Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth

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Abstract. This work deals with basic approximation problems such as direct, inverse and simultaneous theorems of trigonometric approximation of functions of weighted Lebesgue spaces with a variable exponent on weights satisfying a variable Muckenhoupt $A_{p(\cdot)}$ type condition. Several applications of these results help us transfer the approximation results for weighted variable Smirnov spaces of functions defined on sufficiently smooth finite domains of complex plane $\mathbb{C}$.

Keywords. Fractional derivatives, inverse theorems, Jackson theorems, Lebesgue spaces with a variable exponent, weighted fractional moduli of smoothness.

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1 Introduction and main results

For functions of weighted Lebesgue spaces $L_\omega^{p(\cdot)}$ with nonstandard growth, it was proved in [25] that

$$E_n(f)_{p(\cdot),\omega} \leq c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot),\omega}, \quad n+1,\ r = 1,2,3,\dots, \tag{1.1}$$

and its weak inverse

$$\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{C}{n^{2r}} \sum_{\nu=0}^{n} (\nu+1)^{2r-1} E_\nu(f)_{p(\cdot),\omega}, \quad n, r = 1,2,3,\dots, \tag{1.2}$$

holds provided the weight ω and the exponent $p(\cdot)$ are such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $L_\omega^{p(x)}$, where

$$E_n(f)_{p(\cdot),\omega} := \inf_{T\in\mathcal{T}_n} \|f-T\|_{p(\cdot),\omega}, \quad n = 0,1,2,\dots,\ f \in L_\omega^{p(\cdot)},$$

$\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n,

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{0\le h_i\le\delta} \left\| \prod_{i=1}^{r}(I-\sigma_{h_i})f \right\|_{p(\cdot),\omega},$$

$$f \in L_{\omega}^{p(\cdot)}, \quad \delta \ge 0, \quad r = 1,2,3,\ldots,$$

is the modulus of smoothness of degree r ([16]), I is the identity operator and

$$\sigma_h f(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt \quad \text{for } h \in \mathbb{R} \text{ and } x \in \boldsymbol{T} := [-\pi,\pi].$$

Inequalities (1.1), (1.2) and their several consequences were given in [25]. In the recent papers [5] and [1] we considered the weighted fractional moduli of smoothness, i.e., $\Omega_r(f,\cdot)_{p,\omega}$ with $r \in (0,\infty)$, to obtain inequalities of types (1.1) and (1.2) in weighted Orlicz spaces. Fractional smoothness is not a new concept for nonweighted Lebesgue spaces; Butzer [8], Taberski [45], Tikhonov–Simonov [44] and Akgün–Israfilov [4] applied the fractional moduli of smoothness successfully to solve approximation problems in Lebesgue and Smirnov spaces. As a consequence of these facts, defining the weighted fractional moduli of smoothness ([1]), in this work we consider basic approximation problems such as direct, inverse and simultaneous theorems of trigonometric approximation of functions of weighted Lebesgue spaces with variable exponent for weights satisfying a variable Muckenhoupt condition $A_{p(\cdot)}$. Several applications of these results help us to transfer approximation results for weighted Smirnov spaces of functions defined on a finite domain with sufficiently smooth boundary.

Generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent (with nonstandard growth) appeared first in [36] as an example of modular spaces ([17, 35]), and the corresponding Sobolev type spaces have extensive applications in fluid mechanics, differential operators ([12, 38]), elasticity theory, nonlinear Dirichlet boundary value problems ([34]), nonstandard growth and variational calculus ([40]). If $p^*(\boldsymbol{T}) := \operatorname{ess\,sup}_{x\in\boldsymbol{T}} p(x) < \infty$, then $L^{p(\cdot)}$ is a particular case of Musielak–Orlicz spaces [35]. For a constant $p(x) := p$, $1 < p < \infty$, the corresponding generalized Lebesgue spaces $L^{p(\cdot)}$ with nonstandard growth become classical Lebesgue spaces L^p having deep approximation results. The main properties of $L^{p(\cdot)}$ are investigated in [42], [34], [39] and [14]. The boundedness of classical integral transforms on $L^{p(x)}$ and weighted $L^{p(x)}$ is obtained in [32], [40], [10] and [43].

