Ramanujan J (2011) 24: 151–159
DOI 10.1007/s11139-010-9246-1

Commutator subgroups of the power subgroups of some Hecke groups

Recep Sahin · Özden Koruoğlu

Received: 25 August 2009 / Accepted: 18 May 2010 / Published online: 8 December 2010

Abstract Let $q \geq 3$ be a prime and let $H(\lambda_q)$ be the Hecke group associated to q. Let m be a positive integer and $H^m(\lambda_q)$ be the mth power subgroup of $H(\lambda_q)$. In this work, we study the commutator subgroups of the power subgroups $H^m(\lambda_q)$ of $H(\lambda_q)$. Then, we give the derived series for all triangle groups of the form $(0; 2, q, n)$ for n a positive integer, since there is a nice connection between the signatures of the subgroups we studied and the signatures of these derived series.

Keywords Hecke groups · Power subgroup · Commutator subgroup · Derived series

Mathematics Subject Classification (2000) Primary 20H10 · 11F06 · 20D15

1 Introduction

In [5], Erich Hecke introduced the Hecke groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

R. Sahin
Balıkesir Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 10145 Çağış Kampüsü, Balıkesir, Turkey
e-mail: rsahin@balikesir.edu.tr

Ö. Koruoğlu (✉)
Balıkesir Üniversitesi, Necatibey Eğitim Fakültesi, İlkogretim Bölümü, Matematik Egitimi, 10100 Balıkesir, Turkey
e-mail: ozdenk@balikesir.edu.tr

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, for $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case $q \geq 3$ integer and we denote it by $H(\lambda_q)$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q. $H(\lambda_q)$ has a presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_q, \quad [4].$$

It is well known that every Fuchsian group has a presentation of the following type:

Generators	$a_1, b_1, \ldots, a_g, b_g$	(hyperbolic),
	$x_1, \ldots, x_r$	(elliptic),
	$p_1, \ldots, p_t$	(parabolic),
	$h_1, \ldots, h_u$	(hyperbolic boundary),
Relations	$x_j^{m_j} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{t} p_k \prod_{l=1}^{u} h_l = 1,$	

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator of a_i and b_i. We than say that the group has signature $(g; m_1, \ldots, m_r; t; u)$. Here g is the genus of the Riemann surface corresponding to the group and m_i are the integers greater than 1 and they are called the periods of the group. Fuchsian groups including Hecke groups $H(\lambda_q)$ have no hyperbolic boundary elements, therefore we take $u = 0$, and omit it in the signatures.

Note that the Hecke groups $H(\lambda_q)$ can be thought of as triangle groups having an infinity as one of the entries. Coxeter and Moser [3] have shown that the triangle group $(g; k, l, m)$ is finite when $(1/k + 1/l + 1/m) > 1$ and infinite when $(1/k + 1/l + 1/m) \leq 1$. As $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, each is an infinite triangle group. Several of these groups are $H(\lambda_3) = \Gamma = \mathrm{PSL}(2, \mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$.

The extended modular group, denoted by $\overline{H}(\lambda_3) = \Pi = \mathrm{PGL}(2, \mathbb{Z})$, has been defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the modular group $H(\lambda_3)$. Then, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [12, 13], and [14] similar to the extended modular group by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$. Thus the extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_q, \quad (1.1)$$

and Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. The signature of the extended Hecke group $\overline{H}(\lambda_q)$ is $(0; +; [-]; \{(2, q, \infty)\})$. Since the extended Hecke groups $\overline{H}(\lambda_q)$ contain a reflection, they are non-Euclidean crystallographic (NEC) groups.

Let m be a positive integer. Let us define $H^m(\lambda_q)$ to be the subgroup generated by the mth powers of all elements of $H(\lambda_q)$. The subgroup $H^m(\lambda_q)$ is called the mth *power subgroup* of $H(\lambda_q)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

The power subgroups of the modular group $H(\lambda_3)$ have been studied and classified in [8] and [9] by Newman. His results have been generalized to Hecke groups $H(\lambda_q)$, by Cangül and Singerman for $q \geq 3$ prime in [1], by Ikikardes, Koruoğlu and Sahin for $q \geq 4$ even integer in [7] and by Cangül, Sahin, Ikikardes and Koruoğlu for $q \geq 3$ odd integer in [2]. For $q \geq 3$ prime, the power subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ were studied by Sahin, Ikikardes and Koruoğlu in [15, 16] and [17].

