The Minimal Polynomials of 2cos(π/2k) over the Rationals

Article · September 2011

DOI: 10.1063/1.3636731

4 authors, including:

Musa Demirci
Uludag University

Nazli Yildiz Ikikardes
Balikesir University

Ismail naci Cangul
Uludag University

The Minimal Polynomials of $2cos(\pi/2k)$ over the Rationals

Musa Demirci*, Nazlı Y. Ikikardes†, Birsen Ozgur** and I. Naci Cangul‡

*Uludag University, Faculty of Science, Department of Mathematics, Bursa-Turkey, mdemirci@uludag.edu.tr
†Balikesir University, Faculty of Arts and Science, Department of Mathematics, Balikesir-Turkey, nyildiz@balikesir.edu.tr
**Uludag University, Faculty of Science, Department of Mathematics, Bursa-Turkey, birsen2006@gmail.com
‡Uludag University, Faculty of Science, Department of Mathematics, Bursa-Turkey, cangul@uludag.edu.tr

Abstract. The number $\lambda_q = 2\cos\pi/q$, $q \in \mathbf{N}$, $q \geq 3$, appears in the study of Hecke groups which are Fuchsian groups of the first kind, and in the study of regular polyhedra. Here we obtained the minimal polynomial of this number by means of the better known Chebycheff polynomials and the set of roots on the extension $\mathbf{Q}(\lambda_q)$. We follow some kind of inductive method on the number q. The minimal polynomial is obtained for even q.

Keywords: Hecke groups, roots of unity, Chebycheff polynomials, minimal polynomial.
PACS: 2010 MSC: 05E35, 33C45, 33C50, 33D45.

INTRODUCTION

For $n \in \mathbf{N}$, the n-th Chebycheff polynomial $T_n(x)$ is defined by

$$T_n(x) = \cos(n\cos^{-1}x),\ x \in \mathbf{R},\ |x| \leq 1,$$

or

$$T_n(\cos\theta) = \cos(n\theta),\ \theta \in \mathbf{R}.$$

We get a normalisation of T_n given by

$$\begin{aligned} A_n(x) &= 2T_n(x/2) \\ &= 2\cos(n\cos^{-1}(x/2)) \\ &= 2\cos n\theta \end{aligned}$$

where $x = A_1(x) = 2\cos\theta$; $x, \theta \in \mathbf{R}$, $|x| \leq 2$, $n \in \mathbf{N}$.

In this work we search for the minimal polynomial of the algebraic number $\lambda_q = 2\cos(\pi/q)$, $q \in \mathbf{N}$. The number λ_q plays an important role in the theory of Hecke groups, which are the discrete subgroups of $PSL(2,\mathbf{R})$ generated by two linear fractional transformations $R(z) = -1/z$ and $T(z) = z + \lambda_q$, see [1, 3]. λ_q is also used in the geometry of polyhedra.

The polynomials A_n are explicitly given in [1] as

$$A_n(x) = \sum_{i=0}^{[n/2]} (-1)^i \frac{n}{n-i} \binom{n-i}{i} x^{n-2i}$$

where $[\alpha]$ denotes the greatest integer less than or equal to α. Therefore

$$\deg(A_n(x)) = n.$$

The first few A_n's are $A_1(x) = x$, $A_2(x) = x^2 - 2$, $A_3(x) = x^3 - 3x$, $A_4(x) = x^4 - 4x^2 + 2$ and $A_5(x) = x^5 - 5x^3 + 5x$. For convenience we define $A_0(x) = 1$.

In [1], the author obtained the minimal polynomial P_q^* of λ_q by means of the T_n's and A_n's as follows:

$$P_q^*(x) = \begin{cases} 2^{\frac{\varphi(q)-q-1}{2}} \dfrac{A_{\frac{q+1}{2}}(x) + A_{\frac{q-1}{2}}(x)}{\prod\limits_{d\,|\,2q, d \neq 2q, d\,even} \psi_d(x/2)} & if\ q\ is\ odd \\ 2^{\frac{\varphi(2q)-q}{2}} \dfrac{A_{q+1}(x) - A_{q-1}(x)}{\prod\limits_{d\,|\,2q, d \neq 2q, d\,even} \psi_d(x/2)(A_{\frac{q}{2}+1}(x) - A_{\frac{q}{2}-1}(x))} & if\ q\ is\ even \end{cases}$$

Numerical Analysis and Applied Mathematics ICNAAM 2011
AIP Conf. Proc. 1389, 325-328 (2011); doi: 10.1063/1.3636731

where $\psi_d(x)$ denotes the minimal polynomial of $\cos(2\pi/q)$ over $\mathbf{Q}$, and $\varphi(q)$ denotes the Euler function. It was also proven that the degree of $P_q^*(x)$ is $\varphi(2q)/2$.

