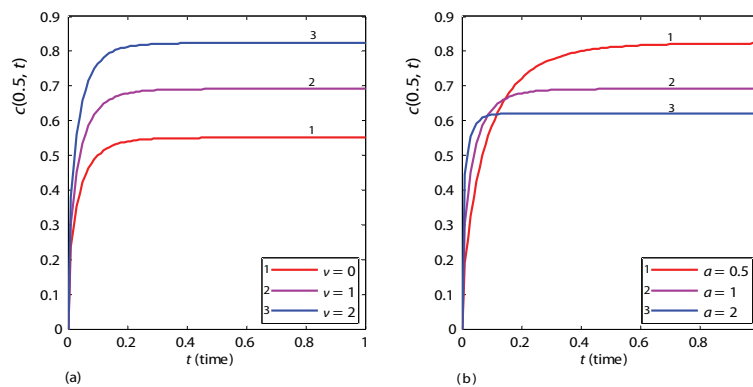(*for color image see journal web site*)

Figure 3. Dependence of the solution with $f(t) = 1$ on variation of: (a) the parameter ν and (b) the parameter a for $\alpha = 0.75$

$$(f * g)(t) = \int\limits_0^t f(\tau)\, g(t - \tau)\mathrm{d}\tau \tag{35}$$

Now, we shall take the inverse Laplace transform of eq. (33) and see that:

Avci, D., *et al.*: The Dirichlet Problem of a Conformable Advection-Diffusion Equation
THERMAL SCIENCE: Year 2017, Vol. 21, No. 1A, pp. 9-18

17

$$\tilde{u}(\xi_k,t) = a\xi_k \int_0^t \tau \, e^{-a\left(\xi_k^2 + \frac{v^2}{4a}\right)\frac{(t-\tau)^\alpha}{\alpha}} \, d\tau \qquad (36)$$

After taking the inverse finite sin-Fourier transform, we obtain:

$$u(x,t) = \frac{2a}{L} \sum_{k=1}^\infty \xi_k \sin(\xi_k x) \int_0^t \tau \, e^{-a\left(\xi_k^2 + \frac{v^2}{4a}\right)\frac{(t-\tau)^\alpha}{\alpha}} \, d\tau \qquad (37)$$

As a result, the fundamental solution is in the following form:

$$c(x,t) = \frac{2a}{L} e^{\frac{v}{2a}x} \sum_{k=1}^\infty \xi_k \sin(\xi_k x) \int_0^t \tau \, e^{-a\left(\xi_k^2 + \frac{v^2}{4a}\right)\frac{(t-\tau)^\alpha}{\alpha}} \, d\tau \qquad (38)$$

The physical structure of the solution is shown in figs. 1(a) and 1(b) for the different values of order α. As expected, a retardation effect occurs in the case of $\alpha < 1$. This also shows the slow anomalous transport of the matter.

Conclusion

In this study, we have proposed a mathematical relation in terms of time-dependent coefficients for an anomalous matter flux. By using this notion, we have introduced the conformable ADE with constant coefficients. For this equation, Dirichlet problem for a line segment $0 < x \le L$ is considered as a similar manner in the non-local formulation in [16]. The fundamental solution of problem is obtained by applying fractional Laplace and finite sin-Fourier transforms. In graphical analysis, we observe that the decreasing of the fractional order of conformable derivative causes the retardation of transport. This case can be occurred by the presence of some internal structures as pores and combs in matters. We have emphasized the results by using two numerical examples. Graphical results are held by MATLAB codes.

References

[1] Kaviany, M., *Principles of Heat Transfer in Porous Media*, Springer, New York, USA, 1995
[2] Baleanu, D., *et al.*, *Fractional Calculus: Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos,* Vol. 3, World Scientific Publishing Co. Pte. Ltd., Singapore, 2012
[3] Chen, W., *et al.*, Anomalous Diffusion Modeling by Fractal and Fractional Derivatives, *Comput. Math. Appl.*, *59* (2010), 5, pp. 1754-1758
[4] Gorenflo, R., Mainardi, F., Fractional Calculus: Integral and Differential Equations of Fractional Order, in: *Fractals and Fractional Calculus in Continuum Mechanics* (Eds. A. Carpinetti, F. Mainardi), Springer-Verlag, New York, USA, 1997, pp. 223-276
[5] Povstenko, Y., Space-Time-Fractional Advection Diffusion Equation in a Plane, in: *Advances in Modeling and Control of Non-Integer Order Systems, Lecture Notes in Electrical Engineering* (Eds. K. J. Latawiec *et al.*), Springer, Switzerland, 2015, 320, pp. 275-284
[6] Metzler, R., Klafter, J., The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamics Approach, *Phys. Rep.*, *339* (2000), 1, pp. 1-77
[7] Benson, D. A., *et al.*, Application of a Fractional Advection-Dispersion Equation, *Water Resour. Res.*, *36* (2000), 6, pp. 1403-1412
[8] Povstenko, Y., Fundamental Solutions to Time-Fractional Advection Diffusion Equation in a Case of Two Space Variables, *Math. Probl. Eng.*, *2014* (2014), ID 705364
[9] Povstenko, Y., Theory of Diffusive Stresses Based on the Fractional Advection-Diffusion Equation, in: *Fractional Calculus: Applications* (Eds. R. Abi Zeid Daou, X. Moreau), Nova Science Publishers, New York, USA, 2015, pp. 227-241
[10] Povstenko, Y., *Fractional Thermoelasticity*, Springer, New York, USA, 2015

