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B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature

Cihan Özgür

Abstract

In this paper, we prove B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature, i.e., relations between the mean curvature, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

Key Words: Riemannian manifold of quasi-constant curvature, B. Y. Chen inequality, Ricci curvature

1. Introduction

In [11], B. Y. Chen and K. Yano introduced the notion of a Riemannian manifold (M, g) of quasi-constant curvature as a Riemannian manifold with the curvature tensor satisfying the condition

$$\begin{aligned} R(X,Y,Z,W) = a\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right] + \\ +b\left[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) + \right. \\ \left. g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)\right], \end{aligned} \tag{1.1}$$

where a, b are scalar functions and T is a 1-form defined by

$$g(X,P) = T(X), \tag{1.2}$$

and P is a unit vector field. It can be easily seen that, if the curvature tensor R is of the form (1.1), then the manifold is conformally flat. If $b = 0$ then the manifold reduces to a space of constant curvature.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold [4] if its Ricci tensor satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where a, b are scalar functions such that $b \neq 0$ and A is a non-zero 1-form such that $g(X,U) = A(X)$ for every vector field X and U is a unit vector field. If $b = 0$ then the manifold reduces to an Einstein manifold. It can be easily seen that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

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One of the basic problems in submanifold theory is to find simple relations between the extrinsic and intrinsic invariants of a submanifold. In [6], [7], [9] and [10], B. Y. Chen established some inequalities in this respect. They are called B. Y. Chen inequalities.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [1]–[3], [12] and [13].

Motivated by the studies of the above authors, in the present paper, we study B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature.

2. Preliminaries

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold N^{n+m}. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$. The Gauss equation is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)) \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where R is the curvature tensor of M.

The mean curvature vector H is given by $H = \frac{1}{n}\operatorname{trace}(h)$. The submanifold M is *totally geodesic* in N^{m+n} if $h = 0$, and *minimal* if $H = 0$ [5].

Using (1.1), the Gauss equation for the submanifold M^n of a Riemannian manifold of quasi-constant curvature is

$$\begin{aligned} R(X,Y,Z,W) &= a\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right] + \\ &+ b\left[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) + \right. \\ &\left. g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)\right] + \\ &+ g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)). \end{aligned} \tag{2.2}$$

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n. For any orthonormal basis $\{e_1, ..., e_m\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We recall the following algebraic Lemma:

Lemma 2.1 *[6] Let $a_1, a_2, ..., a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, *with equality holding if and only if* $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

Ones define [8] the *Ricci curvature* (or k-*Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

Decomposing the vector field P on M uniquely into its tangent and normal components P^T and $P^\perp$, respectively, we have

$$P = P^T + P^\perp. \tag{2.3}$$

3. Chen First Inequality

Recall that the *Chen first invariant* is given by

$$\delta_{M^n}(x) = \tau(x) - \inf \{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(see for example [10]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

Let us define

$$P_\pi = pr_\pi P, \tag{3.1}$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

For submanifolds of a Riemannian manifold of quasi-constant curvature we establish the following optimal inequality, which will call *Chen first inequality.*

Theorem 3.1 *Let* $M^n, n \geq 3$, *be an* n*-dimensional submanifold of an* $(n+m)$*-dimensional Riemannian manifold of quasi-constant curvature* N^{n+m}. *Then we have*

$$\delta_{M^n}(x) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{a}{2}\right] \tag{3.2}$$

$$+b\left[(n-1)\left\|P^T\right\|^2 - \|P_\pi\|^2\right],$$

where π is a 2-plane section of $T_xM^n, x \in M^n$. The equality case of inequality (3.2) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, ..., e_{n+m}\}$ of $T_x^{\perp}M^n$ such that the shape operators of M^n in N^{n+m} at x have the forms

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq m,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n + 1 \leq r \leq n + m$.

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{n+m}\}$ be orthonormal basis of T_xM^n and $T_x^{\perp}M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equations (2.2), (2.3) and (1.2) it follows that

$$a + b\left[g\left(P^T, e_j\right)^2 + g\left(P^T, e_i\right)^2\right] = R(e_i, e_j, e_j, e_i) +$$

$$+g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

By summation after $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2\|H\|^2 = 2b(n-1)\left\|P^T\right\|^2 + (n^2 - n)a, \tag{3.3}$$

where we denote by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - (n^2 - n)a - 2b(n-1)\left\|P^T\right\|^2. \tag{3.4}$$

Then, from (3.3) and (3.4) we get

$$n^2\|H\|^2 = (n-1)\left(\|h\|^2 + \varepsilon\right). \tag{3.5}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.5) we obtain

