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CHEN INEQUALITIES FOR SUBMANIFOLDS OF COMPLEX SPACE FORMS AND SASAKIAN SPACE FORMS ENDOWED WITH SEMI-SYMMETRIC METRIC CONNECTIONS

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ABSTRACT. In this paper we prove Chen inequalities for submanifolds of complex space forms and, respectively, Sasakian space forms, endowed with semi-symmetric metric connections, i.e., relations between the mean curvature associated with the semi-symmetric metric connection, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

1. Introduction. In [**10**], Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Yano studied in [**18**] some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [**11, 12**], Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Nakao [**16**] studied submanifolds of a Riemannian manifold with semi-symmetric connections.

On the other hand, one of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. Chen [**5–9**] established inequalities in this respect, well-known as *Chen inequalities*.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example, see [**2–4, 13, 14, 17**].

Recently, in [**15**] the present authors proved Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection.

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As a natural prolongation of our research, in this paper we will study Chen inequalities for submanifolds in complex, respectively Sasakian space forms, endowed with semi-symmetric metric connections.

2. Preliminaries. Semi-symmetric metric connection. Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \omega(\widetilde{Y})\widetilde{X} - \omega(\widetilde{X})\widetilde{Y}$$

for a 1-form ω, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection*.

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla} g = 0$, then $\widetilde{\nabla}$ is called a semi-symmetric metric connection on N^{n+p}.

Following [**18**], a semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \omega(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})U,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and U is a vector field defined by $g(U,\widetilde{X}) = \omega(\widetilde{X})$, for any vector field $\widetilde{X}$.

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\overset{\circ}{\nabla}$ can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \chi(M),$$
$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \quad X, Y \in \chi(M),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [**16**], h is also symmetric. The Gauss equation for the submanifold M^n into an $(n+p)$-dimensional Riemannian manifold N^{n+p} is

$$\begin{aligned}(2.1)\quad &\overset{\circ}{\widetilde{R}}(X,Y,Z,W)\\ &= \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z), \overset{\circ}{h}(Y,W)) - g(\overset{\circ}{h}(X,W), \overset{\circ}{h}(Y,Z)).\end{aligned}$$

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p}. The curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} can be written as (see [**12**])

$$\begin{aligned}(2.2)\quad &\widetilde{R}(X,Y,Z,W)\\ &= \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W)\\ &\quad - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a (0, 2)-tensor field defined by

$$\alpha(X,Y) = (\overset{\circ}{\widetilde{\nabla}}_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(P)g(X,Y), \text{ for all } X, Y \in \chi(M).$$

Denote by λ the trace of α.

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, \dots, e_m\}$ of the tangent space $T_x M^n$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

Recall that the *Chen first invariant* is given by

$$\delta_M(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n,\ x \in M^n,\ \dim\pi = 2\},$$

(see for example [**9**]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n$, $x \in M^n$ and τ is the scalar curvature at x.

The following algebraic lemma is well-known.

Lemma [**5**]. *Let $a_1, a_2, \dots, a_n, b$ be $(n+1)$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$.

One defines [**7**] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$\operatorname{Ric} L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1}\inf_{L,X} \operatorname{Ric}_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

We will recall the definitions of a complex manifold and of a Sasakian manifold, in particular, of a complex space form and a Sasakian space form and fix the notations at the beginning of the corresponding sections. We consider as an ambient space a complex space form endowed with a semi-symmetric metric connection, respectively a Sasakian space form endowed with a semi-symmetric metric connection.

3. Chen first inequality for submanifolds of complex space forms. Let N^{2m} be a Kaehler manifold and J the canonical almost complex structure. The sectional curvature of N^{2m} in the direction of an invariant 2-plane section by J is called the *holomorphic sectional curvature.*

If the holomorphic sectional curvature is constant $4c$ for all plane sections π of T_xN^{2m} invariant by J for any $x \in N^{2m}$, then N^{2m} is called a complex space form and is denoted by $N^{2m}(4c)$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $N^{2m}(4c)$ is given by

$$\text{(C.2.3)}\quad \begin{aligned} &\overset{\circ}{\widetilde{R}}(X,Y,Z,W)\\ &= c[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - g(JX,Z)g(JY,W)\\ &\quad + g(JX,W)g(JY,Z) - 2g(X,JY)g(Z,JW)]. \end{aligned}$$

If $N^{2m}(4c)$ is a complex space form of constant holomorphic sectional curvature $4c$ with a semi-symmetric metric connection $\widetilde{\nabla}$, then from (2.2) and (C.2.3), the curvature tensor $\widetilde{R}$ of $N^{2m}(4c)$ can be expressed as

$$\begin{aligned} &\text{(C.2.4)}\quad \widetilde{R}(X,Y,Z,W)\\ &= c[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - g(JX,Z)g(JY,W)\\ &\quad + g(JX,W)g(JY,Z) - 2g(X,JY)g(Z,JW)] - \alpha(Y,Z)g(X,W)\\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z). \end{aligned}$$

Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$. For any tangent vector field X to M^n, we put

$$JX = PX + FX,$$

where PX and FX are the tangential and normal components of JX, respectively. We define

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

Following [**1**], we denote by $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(Je_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in $[0, 1]$, independent of the choice of e_1, e_2.