For $q \geq 3$ odd integer and for m positive integer, they proved the following results of which (a) to (g) are given in [2]:

(a) The normal subgroup $H^2(\lambda_q)$ is the free product of two finite cyclic groups of order q, i.e.,

$$H^2(\lambda_q) = \big\langle S, TST \mid S^q = (TST)^q = I\big\rangle \cong \mathbb{Z}_q * \mathbb{Z}_q, \tag{1.2}$$

and the signature of $H^2(\lambda_q)$ is $(0; q^{(2)}, \infty)$.

(b) The normal subgroup $H^q(\lambda_q)$ is the free product of q finite cyclic groups of order 2, in particular,

$$H^q(\lambda_q) = \langle T\rangle * \big\langle STS^{-1}\big\rangle * \big\langle S^2TS^{-2}\big\rangle * \cdots * \big\langle S^{q-1}TS\big\rangle, \tag{1.3}$$

and it has the signature $(0; 2^{(q)}, \infty)$.

(c) If $(m, 2) = 1$ and $(m, q) = 1$, then $H^m(\lambda_q) \cong H(\lambda_q)$.

(d) If $(m, 2) = 2$ and $(m, q) = 1$, then $H^m(\lambda_q) \cong H^2(\lambda_q)$.

(e) If $(m, 2) = 1$ and $(m, q) = q$, then $H^m(\lambda_q) \cong H^q(\lambda_q)$.

(f) The commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$ satisfies

$$H'(\lambda_q) = H^2(\lambda_q) \cap H^q(\lambda_q). \tag{1.4}$$

Also the signature of $H'(\lambda_q)$ is $(\frac{q-1}{2}; \infty)$ and $H'(\lambda_q)$ is a free group.

(g) If $(m, 2) = 2$ and $(m, q) = q$, then $H^m(\lambda_q) \subset H'(\lambda_q)$ and the groups $H^m(\lambda_q)$ are free.

(h) For $p \geq 3$ prime, the commutator subgroup $\overline{H}'(\lambda_p)$ of $\overline{H}(\lambda_p)$ satisfies

$$\overline{H}'(\lambda_p) = \overline{H}^2(\lambda_p) = H^2(\lambda_p), \quad [17]. \tag{1.5}$$

In this paper, we obtain the group structures and the signatures of commutator subgroups of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, for $q \geq 3$ prime. We achieve this by applying standard techniques of combinatorial group theory (the Reidemeister–Schreier method and the permutation method). Then, we give an application related with the derived series for all triangle groups of the form $(0; 2, q, n)$, for $q \geq 3$ prime and n positive integer, since there is a nice connection between the signatures of our studied subgroups and the signatures of these derived series.

2 Commutator subgroups of the power subgroups of some Hecke groups

In this section, firstly, we study the commutator subgroups of the power subgroups $H^2(\lambda_q)$ and $H^q(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, for $q \geq 3$ prime. Now let us give the following theorems.

Theorem 1 *Let $q \geq 3$ be prime.*

(i) $|H^2(\lambda_q) : (H^2)'(\lambda_q)| = q^2$.
(ii) *The group $(H^2)'(\lambda_q)$ is a free group of rank $(q-1)^2$ with basis* $[S, TST]$, $[S, TS^2T]$, $\ldots$, $[S, TS^{q-1}T]$, $[S^2, TST]$, $[S^2, TS^2T], \ldots, [S^2, TS^{q-1}T], \ldots,$ $[S^{q-1}, TST], [S^{q-1}, TS^2T], \ldots, [S^{q-1}, TS^{q-1}T]$.
(iii) *The group $(H^2)'(\lambda_q)$ is of index q in $H'(\lambda_q)$.*
(iv) *For $n \geq 2$, $|H^2(\lambda_q) : (H^2)^{(n)}(\lambda_q)| = \infty$.*

Proof Since $\overline{H}'(\lambda_q) = H^2(\lambda_q)$, we have $(H^2)'(\lambda_q) = \overline{H}''(\lambda_q)$. Then it is easy to see (i)–(iv) from the Theorem 3.4 in [13]. □

Here, using the permutation method, we get also the signature of $(H^2)'(\lambda_q)$ as $(\frac{q^2-3q+2}{2}; \underbrace{\infty, \infty, \ldots, \infty}_{q \text{ times}}) = (\frac{q^2-3q+2}{2}; \infty^{(q)})$.

