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MATHEMATICS

On generalized pseudo Ricci symmetric manifolds admitting semi-symmetric metric connection

Dedicated to the memory of the late Professor M. C. Chaki

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Abstract. The object of the present paper is to investigate the applications of generalized pseudo Ricci symmetric manifolds admitting a semi-symmetric metric connection to the general relativity and cosmology. Also the existence of a generalized pseudo Ricci symmetric manifold is ensured by an example.

Key words: differential geometry, generalized pseudo Ricci symmetric manifold, semi-symmetric metric connection, Ricci tensor, viscous fluid spacetime, energy momentum tensor.

1. INTRODUCTION

The study of Riemannian symmetric manifolds began with the work of Cartan [3]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [12]. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [16], semisymmetric manifold by Szabó [14], pseudosymmetric manifold by Chaki [4], generalized pseudosymmetric manifold by Chaki [6], and weakly symmetric manifolds by Támassy and Binh [15]. Again a Riemannian manifold is said to be Ricci symmetric if the condition $\nabla S = 0$ holds, where S is the Ricci tensor of type (0, 2). A Riemannian manifold is called Ricci recurrent if the condition $\nabla S = A \otimes S$ holds, where A is a non-zero 1-form. Every locally symmetric manifold is Ricci symmetric but not conversely and every recurrent manifold is Ricci recurrent but the converse does not hold, in general. Again every Ricci symmetric manifold is Ricci recurrent but not conversely. In 1988 Chaki [5] introduced the notion of pseudo Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be pseudo Ricci symmetric if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the relation

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \tag{1.1}$$

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where A is a non-zero 1-form such that $g(X,\rho) = A(X)$ for every vector field X and ∇ is the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold is denoted by $(PRS)_n$.

Again Chaki and Koley [7] introduced the notion of generalized pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized pseudo Ricci symmetric manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(X,Y) \tag{1.2}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, D are 1-forms (not simultaneously zero) and $\chi(M^n)$ is the set of all differentiable vector fields on M^n. Such a manifold of dimension n is denoted by $G(PRS)_n$. The 1-forms A, B, D are called the associated 1-forms of the manifold.

Section 2 is concerned with the study of semi-symmetric metric connection along with its physical significance. In general relativity the matter content of the spacetime is described by the energy-momentum tensor T which is to be determined from the physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modelled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat.

The present paper deals with $G(PRS)_n$ admitting a semi-symmetric metric connection. A $G(PRS)_n$ admitting a semi-symmetric metric connection is denoted by $[G(PRS)_n, \widetilde{\nabla}]$. In Section 3 of the paper we investigate the applications of $[G(PRS)_n, \widetilde{\nabla}]$ to the general relativity and cosmology. It is shown that a viscous fluid spacetime obeying Einstein's equation with a cosmological constant is a connected Lorentzian $[G(PRS)_4, \widetilde{\nabla}]$. Consequently, $[G(PRS)_4, \widetilde{\nabla}]$ can be viewed as a model of the viscous fluid spacetime. Also it is observed that in a viscous fluid $[G(PRS)_4, \widetilde{\nabla}]$ spacetime, none of the isotropic pressure and energy density can be a constant and the matter content of the spacetime is a non-thermalized fluid under a certain condition.

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three different phases, namely, the initial phase, the intermediate phase, and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid [9]. The study of $[G(PRS)_4, \widetilde{\nabla}]$ is important because such spacetime represents the intermediate phase in the evolution of the universe. Consequently the investigations of $[G(PRS)_4, \widetilde{\nabla}]$ help us to have a deeper understanding of the global character of the universe including the topology, because the nature of the singularities can be defined from a differential geometric standpoint.

In the last section the existence of $G(PRS)_4$ is ensured by a suitable example.

2. SEMI-SYMMETRIC METRIC CONNECTION

In 1924 Friedmann and Schouten [8] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932 Hayden [10] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by Yano in 1970 [17].

Semi-symmetric metric connection plays an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North Pole, then this displacement is semi-symmetric and metric [13, p. 143]. During the mathematical congress in Moscow in 1934 one evening mathematicians invented the 'Moscow displacement'. The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the street always facing the Kremlin, then this displacement is semi-symmetric and metric [13, p. 143].