Let $\mathcal{P}(\boldsymbol{T})$ be the class of Lebesgue measurable functions $p = p(x) : \boldsymbol{T} \to (1,\infty)$ such that

$$1 < p_*(\boldsymbol{T}) := \operatorname{ess\,inf}_{x\in\boldsymbol{T}} p(x) \le p^* < \infty.$$

We define a class $L_{2\pi}^{p(\cdot)}$ of 2π-periodic measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying

$$\int_{-\pi+c}^{\pi+c} |f(x)|^{p(x)}\,dx < \infty$$

for any real number c and $p \in \mathcal{P}(\boldsymbol{T})$.

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space ([34]) with any of the following equivalent norms:

$$\|f\|_{\boldsymbol{T},p(\cdot)} := \inf_{\alpha>0}\left\{\int_{\boldsymbol{T}}\left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \le 1\right\}$$

and

$$\|f\|^*_{\boldsymbol{T},p(\cdot)} := \sup_{g\in L_{2\pi}^{p'(\cdot)}}\left\{\int_{\boldsymbol{T}}|f(x)g(x)|\,dx : \int_{\boldsymbol{T}}|g(x)|^{p'(x)}\,dx \le 1\right\}, \tag{1.3}$$

where $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$.

Let $\omega : \boldsymbol{T} \to [0,\infty]$ be a 2π periodic weight, i.e., a Lebesgue measurable and a.e. positive function. Denote by $L_\omega^{p(\cdot)}$ the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying $\omega f \in L_{2\pi}^{p(\cdot)}$. Weighted Lebesgue spaces with nonstandard growth $L_\omega^{p(\cdot)}$ are Banach spaces with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{\boldsymbol{T},p(\cdot)}$.

For given $p \in \mathcal{P}(\boldsymbol{T})$ the class of weights ω satisfying the condition ([11])

$$\left\|\omega^{p(x)}\right\|_{A_{p(\cdot)}} := \sup_{B\in\mathcal{B}}\frac{1}{|B|^{p_B}}\left\|\omega^{p(x)}\right\|_{L^1(B)}\left\|\frac{1}{\omega^{p(x)}}\right\|_{B,(p'(\cdot)/p(\cdot))} < \infty$$

is denoted by $A_{p(\cdot)}(\boldsymbol{T})$. Here $p_B := (\frac{1}{|B|}\int_B \frac{1}{p(x)}dx)^{-1}$ and $\mathcal{B}$ is the class of all balls in $\boldsymbol{T}$.

The variable exponent $p(x)$ is said to satisfy the *local log-Hölder continuity condition* if

$$|p(x_1)-p(x_2)| \le \frac{c}{\log(e+1/|x_1-x_2|)} \quad \text{for all } x_1,x_2\in\boldsymbol{T}. \tag{1.4}$$

We denote by $\mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$ the class of $p \in \mathcal{P}(\boldsymbol{T})$ satisfying (1.4).

Let $f \in L_\omega^{p(\cdot)}$ and

$$\mathcal{A}_h f(x) := \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)dt, \quad x\in\boldsymbol{T},$$

be Steklov's mean operator. For $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$ and $f \in L_\omega^{p(\cdot)}$, it was proved in [11] that

$$\begin{array}{l}\text{The Hardy–Littlewood maximal function } \mathcal{M} \text{ is bounded in } L_\omega^{p(\cdot)}\\ \text{if and only if } \omega\in A_{p(\cdot)}(\boldsymbol{T}).\end{array} \tag{1.5}$$

Therefore if $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, then $\mathcal{A}_h$ is bounded in $L_{\omega}^{p(\cdot)}$. Using these facts and setting $x, h \in \boldsymbol{T}$, $0 \leq r$ we define via binomial expansion, for $f \in L_{\omega}^{p(\cdot)}$,

$$\sigma_h^r f(x) := (I - \mathcal{A}_h)^r f(x)$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f(x + u_1 + \cdots + u_k) du_1 \cdots du_k.$$

Since the binomial coefficients $\binom{r}{k}$ satisfy ([41, p. 14])

$$\binom{r}{k} \leq \frac{c(r)}{k^{r+1}}, \quad k \in \mathbb{Z}^+,$$

we get

$$\sum_{k=0}^{\infty} \binom{r}{k} < \infty$$

and therefore

$$\left\| \sigma_h^r f \right\|_{p(\cdot),\omega} \leq c \left\| f \right\|_{p(\cdot),\omega} < \infty \tag{1.6}$$

provided $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$ and $f \in L_{\omega}^{p(\cdot)}$.