Theorem 2 *Let $q \geq 3$ be prime.*

(i) $|H^q(\lambda_q) : (H^q)'(\lambda_q)| = 2^q$.
(ii) *The group $(H^q)'(\lambda_q)$ is a free group of rank $1 + (q-2)2^{q-1}$.*
(iii) *The group $(H^q)'(\lambda_q)$ is of index 2^{q-1} in $H'(\lambda_q)$.*
(v) *For $n \geq 2$, $|H^q(\lambda_q) : (H^q)^{(n)}(\lambda_q)| = \infty$.*

Proof (i) From (1.3), let $k_1 = T$, $k_2 = STS^{-1}$, $k_3 = S^2TS^{-2}, \ldots, k_q = S^{q-1}TS$. The quotient group $H^q(\lambda_q)/(H^q)'(\lambda_q)$ is the group obtained by adding the relation $k_ik_j = k_jk_i$ to the relations of $H^q(\lambda_q)$, for $i \neq j$ and $i, j \in \{1, 2, \ldots, q\}$. Then

$$H^q(\lambda_q)/\big(H^q\big)'(\lambda_q) \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{q \text{ times}}.$$

Therefore, we obtain $|H^q(\lambda_q) : (H^q)'(\lambda_q)| = 2^q$.

(ii) Now we choose $\Sigma = \{I, k_1, k_2, \ldots, k_q, k_1k_2, k_1k_3, \ldots, k_1k_q, k_2k_3, k_2k_4, \ldots, k_2k_q, \ldots, k_{q-1}k_q, k_1k_2k_3, k_1k_2k_4, \ldots, k_1k_2k_q, \ldots, k_1k_2 \ldots k_q\}$ as a Schreier transversal for $(H^q)'$. According to the Reidemeister–Schreier method, we get the generators of $(H^q)'$ as the followings.

There are $C(q,2) = \binom{q}{2}$ generators of the form $k_ik_jk_ik_j$ where $i < j$ and $i, j \in \{1, 2, \ldots, q\}$. There are $2 \times \binom{q}{3}$ generators of the form $k_ik_jk_tk_jk_tk_i$, or $k_ik_jk_tk_ik_tk_j$ where $i < j < t$ and $i, j, t \in \{1, 2, \ldots, q\}$. There are $3 \times \binom{q}{4}$ generators of the form $k_ik_jk_tk_uk_ik_uk_tk_j$, or $k_ik_jk_tk_uk_jk_uk_tk_i$, or $k_ik_jk_tk_uk_tk_uk_jk_i$ where $i < j < t < u$ and $i, j, t, u \in \{1, 2, \ldots, q\}$. Similarly, there are $(q-1) \times \binom{q}{q}$ generators of the form $k_1k_2 \cdots k_qk_1k_qk_{q-1} \cdots k_2$, or $k_1k_2 \cdots k_qk_2k_qk_{q-1} \cdots k_3k_1$, or $\cdots$, or $k_1k_2 \cdots k_qk_{q-1}k_qk_{q-2} \cdots k_2k_1$. In fact, there are total $1 + (q-2)2^{q-1}$ generators obtained by using the theorem of Nielsen in [11].

(iii) Since $|H(\lambda_q) : (H^q)'(\lambda_q)| = q.2^q$ and $|H(\lambda_q) : H'(\lambda_q)| = 2q$, we obtain $|H'(\lambda_q) : (H^q)'(\lambda_q)| = 2^{q-1}$.

(iv) Taking relations and abelianizing we find that the resulting quotient is infinite. It follows that $(H^q)''(\lambda_q)$ has infinite index in $(H^q)'(\lambda_q)$. Further since this has infinite index it follows that the derived series from this point on have infinite index. □

Also, we obtain the signature of $(H^q)'(\lambda_q)$ as $((q-3)2^{q-2}+1; \underbrace{\infty, \infty, \ldots, \infty}_{2^{q-1} \text{ times}}) = ((q-3)2^{q-2}+1; \infty^{(2^{q-1})})$.

Notice that the results in the Theorems 1 and 2 coincide with the results given in [10, Lemma 1, p. 102] for the modular group $H(\lambda_3)$. Lemma 2 of [10] also directly generalizes, giving the following.