We prove the following

Theorem 3.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. We have:*

$$\tau(x) - K(\pi) \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)c - 2\lambda\right] - [6\Theta^2(\pi) - 3\|P\|^2]\frac{c}{2} - \operatorname{trace}(\alpha|_{\pi^\perp}),$$

where π is a 2-plane section of T_xM^n, $x \in M^n$ and $\lambda = \operatorname{trace}\alpha$.

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{n+p}\}$ be orthonormal bases of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from equation (C.2.4) it follows that:

$$\widetilde{R}(e_i, e_j, e_j, e_i) = c[1 + 3g^2(Je_i, e_j)] - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.1}$$

From (3.1) and the Gauss equation with respect to the semi-symmetric metric connection, we get

$$\begin{aligned} &c[1 + 3g^2(Je_i, e_j)] - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ &\quad = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)). \end{aligned}$$

By summation after $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2\|H\|^2 = c\left[n^2 - n + 3\sum_{i,j=1}^{n} g^2(Je_i, e_j)\right] - 2(n-1)\lambda, \tag{3.2}$$

where

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)),$$

$$H = \frac{1}{n}\operatorname{trace} h.$$

We take

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + 2(n-1)\lambda - (n^2 - n + 3\|P\|^2)c. \tag{3.3}$$

Then, from (3.2) and (3.3) we get

$$n^2\|H\|^2 = (n-1)(\|h\|^2 + \varepsilon).$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = H/\|H\|$ and from relation (3.3) we obtain:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i,j=1}^{n}\sum_{r=n+1}^{n+p}(h_{ij}^r)^2 + \varepsilon\right],$$

or equivalently,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon\right].$$

By using the algebraic Lemma (see Section 2), we have from the previous relation

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon.$$

The Gauss equation for $X = Z = e_1$, $Y = W = e_2$ gives

$$\begin{aligned} K(\pi) &= R(e_1, e_2, e_2, e_1) \\ &= c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ &\quad + \sum_{r=n+1}^{p}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\ &\geq c[1 + 3g^2(Je_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ &\quad + \frac{1}{2}\left[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m}(h_{ij}^r)^2 + \varepsilon\right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m} (h_{12}^r)^2 \\
&= c[1+3g^2(Je_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) \\
&+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m}(h_{ij}^r)^2 \\
&+\frac{1}{2}\varepsilon + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m}(h_{12}^r)^2 \\
&= c[1+3g^2(Je_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) \\
&+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=2m}^{2m}\sum_{i,j>2}(h_{ij}^r)^2 \\
&+\frac{1}{2}\sum_{r=2m}^{2m}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \\
&\geq c[1+3g^2(Je_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{\varepsilon}{2},
\end{aligned}
$$

which implies

$$K(\pi) \geq c[1+3g^2(Je_1,e_2)] - \alpha(e_1,e_1) - \alpha(e_2,e_2) + \frac{\varepsilon}{2}.$$

It follows that

$$
\begin{aligned}
K(\pi) \geq -\frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)c - 2\lambda\right] \\
+ [6\Theta^2(\pi) - 3\|P\|^2]\frac{c}{2} + \operatorname{trace}(\alpha|_{\pi^\perp}),
\end{aligned}
$$

which represents the inequality to prove.

Recall the following important result (Proposition 1.2) from [**11**].