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with the metric tensor g and ∇ be the Riemannian connection of the manifold (M^n, g). A linear connection $\widetilde{\nabla}$ on (M^n, g) is said to be semi-symmetric [8] if the torsion tensor τ of the connection $\widetilde{\nabla}$ satisfies

$$\tau(X,Y) = \alpha(Y)X - \alpha(X)Y \tag{2.1}$$

for any vector fields X, Y on M and α is a 1-form associated with the torsion tensor τ of the connection $\widetilde{\nabla}$ given by

$$\alpha(X) = g(X, \rho), \tag{2.2}$$

where ρ is the vector field associated with the 1-form α. The 1-form α is called the associated 1-form of the semi-symmetric connection and the vector field ρ is called the associated vector field of the connection. A semi-symmetric connection $\widetilde{\nabla}$ is called a semi-symmetric metric connection [10] if in addition it satisfies

$$\widetilde{\nabla} g = 0. \tag{2.3}$$

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Riemannian connection ∇ of (M^n, g) is given by [17]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(Y)X - g(X,Y)\rho. \tag{2.4}$$

In particular, if the 1-form α vanishes identically then a semi-symmetric metric connection reduces to the Riemannian connection. The covariant differentiation of a 1-form ω with respect to $\widetilde{\nabla}$ is given by [17]

$$(\widetilde{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) + \omega(X)\alpha(Y) - \omega(\rho)g(X,Y). \tag{2.5}$$

If R and $\widetilde{R}$ are respectively the curvature tensor of the Levi-Civita connection ∇ and the semi-symmetric metric connection $\widetilde{\nabla}$ then we have [17]

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - P(Y,Z)X + P(X,Z)Y - g(Y,Z)LX + g(X,Z)LY, \tag{2.6}$$

where P is a tensor field of type (0, 2) given by

$$P(X,Y) = g(LX,Y) = (\nabla_X \alpha)(Y) - \alpha(X)\alpha(Y) + \frac{1}{2}\alpha(\rho)g(X,Y) \tag{2.7}$$

for any vector fields X and Y. From (2.6) it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-2)P(Y,Z) - ag(Y,Z), \tag{2.8}$$

where $\widetilde{S}$ and S denote respectively the Ricci tensor with respect to $\widetilde{\nabla}$ and ∇, $a = traceP$. The tensor P of type $(0,2)$ given in (2.7) is not symmetric in general and hence from (2.8) it follows that the Ricci tensor $\widetilde{S}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ is not so either. But if we consider that the 1-form α, associated with the torsion tensor τ, is closed then it can be easily shown that the relation

$$(\nabla_X \alpha)(Y) = (\nabla_Y \alpha)(X) \tag{2.9}$$

holds for all vector fields X, Y and hence the tensor $P(X,Y)$ is symmetric. Consequently, the Ricci tensor $\widetilde{S}$ is symmetric. Conversely, if $P(X,Y)$ is symmetric then from (2.7) it follows that the 1-form α is closed. This leads to the following:

Proposition 2.1 [1]. *Let (M^n, g) $(n > 2)$ be a Riemannian manifold admitting a semi-symmetric metric connection $\widetilde{\nabla}$. Then the Ricci tensor $\widetilde{S}$ of $\widetilde{\nabla}$ is symmetric if and only if the* 1*-form α, associated with the torsion tensor τ, is closed.*

Contracting (2.8) with respect to Y and Z, it can be easily found that

$$\widetilde{r} = r - 2(n-1)a, \tag{2.10}$$

where $\widetilde{r}$ and r denote respectively the scalar curvature with respect to $\widetilde{\nabla}$ and ∇.