Here we give a relatively simpler formula for P_q^*. Similarly to this, the first author found the minimal polynomial of $\cos 2\pi/q$, in [2, 3].

The two cases of odd and even q show differences and here we will be dealing with the latter one:

THE EVEN q CASE

$P_q(x) = A_{q/2}(x)$ is defined. Then $\deg P_q(x) = q/2$. First of all we have

Theorem 1 *Let q be even. Then the set of the roots of P_q is*

$$W_q = \{2\cos\pi/q = \lambda_q, 2\cos 3\pi/q, 2\cos 5\pi/q, ..., 2\cos(q-1)\pi/q\}.$$

Proof $\zeta^q = -1 = e^{i\pi}$, so $\zeta_k = e^{i(\pi/q+2k\pi/q)}$, $k = 1,2,...,q-1$. But $e^{i\pi/q}.e^{i(2q-1)\pi/q} = 1$, $e^{i3\pi/q}.e^{i(2q-3)\pi/q} = 1, ..., e^{i(q-2)\pi/q}.e^{i(q+2)\pi/q} = 1$ and $e^{i\pi/q} + e^{-i\pi/q} = \lambda_q, e^{i3\pi/q} + e^{-i3\pi/q} = 2\cos 3\pi/q, ..., e^{i(q-1)\pi/q} + e^{i(q+1)\pi/q} = 2\cos(q-1)\pi/q$. Therefore

$$P_q(x) = (x - 2\cos\pi/q)(x - 2\cos 3\pi/q)...(x - 2\cos(q-1)\pi/q).$$

Theorem 2 *Let q be even and d be chosen so that q/d is odd. Then $W_d \subset W_q$ and $P_d \mid P_q$.*

Proof $W_d = \{2\cos\pi/d, 2\cos 3\pi/d, 2\cos 5\pi/d, ..., 2\cos(d-1)\pi/d\}$ and

$$W_q = \{2\cos\pi/kd, 2\cos 3\pi/kd, 2\cos 5\pi/kd, ..., 2\cos(kd-1)\pi/kd\}.$$

We want to show that all elements of W_d are also in W_q. As the coefficients of the angles π/d and π/kd are successive odd numbers, we only need to show that the first one and the last one of W_d are in W_q. First, $2\cos\pi/d = 2\cos k\pi/kd \in W_q$ as $1 < k < kd-1$. Secondly $2\cos(d-1)\pi/d = 2\cos k(d-1)\pi/kd \in W_q$ as $1 < k(d-1) < kd-1$, implying the result.

It is remarkable that when d is chosen so that q/d is also even, we do not have $W_d \subset W_q$. Therefore P_d will not divide P_q in that case. Hence we cannot divide P_q by P_d for all proper divisors d of q as we did in the odd q case.

Let us now consider several cases: Let $q = 2^n$. Then

Theorem 3 *If $q = 2^n$, $n \in \mathbf{N}$, then $P_q^* = P_q$, i.e. $P_q(x)$ is irreducible over $\mathbf{Q}$.*

Proof First, $\deg P_{2^n}^*(x) = \frac{\varphi(2^{n+1})}{2} = 2^{n-1}$ and $\deg P_{2^n}(x) = \deg A_{2^{n-1}}(x) = 2^{n-1}$. Secondly, both polynomials are monic and the roots of $P_q^*(x)$ are amongst the roots of $P_q(x)$, establishing equality.

Theorem 4 *If $q = 2p$ then $P_q^*(x) = \frac{P_q(x)}{P_2(x)}$.*

Proof $\deg P_q^*(x) = \frac{\varphi(2^2.p)}{2} = \varphi(p) = p-1$ and $\deg \frac{P_q(x)}{P_2(x)} = \frac{q}{2} - \frac{2}{2} = p-1$.

Theorem 5 *If $q = 2.p^n$ then $P_q^*(x) = \frac{P_q(x)}{P_{2p^{n-1}}(x)}$.*

Proof As $2p^{n-1} \mid 2p^n$, $W_{2p^{n-1}} \subset W_{2p^n}$. Now

$$\deg P^*_{2p^n}(x) = \frac{\varphi(2^2.p^n)}{2} = \varphi(p^n) = p^n - p^{n-1} = p^{n-1}(p-1)$$

and

$$\deg \frac{P_q(x)}{P_{2p^{n-1}}(x)} = \frac{2p^n}{2} - \frac{2p^{n-1}}{2} = p^n - p^{n-1}.$$

As the $P_q(x)$ polynomials are monic, both sides must be equal.