Corollary 3 *We have*

$$H'(\lambda_q) = \big(H^2\big)'(\lambda_q)\big(H^q\big)'(\lambda_q).$$

Proof Since $(H^2)'(\lambda_q)$ and $(H^q)'(\lambda_q)$ are normal subgroups of $H'(\lambda_q)$, we have the chains

$$H'(\lambda_q) \supset \big(H^2\big)'(\lambda_q)\big(H^q\big)'(\lambda_q) \supset \big(H^2\big)'(\lambda_q) \quad \text{and}$$
$$H'(\lambda_q) \supset \big(H^2\big)'(\lambda_q)\big(H^q\big)'(\lambda_q) \supset \big(H^q\big)'(\lambda_q).$$

Since $|H'(\lambda_q) : (H^2)'(\lambda_q)(H^q)'(\lambda_q)|$ divides q and $|H'(\lambda_q) : (H^2)'(\lambda_q)(H^q)'(\lambda_q)|$ divides 2^{q-1}, we get $|H'(\lambda_q) : (H^2)'(\lambda_q)(H^q)'(\lambda_q)| = 1$ as $(q, 2^{q-1}) = 1$. Then we obtain $H'(\lambda_q) = (H^2)'(\lambda_q)(H^q)'(\lambda_q)$. □

Here we give an example related with the Theorem 2.

Example 1 Let $q = 5$. Then $|H^5(\lambda_5) : (H^5)'(\lambda_5)| = 32$. We choose $\Sigma = \{I, k_1, k_2, k_3, k_4, k_5, k_1k_2, k_1k_3, k_1k_4, k_1k_5, k_2k_3, k_2k_4, k_2k_5, k_3k_4, k_3k_5, k_4k_5, k_1k_2k_3, k_1k_2k_4, k_1k_2k_5, k_1k_3k_4, k_1k_3k_5, k_1k_4k_5, k_2k_3k_4, k_2k_3k_5, k_2k_4k_5, k_3k_4k_5, k_1k_2k_3k_4, k_1k_2k_3k_5, k_1k_2k_4k_5, k_1k_3k_4k_5, k_2k_3k_4k_5, k_1k_2k_3k_4k_5\}$ as a Schreier transversal for $(H^5)'$. Using the Reidemeister–Schreier method, we get the generators of $(H^q)'$ as the following. There are 10 generators of the form,

$$\begin{array}{lllll} k_1k_2k_1k_2, & k_1k_3k_1k_3, & k_1k_4k_1k_4, & k_1k_5k_1k_5, & k_2k_3k_2k_3, \\ k_2k_4k_2k_4, & k_2k_5k_2k_5, & k_3k_4k_3k_4, & k_3k_5k_3k_5, & k_4k_5k_4k_5, \end{array}$$

20 generators of the form,

$$\begin{array}{llll} k_1k_2k_3k_2k_3k_1, & k_1k_2k_3k_1k_3k_2, & k_1k_2k_4k_2k_4k_1, & k_1k_2k_4k_1k_4k_2, \\ k_1k_2k_5k_2k_5k_1, & k_1k_2k_5k_1k_5k_2, & k_1k_3k_4k_3k_4k_1, & k_1k_3k_4k_1k_4k_3, \\ k_1k_3k_5k_3k_5k_1, & k_1k_3k_5k_1k_5k_3, & k_1k_4k_5k_4k_5k_1, & k_1k_4k_5k_1k_5k_4, \\ k_2k_3k_4k_3k_4k_2, & k_2k_3k_4k_2k_4k_3, & k_2k_3k_5k_3k_5k_2, & k_2k_3k_5k_2k_5k_3, \\ k_2k_4k_5k_4k_5k_2, & k_2k_4k_5k_2k_5k_4, & k_3k_4k_5k_4k_5k_3, & k_3k_4k_5k_3k_5k_4, \end{array}$$