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i,j=1}^{n} \sum_{r=n+1}^{n+m} (h_{ij}^r)^2 + \varepsilon\right),$$

or equivalently,

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \\ + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^{r})^2 + \varepsilon\}. \tag{3.6}$$

By using Lemma 2.1 we have from (3.6),

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^{r})^2 + \varepsilon. \tag{3.7}$$

Gauss equation for $X = W = e_1, Y = Z = e_2$ gives

$$K(\pi) = R(e_1,e_2,e_2,e_1) = a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] + \sum_{r=n+1}^{m}[h_{11}^{r}h_{22}^{r} - (h_{12}^{r})^2] \geq$$

$$\geq a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] + \frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^{r})^2 + \varepsilon] +$$

$$+ \sum_{r=n+2}^{n+m} h_{11}^{r}h_{22}^{r} - \sum_{r=n+1}^{n+m}(h_{12}^{r})^2 = a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] +$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^{r})^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{n+m} h_{11}^{r}h_{22}^{r} - \sum_{r=n+1}^{n+m}(h_{12}^{r})^2 =$$

$$= a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+m}\sum_{i,j>2}(h_{ij}^{r})^2 +$$

$$+\frac{1}{2}\sum_{r=n+2}^{n+m}(h_{11}^{r} + h_{22}^{r})^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \geq$$

$$\geq a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq a + b\left[g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2\right] + \frac{\varepsilon}{2}. \tag{3.8}$$

From (3.1) it follows that

$$g\left(P^T,e_1\right)^2 + g\left(P^T,e_2\right)^2 = \|P_\pi\|^2.$$

Using (3.4) we get from (3.8)

$$K(\pi) \geq \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{a}{2}\right] + b\left[\|P_\pi\|^2 - (n-1)\left\|P^T\right\|^2\right],$$

which represents the inequality to prove.

The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^{r} = 0, \quad \forall i \neq j, i, j > 2, r = n+1, ..., n+m,$$

$$h_{11}^{r} + h_{22}^{r} = 0, \quad \forall r = n+2, ..., n+m,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^{r}, b = h_{22}^{r}, \mu = h_{33}^{n+1} = ... = h_{nn}^{n+1}$. It follows that the shape operators take the desired forms. □

Corollary 3.2 *Under the same assumptions as in Theorem* 3.1*, if P is tangent to M^n, we have*

$$\delta_{M^n}(x) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{a}{2}\right] + b\left[n-1-\|P_\pi\|^2\right].$$

If P is normal to M^n, we have

$$\delta_{M^n}(x) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{a}{2}\right].$$

4. k-Ricci curvature

We first state a relationship between the sectional curvature of a submanifold M^n of a space of quasi-constant curvature and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

Theorem 4.1 *Let $M^n, n \geq 3,$ be an n-dimensional submanifold of an $(n+m)$-dimensional space of quasi-constant curvature N^{n+m}. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n}\left\|P^T\right\|^2. \tag{4.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (3.3) is equivalent with

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - (n^2-n)a - 2b(n-1)\left\|P^T\right\|^2. \tag{4.2}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+m}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{4.3}$$

$$A_{e_r} = (h_{ij}^r),\ i, j = 1, ..., n; r = n+2, ..., n+m, \mathrm{trace}\, A_r = 0. \tag{4.4}$$

From (4.2), we get

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \\ &\quad -n(n-1)a - 2b(n-1) \left\|P^T\right\|^2 . \end{aligned} \tag{4.5}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \tag{4.6}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 .$$

We have from (4.5)

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - n(n-1)a - 2b(n-1) \left\|P^T\right\|^2 \tag{4.7}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n} \left\|P^T\right\|^2 ,$$

this proves the theorem. □

Corollary 4.2 *Under the same assumptions as in Theorem* 4.1*, if P is tangent to M^n, we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n}.$$

If P is normal to M^n, we have

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a.$$

Using Theorem 4.1, we obtain the following:

Theorem 4.3 *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold of quasi-constant curvature N^{n+m}. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - a - \frac{2b}{n}\left\|P^T\right\|^2. \tag{4.8}$$

Proof. Let $\{e_1, ...e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1...i_k}) = \frac{1}{2}\sum_{i\in\{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i),$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<...<i_k\leq n} \tau(L_{i_1...i_k}).$$

From (4.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(p),$$

which implies (4.8). □

Corollary 4.4 *Under the same assumptions as in Theorem 4.3, if P is tangent to M^n, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - a - \frac{2b}{n}.$$

If P is normal to M^n, we have

$$\|H\|^2(p) \geq \Theta_k(p) - a.$$

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Cihan ÖZGÜR
University of Balıkesir,
Department of Mathematics,
10145, Cagis, Balıkesir-TURKEY
e-mail: cozgur@balikesir.edu.tr

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