We now state this result in an alternative way for our purpose of generalisation:

Theorem 6 *If $q = 2.p^n$ then*

$$P^*_q(x) = \frac{P_q(x)}{P^*_2(x)P^*_{2p}(x)...P^*_{2p^{n-2}}(x)P^*_{2p^{n-1}}(x)},$$

i.e. $P^*_q(x)$ is obtained by dividing $P_q(x)$ by all $P^*_d(x)$ such that q/d is odd.

Proof By the last result

$$P^*_2(x)P^*_{2p}(x)...P^*_{2p^{n-2}}(x)P^*_{2p^{n-1}}(x) = P_2(x)\frac{P_{2p}(x)}{P_2(x)}...\frac{P_{2p^{n-2}}(x)}{P_{2p^{n-3}}(x)}\frac{P_{2p^{n-1}}(x)}{P_{2p^{n-2}}(x)} = P_{2p^{n-1}}(x).$$

Theorem 7 *If $q = 2^m p$ then $P^*_q(x) = \frac{P_q(x)}{P_{2^m}(x)}$.*

Proof Again $W_{2^m} \subset W_{2^m p}$. As all polynomials are monic we only need to show that the degrees of the left and right hand sides are equal establishing the equality.

$$\deg P^*_q(x) = \frac{\varphi(2^{m+1}p)}{2} = 2^{m-1}(p-1)$$

and

$$\deg \frac{P_q(x)}{P_{2^m}(x)} = \frac{2^m p}{2} - \frac{2^m}{2} = 2^{m-1}(p-1).$$

Theorem 8 *If $q = 2^m p^n$ then $P^*_q(x) = \frac{P_q(x)}{P_{2^m p^{n-1}}(x)}$.*

Proof As $2^m p^{n-1} \mid 2^m p^n$, $W_{2^m p^{n-1}} \subset W_{2^m p^n}$. Now

$$\deg P^*_{2^m p^n}(x) = \frac{\varphi(2^{m+1}p^n)}{2} = 2^{m-1}(p^n - p^{n-1})$$

and

$$\deg \frac{P_q(x)}{P_{2^m p^{n-1}}(x)} = \frac{2^m p^n}{2} - \frac{2^m p^{n-1}}{2} = 2^{m-1}(p^n - p^{n-1}).$$

As all polynomials are monic, both sides are equal.

Alternatively

Theorem 9 *If $q = 2^m p^n$ then*

$$P^*_q(x) = \frac{P_q(x)}{P^*_{2^m}(x)P^*_{2^m p}(x)...P^*_{2^m p^{n-1}}(x)}, \text{ i.e.}$$

$P^*_q(x)$ is obtained by dividing $P_q(x)$ by all $P_d(x)$ such that q/d is odd.

Proof By the last result

$$P^*_{2^m}(x).P^*_{2^m p}(x)...P^*_{2^m p^{n-1}}(x) = P_{2^m}(x)\frac{P_{2^m p}(x)}{P_{2^m}(x)}...\frac{P_{2^m p^{n-1}}(x)}{P_{2^m p^{n-2}}(x)}$$

i.e. $P^*_q(x) = \frac{P_q(x)}{P_{2^m p^{n-1}}(x)}$ which is true by the last result.

Let us now obtain the generalisation of all these results. Recall that for even q, $P_d|P_q$ if q/d is odd. Considering this, we conclude that we must divide P_q by all P_d so that q/d is odd. Therefore

Theorem 10 *Let q be even. Then the minimal polynomial $P^*_q(x)$ of λ_q over $\mathbf{Q}$ is given by*

$$P^*_q(x) = \frac{P_q(x)}{\prod\limits_{d|q,q/d \ \ odd} P^*_d(x)}.$$

Acknowledgement The fourth author is supported by the Commission of Scientific Research Projects of Uludag University, Project numbers 2006/40, 2008/31 and 2008/54.

REFERENCES

1. I.N. Cangül. *Normal Subgroups of Hecke Groups.* PhD Thesis, Southampton, 1994.
2. I.N. Cangül. *The Minimal Polynomials of* $\cos(2\pi/n)$ *over* $\mathbf{Q}$. Problemy Matematyczne, **15**(1994), 57–62.
3. I.N. Cangül. *Normal Subgroups of Hecke Groups and Regular Maps.* Math. Proc. Camb. Phil. Soc., **123**(1998), 59–74