15 generators of the form,

$$\begin{array}{lll} k_1k_2k_3k_4k_1k_4k_3k_2, & k_1k_2k_3k_4k_2k_4k_3k_1, & k_1k_2k_3k_4k_3k_4k_2k_1, \\ k_1k_2k_3k_5k_1k_5k_3k_2, & k_1k_2k_3k_5k_2k_5k_3k_1, & k_1k_2k_3k_5k_3k_5k_2k_1, \\ k_1k_2k_4k_5k_1k_5k_4k_2, & k_1k_2k_4k_5k_2k_5k_4k_1, & k_1k_2k_4k_5k_4k_5k_2k_1, \\ k_1k_3k_4k_5k_1k_5k_4k_3, & k_1k_3k_4k_5k_3k_5k_4k_1, & k_1k_3k_4k_5k_4k_5k_3k_1, \\ k_2k_3k_4k_5k_2k_5k_4k_3, & k_2k_3k_4k_5k_3k_5k_4k_2, & k_2k_3k_4k_5k_4k_5k_3k_2, \end{array}$$

and 4 generators of the form

$$\begin{array}{ll} k_1k_2k_3k_4k_5k_1k_5k_4k_3k_2, & k_1k_2k_3k_4k_5k_2k_5k_4k_3k_1, \\ k_1k_2k_3k_4k_5k_3k_5k_4k_2k_1, & k_1k_2k_3k_4k_5k_4k_5k_3k_2k_1. \end{array}$$

Therefore, the group $(H^5)'(\lambda_5)$ is a free group of rank 49.

Using the Theorems 1 and 2, we have the following results.

Corollary 4 *Let $q \geq 3$ be a prime and let m be a positive integer. Then*

(i) *If $(m,2)=1$ and $(m,q)=1$, then $H^m(\lambda_q) \cong H(\lambda_q)$ and so $(H^m)'(\lambda_q) \cong H'(\lambda_q)$. In this case, the series of the signatures of $H(\lambda_q)$, $H^m(\lambda_q)$ and $(H^m)'(\lambda_q)$, respectively, is*

$$\begin{aligned} &H(\lambda_q)(0;2,q,\infty) = H^m(\lambda_q)(0;2,q,\infty), \\ &H(\lambda_q)(0;2,q,\infty) \supset \big(H^m\big)'(\lambda_q)\left(\frac{q-1}{2};\infty\right). \end{aligned} \tag{2.1}$$

(ii) *If $(m,2)=2$ and $(m,q)=1$, then $(H^m)'(\lambda_q) \cong (H^2)'(\lambda_q)$. In this case, the series of the signatures of $H(\lambda_q)$, $H^m(\lambda_q)$ and $(H^m)'(\lambda_q)$, respectively, is*

$$\begin{aligned} H(\lambda_q)(0;2,q,\infty) &\supset H^m(\lambda_q)\big(0;q^{(2)},\infty\big) \\ &\supset \big(H^m\big)'(\lambda_q)\left(\frac{q^2-3q+2}{2};\infty^{(q)}\right). \end{aligned} \tag{2.2}$$

(iii) *If $(m,2)=1$ and $(m,q)=q$, then $(H^m)'(\lambda_q) \cong (H^q)'(\lambda_q)$. In this case, the series of the signatures of $H(\lambda_q)$, $H^m(\lambda_q)$ and $(H^m)'(\lambda_q)$, respectively, is*

$$\begin{aligned} H(\lambda_q)(0;2,q,\infty) &\supset H^m(\lambda_q)\big(0;2^{(q)},\infty\big) \\ &\supset \big(H^m\big)'(\lambda_q)\big((q-3)2^{q-2}+1;\infty^{(2^{q-1})}\big). \end{aligned} \tag{2.3}$$

After Corollary 4, we are only left to consider the case where $(m,2)=2$ and $(m,q)=q$. In this case, the factor group $H(\lambda_q)/H^m(\lambda_q)$ is the infinite group. Therefore we cannot say much about $H^m(\lambda_q)$ apart from the fact that they are all normal subgroups with torsion.

3 The relationships between the derived series of the triangle groups $(0;2,q,n)$ and the signatures of some subgroups of $H(\lambda_q)$

Now let us give the relationship between the derived series for all triangle groups of the form $(0;2,q,n)$ for $q \geq 3$ prime and n positive integer and the signatures of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$ and their commutator subgroups. The derived series for all triangle groups of the form $(0;2,3,n)$ were studied by Zomorrodian in [18]. Here we consider the cases $q \geq 3$ prime. All our findings coincide with the ones given in [18] for $q=3$.