Definition 2.1. *A Riemannian manifold (M^n, g) $(n > 2)$ is called generalized pseudo Ricci symmetric manifold admitting semi-symmetric metric connection if its Ricci tensor $\widetilde{S}$ of type $(0,2)$ is not identically zero and satisfies the condition*

$$(\widetilde{\nabla}_X \widetilde{S})(Y,Z) = 2\widetilde{A}(X)\widetilde{S}(Y,Z) + \widetilde{B}(Y)\widetilde{S}(Z,X) + \widetilde{D}(Z)\widetilde{S}(X,Y), \tag{2.11}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{D}$ are distinct 1*-forms* (*not simultaneously zero*) *and $\widetilde{\nabla}$ denotes the semi-symmetric metric connection.*

The 1-forms $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{D}$ are known as the associated 1-forms of the manifold. Such an n-dimensional manifold is denoted by $[G(PRS)_n, \widetilde{\nabla}]$. In view of (2.4) and (2.8), it follows from (2.11) that

$$\begin{aligned}(\nabla_X S)(Y,Z) - 2\widetilde{A}(X)S(Y,Z) - [\alpha(Y) + \widetilde{B}(Y)]S(Z,X) - [\alpha(Z) + \widetilde{D}(Z)]S(X,Y) \\ = (n-2)[(\nabla_X P)(Y,Z) + P(\rho,Z)g(X,Y) + P(Y,\rho)g(X,Z) \\ - 2\widetilde{A}(X)P(Y,Z) - \widetilde{B}(Y)P(Z,X) - \widetilde{D}(Z)P(X,Y) \\ - \alpha(Y)P(X,Z) - \alpha(Z)P(Y,X)] - [2a\widetilde{A}(X) - da(X)]g(Y,Z) \\ - [a\widetilde{B}(Y) + \alpha(QY)]g(Z,X) - [a\widetilde{D}(Z) + \alpha(QZ)]g(X,Y),\end{aligned} \tag{2.12}$$

where Q is the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$.

In particular, if the 1-form α vanishes identically, then (2.12) reduces to (1.2) with $\widetilde{A} = A$, $\widetilde{B} = B$ and $\widetilde{D} = D$. Hence the manifold $G(PRS)_n$ is a particular case of $[G(PRS)_n, \widetilde{\nabla}]$. Also the manifold $G(PRS)_n$ could be a $[G(PRS)_n, \widetilde{\nabla}]$ when it admits a semi-symmetric metric connection $\widetilde{\nabla}$ different from the Riemannian connection ∇. We now prove the following Lemma.

Lemma 2.1. *If in a $[G(PRS)_n, \widetilde{\nabla}]$ the* 1*-form α, associated with the torsion tensor τ, is closed, then its Ricci tensor $\widetilde{S}$ is of the form*:

$$\widetilde{S} = \widetilde{r}\gamma \otimes \gamma, \tag{2.13}$$

where γ is a non-zero 1*-form defined by $\gamma(X) = g(X, \mu)$, μ being a unit vector field.*

Proof. Interchanging Y and Z in (2.11) we obtain

$$(\widetilde{\nabla}_X \widetilde{S})(Z,Y) = 2\widetilde{A}(X)\widetilde{S}(Z,Y) + \widetilde{B}(Z)\widetilde{S}(Y,X) + \widetilde{D}(Y)\widetilde{S}(X,Z). \tag{2.14}$$

Since the 1-form α, associated with the torsion tensor τ, is closed, from Proposition 2.1 it follows that the Ricci tensor $\widetilde{S}$ is symmetric. Subtracting the last relation from (2.11) we get

$$[\widetilde{B}(Y) - \widetilde{D}(Y)]\widetilde{S}(X,Z) = [\widetilde{B}(Z) - \widetilde{D}(Z)]\widetilde{S}(X,Y), \tag{2.15}$$

where the symmetry property of $\widetilde{S}$ has been used.

Let us consider $\widetilde{E}(X) = g(X,\nu) = \widetilde{B}(X) - \widetilde{D}(X)$ for all vector fields X and ν is a vector field associated with the 1-form $\widetilde{E}$. Then the above relation reduces to

$$\widetilde{E}(Y)\widetilde{S}(Z,X) = \widetilde{E}(Z)\widetilde{S}(X,Y). \tag{2.16}$$