Proposition 3.2. *The mean curvature H of M^n with respect to the semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field U is tangent to M^n.*

Remark. According to formula (7) from [**16**] (see also Proposition 3.2), it follows that $h = \overset{\circ}{h}$ if U is tangent to M^n. In this case the inequality proved in Theorem 3.1 becomes

$$\tau(x) - K(\pi) \le \frac{n-2}{2}\left[\frac{n^2}{n-1}\left\|\overset{\circ}{H}\right\|^2 + (n+1)c - 2\lambda\right] - [6\Theta^2(\pi) - 3\|P\|^2]\frac{c}{2} - \text{trace}\,(\alpha|_{\pi^\perp}),$$

Theorem 3.3. *Under the same assumptions as in Theorem* 3.1*, if the vector field U is tangent to M^n, then the equality case of inequality from Theorem* 3.1 *holds at a point $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1, e_2, \dots e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{2m}(4c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \le i \le 2m,$$

where $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \le i,\ j \le n$ and $n+2 \le r \le 2m$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \text{for all } i \ne j, i, j > 2,$$
$$h_{ij}^r = 0, \quad \text{for all } i \ne j, i, j > 2,\ r = n+1, \dots, 2m,$$
$$h_{11}^r + h_{22}^r = 0, \quad \text{for all } r = n+2, \dots, 2m,$$
$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \text{for all } j > 2,$$
$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

We may choose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$, and we denote by $a = h_{11}^r$, $b = h_{22}^r$, $\mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$. It follows that the shape operators take the desired forms.

4. Ricci curvature for submanifolds of complex space forms. In this section we prove relationships between the Ricci curvature of a submanifold M^n of a complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature, endowed with a semi-symmetric metric connection, and the squared mean curvature $\|H\|^2$. We suppose that the vector field U is tangent to M^n.

Theorem 4.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$ endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M^n. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - c - \frac{3c}{n(n-1)}\|P\|^2. \tag{4.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis of T_xM^n. The relation (3.2) is equivalent with

$$n^2\|H\|^2 = 2\tau + \|h\|^2 + 2(n-1)\lambda - c[n^2 - n + 3\|P\|^2]. \tag{4.2}$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, \dots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_{e_r} = (h^r_{ij}), \quad i,j = 1, \dots, n; \quad r = n+2, \dots, 2m, \ \operatorname{trace} A_{e_r} = 0.$$

From (4.2), we get

$$\begin{aligned} &n^2\|H\|^2 \\ &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h^r_{ij})^2 + 2(n-1)\lambda - c[n^2 - n + 3\|P\|^2]. \end{aligned} \tag{4.3}$$

On the other hand, since

$$0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2. \tag{4.4}$$

We have from (4.3)

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 + 2(n-1)\lambda - c[n^2 - n + 3\|P\|^2],$$

i.e., (4.1).

Using Theorem 4.1, we obtain the following

Theorem 4.2. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional complex space form $N^{2m}(4c)$ of constant holomorphic sectional curvature $4c$ endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, such that the vector field U is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(x) \geq \Theta_k(p) + \frac{2}{n}\lambda - c - \frac{3c}{n(n-1)}\|P\|^2. \tag{4.5}$$

Proof. Let $\{e_1, \dots e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1\dots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\dots,i_k\}} \mathrm{Ric}_{L_{i_1\dots i_k}}(e_i), \tag{4.6}$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1\dots i_k}). \tag{4.7}$$

From (4.1), (4.6) and (4.7), one derives

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(x), \tag{4.8}$$

which implies (4.5).

5. Chen first inequality for submanifolds of Sasakian space forms. A $(2m+1)$-dimensional Riemannian manifold (N^{2m+1}, g) has an almost contact metric structure if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying:

$$\varphi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(X,\xi) = \eta(X),$$

for any vector fields X, Y on TN. Let Φ denote the fundamental 2-form in N^{2m+1}, given by $\Phi(X,Y) = g(X, \varphi Y)$, for all X, Y on TN. If $\Phi = d\eta$, then N^{2m+1} is called a *contact metric manifold*. The structure of N^{2m+1} is called *normal* if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A *Sasakian manifold* is a normal contact metric manifold.

A plane section π in T_pN^{2m+1} is called a *φ-section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a φ-section is called a *φ-sectional curvature*. A Sasakian manifold with constant φ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $N^{2m+1}(c)$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $N^{2m+1}(c)$ is expressed by

(S.2.5)
$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \frac{c+3}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] + \frac{c-1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) + g(X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W) + 2g(X,\varphi Y)g(\varphi Z,W)],$$

for vector fields X, Y, Z, W on $N^{2m+1}(c)$.

If $N^{2m+1}(c)$ is a $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, from (2.2) and (S.2.5) it follows that the curvature tensor $\widetilde{\nabla}$ of $N^{2m+1}(c)$ can be expressed as

(S.2.6)
$$\widetilde{R}(X,Y,Z,W) = \frac{c+3}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] + \frac{c-1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) + g(X,\varphi Z)g(\varphi Y,W)g(Y,\varphi Z)g(\varphi X,W) + 2g(X,\varphi Y)g(\varphi Z,W)] - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).$$

Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature $N^{n+p}(c)$ of constant sectional curvature c. For any tangent vector field X to M^n, we put

$$\varphi X = PX + FX,$$

where PX and FX are tangential and normal components of φX, respectively, and we decompose

$$\xi = \xi^{\top} + \xi^{\perp},$$

where $\xi^{\top}$ and $\xi^{\perp}$ denotes the tangential and normal parts of ξ.