Indeed, by adding relation $(TS)^n$ to the existing two relations, all triangle groups of the form $(0;2,q,n)$ become one relator quotient groups of the Hecke groups $H(\lambda_q)$. Thus if Γ is a Fuchsian group with signature $(0;2,q,n)$, then Γ has the following presentation

$$\Gamma \cong H(\lambda_q)/R(T,S) = \langle T,S \mid T^2 = S^q = (TS)^n = I\rangle, \tag{3.1}$$

where $R(T,S) = (TS)^n$ for any value of the positive integer n.

The quotient group Γ/Γ' is the group obtained by adding the relation $TS = ST$ to the relations of Γ. Then Γ/Γ' has a presentation

$$\Gamma/\Gamma' \cong \langle t,s \mid t^2 = s^q = (ts)^n = I,\ ts = st\rangle, \tag{3.2}$$

where t and s are the images of T and S, respectively, under the homomorphism of Γ to Γ/Γ'. Now we give the following example:

Example 2 Let Γ be a Fuchsian group with signature $(0;2,q,n)$. Let $\ell(\Gamma)$ denote the drive length of Γ and $\Gamma \rhd \Gamma' \rhd \Gamma'' \rhd \cdots \rhd \Gamma^{(k)} \rhd \cdots$ be its derived series. Then

(i) If $(n,2)=1$ and $(n,q)=1$, then by using (3.2), we find $t=s=I$ from the relations $(ts)^n = (ts)^{2q} = I$. Thus $\Gamma = \Gamma'$, i.e. Γ is the perfect group. Therefore $\Gamma = \Gamma' = \Gamma'' = \cdots = \Gamma^{(k)} = \cdots$. Consequently, we get $\ell(\Gamma) = \infty$.

(ii) If $(n,2)=2$ and $(n,q)=1$, then we have $s=I$, since $s^2 = s^q = I$. Thus we get $\Gamma/\Gamma' \cong \mathbb{Z}_2$. Using the Reidemeister–Schreier method and the permutation method, we find the derived series of Γ as the following:

$$\Gamma(0;2,q,2r) \supset \Gamma' = \Gamma(0;q,q,r) \supset \Gamma'' = \Gamma(0;\underbrace{r,r,\ldots,r}_{q\ \text{times}})$$

$$\supset \Gamma''' = \Gamma\left(\frac{(q-2)r^{q-1} - qr^{q-2} + 2}{2}; -\right). \tag{3.3}$$

Here, the corresponding factor groups are $\mathbb{Z}_2$, $\mathbb{Z}_q$ and $\underbrace{\mathbb{Z}_r \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r}_{(q-1)\ \text{times}}$. In fact, there are infinitely many automorphism groups covered by Γ which are residually soluble (Γ''' and all the terms following Γ''' in the series). Thus we find $\ell(\Gamma) = 4$.

(iii) If $(n,2)=1$ and $(n,q)=q$, then we have $t=I$, since $t^2=t^q=I$. Thus we get $\Gamma/\Gamma'\cong\mathbb{Z}_3$. Using the Reidemeister–Schreier method and the permutation method, we find the derived series of Γ as the following:

$$\Gamma(0;2,q,qr)\supset\Gamma(0;\underbrace{2,2,\ldots,2}_{q\text{ times}},r)\supset\Gamma\big((q-4)2^{q-3}+1;\underbrace{r,r,\ldots,r}_{2^{(q-1)}\text{ times}}\big)$$
$$\supset\Gamma\big(2^{q-3}r^{2^{q-1}-2}\big((q-2)r-2\big)+1;-\big). \tag{3.4}$$

Here, the corresponding factor groups are $\mathbb{Z}_q$, $\underbrace{\mathbb{Z}_2\times\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2}_{(q-1)\text{ times}}$ and $\underbrace{\mathbb{Z}_r\times\mathbb{Z}_r\times\cdots\times\mathbb{Z}_r}_{2^{(q-1)}-1\text{ times}}$. There are infinitely many automorphism groups covered by Γ which are residually soluble (Γ''' and all the terms following Γ''' in the series). Thus we find $\ell(\Gamma)=4$.