Contracting (2.16) with respect to X and Z we have

$$\widetilde{r}\widetilde{E}(Y) = \widetilde{E}(\widetilde{Q}Y), \tag{2.17}$$

where $\widetilde{Q}$ is the Ricci operator associated with the Ricci tensor $\widetilde{S}$, i.e., $\widetilde{S}(X,Y) = g(\widetilde{Q}X,Y)$. Also from (2.16) we have

$$\widetilde{E}(\nu)\widetilde{S}(X,Y) = \widetilde{E}(Y)\widetilde{S}(X,\nu) = \widetilde{E}(Y)g(\widetilde{Q}X,\nu) = \widetilde{E}(Y)\widetilde{E}(\widetilde{Q}X), \tag{2.18}$$

which, in view of (2.17), yields

$$\begin{aligned} \widetilde{S}(X,Y) &= \frac{\widetilde{r}}{\widetilde{E}(\nu)}\widetilde{E}(X)\widetilde{E}(Y) \\ &= \widetilde{r}\gamma(X)\gamma(Y), \end{aligned} \tag{2.19}$$

where $\gamma(X) = g(X,\mu) = \frac{1}{\sqrt{\widetilde{E}(\nu)}}\widetilde{E}(X)$, μ being a unit vector field associated with the 1-form γ. Hence the lemma is proved.

Using (2.13) and (2.10) in (2.8) we obtain

$$S(Y,Z) = ag(Y,Z) + [r-2(n-1)a]\gamma(Y)\gamma(Z) + (n-2)P(Y,Z), \tag{2.20}$$

provided that the 1-form α, associated with the torsion tensor τ, is closed. The unit vector field μ associated to the 1-form γ is called the generator of $[G(PRS)_n,\widetilde{\nabla}]$ and P is known as the structure tensor of $[G(PRS)_n,\widetilde{\nabla}]$.

A non-zero vector V on a manifold M is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g(V,V) < 0$ (resp. ≤ 0, $= 0$, > 0) [2,12].

Since μ is a unit vector field on the connected and non-compact Riemannian manifold $M = [G(PRS)_n,\widetilde{\nabla}]$ with metric tensor g, it can be easily shown [12, p. 148] that $\widetilde{g} = g - 2\gamma\otimes\gamma$ is a Lorentz metric on M. Also, μ becomes timelike so the resulting Lorentz manifold is time-orientable.

3. GENERAL RELATIVISTIC VISCOUS FLUID $[G(PRS)_4,\widetilde{\nabla}]$ SPACETIME

A viscous fluid spacetime is a connected Lorentz manifold (M^4,g) with signature (-, +, +, +). In general relativity the key role is played by Einstein's equation

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{3.1}$$

for all vector fields X, Y, where S is the Ricci tensor of type (0, 2), r is the scalar curvature, λ is the cosmological constant, k is the gravitational constant, and T is the energy-momentum tensor of type (0, 2). Let us consider the energy-momentum tensor T of a viscous fluid spacetime of the following form [11]

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + P(X,Y), \tag{3.2}$$

where σ, p are the energy density and isotropic pressure, respectively, and P denotes the anisotropic pressure tensor of the fluid, μ is the unit timelike vector field, called flow vector field of the fluid associated with the 1-form γ given by $g(X,\mu) = \gamma(X)$ for all X. Then by virtue of (3.2), (3.1) can be written as

$$S(X,Y) = \left(\frac{r}{2} + kp - \lambda\right)g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y) + kP(X,Y), \tag{3.3}$$

which, in view of (2.20), shows that the spacetime under consideration is a $[G(PRS)_4, \widetilde{\nabla}]$ with μ as the unit timelike flow vector field of the fluid and P as the anisotropic pressure tensor, where the 1-form α, associated with the torsion tensor τ, corresponding to the semi-symmetric metric connection $\widetilde{\nabla}$, is closed. Hence we can state the following:

Theorem 3.1. *A viscous fluid spacetime obeying Einstein's equation with a cosmological constant is a connected non-compact Lorentzian* $[G(PRS)_4, \widetilde{\nabla}]$ *with closed* 1*-form* α, *associated with the torsion tensor* τ, *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$, *the generator* μ *as the flow vector field of the fluid and the structure tensor P as the anisotropic pressure tensor of the fluid.*

Now by virtue of (3.3), (2.20) yields

$$\left(\frac{r}{2} + kp - \lambda - a\right) g(X,Y) + (k\sigma + kp + 6a - r)\gamma(X)\gamma(Y) + (k-2)P(X,Y) = 0. \tag{3.4}$$