Recall that $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π, is a real number in $[0, 1]$, independent of the choice of e_1, e_2.

For submanifolds of Sasakian space forms endowed with a semi-symmetric metric connection we establish the following optimal inequality.

Theorem 5.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. We have*:

$$(5.1)\quad \begin{aligned}\tau(x) - K(\pi)& \\ &\leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right]\\ &\quad + \frac{c-1}{8}[3\|P\|^2 - 6\Theta^2(\pi) - 2(n-1)\|\xi^{\top}\|^2 + 2\|\xi_{\pi}\|^2]\\ &\quad - \operatorname{trace}(\alpha|_{\pi^{\perp}}),\end{aligned}$$

where π is a 2-plane section of T_xM^n, $x \in M^n$.

Proof. From [**16**], the Gauss equation with respect to the semi-symmetric metric connection is

$$(5.2)\quad \begin{aligned}&\widetilde{R}(X, Y, Z, W)\\ &= R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W)).\end{aligned}$$

Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2m+1}\}$ be orthonormal bases of T_xM^n and $T_x^{\perp}M^n$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from equation (S.2.6) it follows that:
(5.3)

$$\begin{aligned}\widetilde{R}(e_i, e_j, e_j, e_i) = \frac{c+3}{4} + \frac{c-1}{4}[-\eta(e_i)^2 - \eta(e_j)^2 + 3g^2(Pe_j, e_i)]\\ - \alpha(e_i, e_i) - \alpha(e_j, e_j).\end{aligned}$$

From (5.2) and (5.3) we get

$$\begin{aligned}\frac{c+3}{4}+\frac{c-1}{4}[-\eta(e_i)^2-\eta(e_j)^2+3g^2(Pe_j,e_i)]&\\ -\alpha(e_i,e_i)-\alpha(e_j,e_j)&\\ =R(e_i,e_j,e_j,e_i)+g(h(e_i,e_j),h(e_i,e_j))&\\ -g(h(e_i,e_i),h(e_j,e_j)).&\end{aligned}$$

By summation after $1 \leq i,\, j \leq n$, it follows from the previous relation that

$$\begin{aligned}2\tau+\|h\|^2-n^2\|H\|^2 &= 2(n-1)\lambda+(n^2-n)\frac{c+3}{4}\\ &\quad+\frac{c-1}{4}[-2(n-1)\|\xi^\top\|^2+3\|P\|^2].\end{aligned}\tag{5.4}$$

We take

$$\begin{aligned}\varepsilon = 2\tau-\frac{n^2(n-2)}{n-1}\lambda-(n^2-n)\frac{c+3}{4}&\\ -\frac{c-1}{4}[-2(n-1)\|\xi^\top\|^2+3\|P\|^2].&\end{aligned}\tag{5.5}$$

Then, from (5.4) and (5.5), we get

$$n^2\|H\|^2=(n-1)(\|h\|^2+\varepsilon).\tag{5.6}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim\pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = H/\|H\|$, and from the relation (5.6) we obtain:

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2=(n-1)\left(\sum_{i,j=1}^n\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2+\varepsilon\right),$$

or equivalently,

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2=(n-1)\left[\sum_{i=1}^n(h_{ii}^{n+1})^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{i,j=1}^n\sum_{r=n+2}^{2Xm+1}(h_{ij}^r)^2+\varepsilon\right].$$

By using the algebraic Lemma we have from the previous relation,

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon.$$

If we denote by $\xi_\pi = pr_\pi \xi$ we can write

$$-\eta(e_1)^2 - \eta(e_2)^2 = -\|\xi_\pi\|^2.$$

The Gauss equation for $X = Z = e_1$, $Y = W = e_2$ gives

$$\begin{aligned}
K(\pi) &= R(e_1, e_2, e_2, e_1)\\
&= \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2)\\
&\quad + \sum_{r=n+1}^{p}[h_{11}^r h_{22}^r - (h_{12}^r)^2]\\
&\geq \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2)\\
&\quad + \frac{1}{2}\Big[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon\Big]\\
&\quad + \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2\\
&= \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2)\\
&\quad + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2\\
&\quad + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m:1}(h_{12}^r)^2\\
&= \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)] - \alpha(e_1, e_1) - \alpha(e_2, e_2)\\
&\quad + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2\\
&\quad + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon
\end{aligned}$$

$$\geq \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)] - \alpha(e_1, e_1)$$
$$- \alpha(e_2, e_2) + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq \frac{c+3}{4} + \frac{c-1}{4}[-\|\xi_\pi\|^2 + 3g^2(Pe_1, e_2)]$$
$$- \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{\varepsilon}{2}.$$