(iv) If $(n,2)=2$ and $(n,q)=q$, then we get $t^2=s^q=I$. Thus we get $\Gamma/\Gamma'\cong\mathbb{Z}_{2q}$. Using the Reidemeister–Schreier method and the permutation method, Γ' is a Fuchsian group generated by $z=(TS)^{2q}, a_1=S^{q-1}TST, a_2=S^{q-2}TS^2T,\ldots,a_{q-1}=STS^{q-1}T$. Here the only element of finite order is $z=(TS)^{2q}$ and its order is $n/(2q)$. Using the permutation method, Γ' has signature $(\frac{q-1}{2};r)$, since it is of index $2q$ in Γ. Then the second derived group Γ'' is of infinite index in Γ. Using the theorem of Hoare–Karrass–Solitar in [6, p. 65], we can find the following series:

$$\Gamma(0;2,q,2qr)\supset\Gamma'=\Gamma\left(\frac{q-1}{2};r\right)\supset\Gamma''\supset\cdots. \tag{3.5}$$

Here Γ' is a free product of a finite and $(q-1)$ infinite cyclic groups and Γ'' is a free group. Also the corresponding factor groups are $\mathbb{Z}_{2q}$ and $\underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{(q-1)\text{ times}}$.

Therefore, we find $\ell(\Gamma)=3$.

Notice that there are similar results between the derived series for all triangle groups Γ of the form $(0;2,q,n)$ and the series of the signatures of the power subgroups of the Hecke groups $H(\lambda_q)$ and their commutator subgroups. Especially, there are similarities between (2.1), (2.2), (2.3) and (3.5), (3.3), (3.4), respectively. Of course, there are some differences in these signatures, since $(TS)^n=I$ in Γ and $(TS)^\infty=I$ in $H(\lambda_q)$.

References

1. Cangül, I.N., Singerman, D.: Normal subgroups of Hecke groups and regular maps. Math. Proc. Camb. Philos. Soc. **123**, 59–74 (1998)
2. Cangül, I.N., Sahin, R., Ikikardes, S., Koruoğlu, Ö.: Power subgroups of some Hecke groups. II. Houst. J. Math. **33**(1), 33–42 (2007)
3. Coxeter, H.S.M., Moser, W.O.J.: Generators and Relations for Discrete Groups. Springer, Berlin (1957)

4. Evans, R.J.: A new proof on the free product structure of Hecke groups. Publ. Math. Debr. **22**(1–2), 41–42 (1975)
5. Hecke, E.: Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. Math. Ann. **112**, 664–699 (1936)
6. Hoare, A.H.M., Karrass, A., Solitar, D.: Subgroups of infinite index in Fuchsian groups. Math. Z. **125**, 59–69 (1972)
7. Ikikardes, S., Koruoğlu, Ö., Sahin, R.: Power subgroups of some Hecke groups. Rocky Mt. J. Math. **36**(2), 497–508 (2006)
8. Newman, M.: The structure of some subgroups of the modular group. Ill. J. Math. **6**, 480–487 (1962)
9. Newman, M.: Free subgroups and normal subgroups of the modular group. Ill. J. Math. **8**, 262–265 (1964)
10. Newman, M., Smart, J.R.: Note on a subgroup of the modular group. Proc. Am. Math. Soc. **14**, 102–104 (1963)
11. Nielsen, J.: Kommutatorgruppen für das freie Produkt von zyklischen Gruppen. Mat. Tidsskr. B, 49–56 (1948) (Danish)
12. Ozgür, N.Y., Sahin, R.: On the extended Hecke groups $\overline{H}(\lambda_q)$. Turk. J. Math. **27**, 473–480 (2003)
13. Sahin, R., Bizim, O.: Some subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$. Acta Math. Sci., Ser. B, Engl. Ed. **23**(4), 497–502 (2003)
14. Sahin, R., Bizim, O., Cangül, I.N.: Commutator subgroups of the extended Hecke groups. Czech. Math. J. **54**(1), 253–259 (2004)
15. Sahin, R., Ikikardes, S., Koruoğlu, Ö.: On the power subgroups of the extended modular group $\overline{\Gamma}$. Turk. J. Math. **29**, 143–151 (2004)
16. Sahin, R., Koruoğlu, Ö., Ikikardes, S.: On the extended Hecke groups $\overline{H}(\lambda_5)$. Algebra Colloq. **13**(1), 17–23 (2006)
17. Sahin, R., Ikikardes, S., Koruoğlu, Ö.: Some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$. Rocky Mt. J. Math. **36**(3), 1033–1048 (2006)
18. Zomorrodian, R.: Residual solubility of Fuchsian groups. Ill. J. Math. **51**(3), 697–703 (2007)