Setting $X = Y = \mu$ in (3.4) we obtain by virtue of (2.10) that

$$\sigma = \frac{1}{2k}[3\widetilde{r} - 2\lambda + 4a - 2(k-2)b], \tag{3.5}$$

where $b = P(\mu, \mu)$. Again, contracting (3.4) we find by virtue of (2.10) that

$$p = \frac{1}{6k}[6\lambda - 3\widetilde{r} - 2(k+4)a - 2(k-2)b]. \tag{3.6}$$

This leads to the following:

Theorem 3.2. *In a viscous fluid* $[G(PRS)_4, \widetilde{\nabla}]$ *spacetime, with closed* 1*-form* α *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$, *obeying Einstein's equation with a cosmological constant* λ *none of the isotropic pressure and energy density can be a constant.*

Now suppose that $\sigma > 0$ and $p > 0$. Then we have from (3.5) and (3.6) that

$$\lambda < \frac{3\widetilde{r} + 4a - 2(k-2)b}{2} \quad \text{and} \quad \lambda > \frac{3\widetilde{r} + 2(k+4)a + 2(k-2)b}{6}$$

and hence

$$\frac{3\widetilde{r} + 2(k+4)a + 2(k-2)b}{6} < \lambda < \frac{3\widetilde{r} + 4a - 2(k-2)b}{2} \tag{3.7}$$

and

$$\widetilde{r} > \frac{k-2}{3}(a + 4b). \tag{3.8}$$

This leads to the following:

Theorem 3.3. *In a viscous fluid* $[G(PRS)_4, \widetilde{\nabla}]$ *spacetime, with closed* 1*-form* α *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$, *obeying Einstein's equation, the cosmological constant* λ *satisfies the relation* (3.7) *and the scalar curvature* $\widetilde{r}$ *satisfies the relation* (3.8) *provided that the energy density and isotropic pressure are both positive.*

We now investigate whether a viscous fluid $[G(PRS)_4, \widetilde{\nabla}]$ spacetime with μ as the unit timelike flow vector field can admit heat flux or not. Therefore, if possible, let the energy-momentum tensor T be of the following form [11]

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\eta(Y) + \gamma(Y)\eta(X), \tag{3.9}$$

where $\eta(X) = g(X,V)$ for all vector fields X; V being the heat flux vector field; σ, p are the energy density and isotropic pressure, respectively. Thus we have $g(\mu,V) = 0$, i.e., $\eta(\mu) = 0$. Hence by virtue of the last relation, (2.10) and (2.20), (3.1) yields

$$2P(X,Y) + \left(\lambda - \frac{\widetilde{r}}{2} - kp - 2a\right) g(X,Y) + [\widetilde{r} - k(p+\sigma)]\gamma(X)\gamma(Y)$$
$$-k[\gamma(X)\eta(Y) + \gamma(Y)\eta(X)] = 0. \tag{3.10}$$

Setting $Y = \mu$ in (3.10) we obtain

$$2P(X,\mu) + \left(\lambda - \frac{3}{2}\widetilde{r} + k\sigma - 2a\right)\gamma(X) + k\eta(X) = 0. \tag{3.11}$$

Putting $X = \mu$ in (3.11) we obtain

$$2b - \left(\lambda - \frac{3}{2}\widetilde{r} + k\sigma - 2a\right) = 0. \tag{3.12}$$

Using the last relation in (3.11) we obtain

$$\eta(X) = -\frac{2}{k}[P(X,\mu) + b\gamma(X)], \quad \text{since } k \neq 0. \tag{3.13}$$

Thus we have the following:

Theorem 3.4. *A viscous fluid* $[G(PRS)_4, \widetilde{\nabla}]$ *spacetime, with closed* 1*-form* α *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$*, obeying Einstein's equation with a cosmological constant* λ *admits heat flux given by* (3.13) *unless* $P(X,\mu) + b\gamma(X) \neq 0$ *for all* X.