[11] Liu, F, *et al.*, Time-Fractional Advection-Dispersion Equation, *J. Appl. Math. Comput.*, *13* (2003), 1-2, pp. 233-245

[12] Huang, F, Liu, F., The Time Fractional Diffusion Equation and the Advection-Dispersion Equation. *AN-ZIAM J.*, *46* (2005), 3, pp. 317-330

[13] Hristov, J., Approximate Solutions to Time-Fractional Models by Integral Balance Approach, in: *Fractional Dynamics*, (Eds. C. Cattani, H. M. Srivastava, Xia-Jun Yang), De Gruyter Open, Warsaw, 2015 , Chapter 5, pp. 78-109

[14] Hristov, J., Diffusion Models with Weakly Singular Kernels in the Fading Memories: How the Integral-Balance Method can be Applied? *Thermal Science*, *19* (2015), 3, pp. 947-957

[15] Wei, S., *et al.*, Implicit Local Radial Basis Function Method for Solving Two-Dimensional Time Fractional Diffusion Equations, *Thermal Science*, *19* (2015), Suppl. 1, pp. S59-S67

[16] Povstenko, Y., Klekot, J., The Dirichlet Problem for the Time-Fractional Advection-Diffusion Equation in a Line Segment, *Boundary Value Problems*, *89* (2016), Apr., pp. 1-9

[17] Khalil, R., *et al.*, M., A New Definition of Fractional Derivative, *J. Comput. Appl. Math.*, *264* (2014), July, pp. 65-70

[18] Abdeljawad, T., On Conformable Fractional Calculus, *J. Comput. Appl. Math.*, *279* (2015), May, pp. 57-66

[19] Atangana, A., *et al.*, New Properties of Conformable Derivative, *Open Math.*, *13* (2015), 1, pp. 889-898

[20] Abu Hammad, I., Khalil, R., Fractional Fourier Series with Applications, Am. J. Comput. Appl. Math., *4* (2014), 6, pp. 187-191

[21] Abu Hammad, I., Khalil, R., Conformable Fractional Heat Differential Equation, *Int. J. Pure Appl. Math.*, *94* (2014), 2, pp. 215-221

[22] Khalil, R., Abu-Shaab, H., Solution of some Conformable Fractional Differential Equations, *Int. J. Pure Appl. Math.*, *103* (2015), 4, pp. 667-673

[23] Avcı, D., *et al.*, Conformable Heat Problem in A Cylinder, *Proceedings*, International Conference on Fractional Differentiation and its Applications, Novi Sad, Serbia, 2016, pp. 572-58

[24] Iyiola, O. S., Nwaeze, E. R., Some New Results on the New Conformable Fractional Calculus with Application Using D'Alembert Approach, *Progr. Fract. Differ. Appl.*, *2* (2016), 2, pp.1-7

[25] Neamaty, A., *et al.*, On the Determination of the Eigenvalues for Airy Fractional Differential Equation with Turning Point, *TJMM*, *7* (2015), 2, pp. 149-153

[26] Eroglu, B. B. I., *et al.*, Optimal Control Problem for a Conformable Fractional Heat Equation, ICCESEN 2016, Antalya, Turkey, 2016

[27] Atangana, A., *Derivative with a New Parameter: Theory, Methods and Applications*, Elsevier, London, UK, 2015

[28] Jaiswal, D. K., *et al.*, Analytical Solution to the One-Dimensional Advection-Diffusion Equation with Temporally Dependent Coefficients, *Journal of Water Resource and Protection*, *3* (2011), 1, pp. 76-84

[29] Banks, D. S., Fradin, C., Anomalous Diffusion of Proteins Due to Molecular Crowding, *Biophysical Journal*, *89* (2005), 5, pp. 2960-2971

[30] Wu, J., Berland, K.M., Propagators and Time-Dependent Diffusion Coefficients for Anomalous Diffusion, *Biophysical Journal*, *95* (2008), 4, pp. 2049-2052

[31] Sneddon, I. N., *The Use of Integral Transforms*, McGraw-Hill, New York, USA, 1972