From (5.5) it follows

$$K(\pi) \geq \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right]$$
$$- \frac{c-1}{8}[3\|P\|^2 - 6\Theta^2(\pi) - 2(n-1)\|\xi^\top\|^2 + 2\|\xi_\pi\|^2]$$
$$+ \operatorname{trace}(\alpha|_{\pi^\perp}),$$

which represents the inequality to prove.

Corollary. *Under the same assumptions as in Theorem* 5.1*, if* ξ *is tangent to* M^n*, we have*

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right]$$
$$+ \frac{c-1}{8}[3\|P\|^2 - 6\Theta^2(\pi) - 2(n-1) + 2\|\xi_\pi\|^2]$$
$$- \operatorname{trace}(\alpha|_{\pi^\perp}).$$

If ξ *is normal to* M^n*, we have*

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c+3}{8} - \lambda\right]$$
$$- \operatorname{trace}(\alpha|_{\pi^\perp}).$$

Remark. According to formula (7) from [**16**] (see also Proposition 3.2), it follows that $h = \overset{\circ}{h}$ if U is tangent to M^n. In this case inequality

(5.1) becomes

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{c+3}{8} - \lambda\right] + \frac{c-1}{8}[3\|P\|^2 - 6\Theta^2(\pi) - 2(n-1)\|\xi^\top\|^2 + 2\|\xi_\pi\|^2] - \text{trace}\,(\alpha|_{\pi^\perp}).$$

Theorem 5.2. *If the vector field U is tangent to M^n, then the equality case of inequality* (5.1) *holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, \dots, e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{2m+1}(c)$ at x have the following forms*:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \qquad a+b=\mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m+1,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+2 \leq r \leq 2m+1$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves equality in all the previous inequalities and we have equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \text{for all } i \neq j, i, j > 2,$$
$$h_{ij}^r = 0, \quad \text{for all } i \neq j, i, j > 2, r = n+1, \dots, 2m+1,$$
$$h_{11}^r + h_{22}^r = 0, \quad \text{for all } r = n+2, \dots, 2m+1,$$
$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \text{for all } j > 2,$$
$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

We may choose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$, and we denote by $a = h_{11}^r$, $b = h_{22}^r$, $\mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$. It follows that the shape operators take the desired forms.

6. Ricci curvature for submanifolds of Sasakian space forms. We first state a relationship between the sectional curvature of a submanifold M^n of a Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

In this section we suppose that the vector field U is tangent to M^n.

Theorem 6.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional real space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M^n. Then we have*

$$(6.1)\quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - \frac{c+3}{4} - \frac{c-1}{4n(n-1)}[-2(n-1)\|\xi^\top\|^2 + \|P\|^2].$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ be orthonormal bases of T_xM^n. Relation (5.4) is equivalent to

$$(6.2)\quad n^2\|H\|^2$$
$$= 2\tau + \|h\|^2 + 2(n-1)\lambda - (n^2-n)\frac{c+3}{4} - \frac{c-1}{4}[-2(n-1)\|\xi^\top\|^2 + 3\|P\|^2].$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, \dots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_{e_r} = (h_{ij}^r), i, j = 1, \dots, n; r = n+2, \dots, 2m+1,\ \operatorname{trace} A_{e_r} = 0.$$

From (6.2), we get

(6.3)
$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + 2(n-1)\lambda - (n^2-n)\frac{c+3}{4} - \frac{c-1}{4}[-2(n-1)\|\xi^\top\|^2 + 3\|P\|^2],$$

which implies

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 + 2(n-1)\lambda - (n^2-n)\frac{c+3}{4} - \frac{c-1}{4}[-2(n-1)\|\xi^\top\|^2 + \|P\|^2],$$

because $\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2$ (see (4.4)).

The last inequality represents (6.1).

Using Theorem 6.1, we obtain the following

Theorem 6.2. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional Sasakian space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, such that the vector field U is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

(6.4)
$$\|H\|^2(x) \geq \Theta_k(x) + \frac{2}{n}\lambda - \frac{c+3}{4} - \frac{c-1}{4n(n-1)}[-2(n-1)\|\xi^\top\|^2 + \|P\|^2].$$

Proof. It follows immediately from (6.1) and (4.8). ◻

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