From (3.13) it follows that

$$V = -\frac{2}{k}(L + bI)\mu, \tag{3.14}$$

where $P(X,Y) = g(LX,Y)$ for all vector fields X, Y. This implies that $V = 0$ if and only if $-b$ is the eigenvalue of P corresponding to the eigenvector μ. Hence we can state the following:

Theorem 3.5. *A viscous fluid* $[G(PRS)_4, \widetilde{\nabla}]$ *spacetime, with closed* 1*-form* α *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$*, cannot admit heat flux if and only if* $-b$ *is the eigenvalue of* P *corresponding to the eigenvector* μ.

4. EXAMPLE OF $G(PRS)_4$

This section deals with an example of $G(PRS)_4$.

Example. Let $M^4 = \{(x^1,x^2,x^3,x^4) \in \mathbf{R}^4\}$ be an open subset of $\mathbf{R}^4$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = f(x^1,x^3)(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2$$
$$(i,j = 1,2,..,4), \tag{4.1}$$

where f is a continuously differentiable function of x^1 and x^3 such that $f_{\cdot 33} \neq 0$, $f_{\cdot 331} \neq 0$, and $f_{\cdot 333} \neq 0$, and '$\cdot$' denotes the partial differentiation with respect to the coordinates. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, and their covariant derivatives are given by

$$\Gamma^3_{11} = -\frac{1}{2}f_{\cdot 3} = -\Gamma^2_{13}, \; \Gamma^2_{11} = \frac{1}{2}f_{\cdot 1},$$

$$R_{1331} = \frac{1}{2}f_{\cdot 33},\ S_{11} = \frac{1}{2}f_{\cdot 33},\ S_{11,1} = \frac{1}{2}f_{\cdot 331},\ S_{11,3} = \frac{1}{2}f_{\cdot 333}$$

and the components which can be obtained from these by the symmetry properties, where ',' denotes the covariant differentiation with respect to the metric tensor g. Therefore our M^4 with the considered metric is a Riemannian manifold which is neither Ricci symmetric nor Ricci recurrent and is of vanishing scalar curvature.

Now we shall show that this M^4 is $G(PRS)_4$, i.e., it satisfies (1.2). We consider the 1-forms

$$\begin{aligned} A_i(x) &= \frac{f_{\cdot 331}}{2f_{\cdot 33}} \quad \text{for} \quad i=1 \\ &= \frac{f_{\cdot 333}}{2f_{\cdot 33}} \quad \text{for} \quad i=3 \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

$$\begin{aligned} B_i(x) &= \frac{1}{2} \quad \text{for} \quad i=1 \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

$$\begin{aligned} D_i(x) &= -\frac{1}{2} \quad \text{for} \quad i=1 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

at any point $x \in M$. Then in our M^4, (1.2) reduces to the following equations

$$S_{11,1} = 2A_1 S_{11} + B_1 S_{11} + D_1 S_{11}, \tag{4.2}$$

$$S_{11,3} = 2A_3 S_{11} + B_1 S_{31} + D_1 S_{13}, \tag{4.3}$$

since for the cases other than (4.2) and (4.3), the components of each term of (1.2) vanish identically and the relation (1.2) holds trivially.

$$\begin{aligned} \text{Now the right hand side of (4.2)} &= (2A_1 + B_1 + D_1)S_{11} \\ &= \frac{1}{2}f_{\cdot 331} \\ &= \text{left hand side of (4.2).} \end{aligned}$$

Similarly it can be proved that the relation (4.3) is true. Hence our considered M^4 equipped with the considered metric given by (4.1) is a $G(PRS)_4$. Thus we can state the following:

Theorem 4.1. *Let* (M^4, g) *be a Riemannian manifold endowed with the metric* (4.1). *Then* (M^4, g) *is a generalized pseudo Ricci symmetric manifold with vanishing scalar curvature which is neither Ricci symmetric nor Ricci recurrent.*

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Poolsümmeetrilist meetrilist seost järgivatest üldistatud pseudo Ricci sümmeetrilistest muutkondadest

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Artikli eesmärgiks on uurida poolsümmeetrilist meetrilist seost järgivate üldistatud pseudo Ricci sümmeetriliste muutkondade rakendamist üldrelatiivsusteoorias ja kosmoloogias. Ühtlasi on näite abil veendutud üldistatud pseudo Ricci sümmeetriliste muutkondade olemasolus.