For $0 \leq r$ we can now define ([48]) the *fractional moduli of smoothness of the index r* for $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$ and $f \in L_{\omega}^{p(\cdot)}$ as

$$\Omega_r(f, \delta)_{p(\cdot),\omega} := \sup_{0 < h_i, t \leq \delta} \left\| \prod_{i=1}^{[r]} (I - \mathcal{A}_{h_i}) \sigma_t^{r-[r]} f \right\|_{p(\cdot),\omega}, \quad \delta \geq 0,$$

where $\Omega_0(f, \delta)_{p(\cdot),\omega} := \| f \|_{p(\cdot),\omega}$, $\prod_{i=1}^{0} (I - \mathcal{A}_{h_i}) \sigma_t^r f := \sigma_t^r f$ for $0 < r < 1$, and $[r]$ denotes the integer part of the nonnegative real number r.

We have by (1.6) that

$$\Omega_r(f, \delta)_{p(\cdot),\omega} \leq c \, \| f \|_{p(\cdot),\omega}$$

where $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, $f \in L_{\omega}^{p(\cdot)}$ and the constant $c > 0$ depends only on r and p.

Remark 1.1. The modulus of smoothness $\Omega_r(f, \delta)_{p(\cdot),\omega}$, $r \in \mathbb{R}^+$, has the following properties for $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$ and $f \in L_{\omega}^{p(\cdot)}$:

(i) $\Omega_r(f, \delta)_{p(\cdot),\omega}$ is a nonnegative and nondecreasing function of $\delta \geq 0$,

(ii) $\Omega_r(f_1 + f_2, \cdot)_{p(\cdot),\omega} \leq \Omega_r(f_1, \cdot)_{p(\cdot),\omega} + \Omega_r(f_2, \cdot)_{p(\cdot),\omega}$,

(iii) $\lim_{\delta \to 0+} \Omega_r(f, \delta)_{p(\cdot),\omega} = 0$.

If $p \in \mathcal{P}^{\log}_{\pm}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, then $\omega^{p(x)} \in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense in $L^{p(\cdot)}_{\omega}$ ([32]). Therefore approximation problems make sense in $L^{p(\cdot)}_{\omega}$. On the other hand, if $p \in \mathcal{P}^{\log}_{\pm}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}$, then $L^{p(\cdot)}_{\omega} \subset L^1(\boldsymbol{T})$.

For given $f \in L^1(\boldsymbol{T})$, let

$$f(x) \backsim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k(f)e^{ikx} \tag{1.7}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$$

be respectively the *Fourier* and the *conjugate Fourier series* of f. We set

$$L^1_0(\boldsymbol{T}) := \left\{f \in L^1(\boldsymbol{T}) : c_0(f) = 0 \text{ for the series in (1.7)}\right\}.$$

Let $\alpha \in \mathbb{R}^+$ be given. We define the *fractional derivative* of a function $f \in L^1_0(\boldsymbol{T})$ as

$$f^{(\alpha)}(x) := \sum_{k=-\infty}^{\infty} c_k(f)(ik)^{\alpha} e^{ikx}$$

provided the right-hand side, where $(ik)^{\alpha} := |k|^{\alpha} e^{(1/2)\pi i\alpha \operatorname{sign} k}$, exists as principal value. We say that a function $f \in L^{p(\cdot)}_{\omega}$ has the *fractional derivative of degree* $\alpha \in \mathbb{R}^+$ if there exists a function $g \in L^{p(\cdot)}_{\omega}$ such that its Fourier coefficients satisfy $c_k(g) = c_k(f)(ik)^{\alpha}$. In that case, we write $f^{(\alpha)} = g$.

For $p \in \mathcal{P}(\boldsymbol{T})$ and $\alpha > 0$, let $W^{\alpha}_{p(\cdot),\omega}$ be the class of functions $f \in L^{p(\cdot)}_{\omega}$ such that $f^{(\alpha)} \in L^{p(\cdot)}_{\omega}$. Then $W^{\alpha}_{p(\cdot),\omega}$ becomes a Banach space with the norm

$$\|f\|_{W^{\alpha}_{p(\cdot),\omega}} := \|f\|_{p(\cdot),\omega} + \|f^{(\alpha)}\|_{p(\cdot),\omega}.$$

The main results of this work are as follows.

Theorem 1.2. *If* $p \in \mathcal{P}^{\log}_{\pm}(\boldsymbol{T})$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\boldsymbol{T}) \quad \textit{for some } \ p_0 \in (1, p_*(\boldsymbol{T})),$$

$\alpha \in \mathbb{R}^+$ *and* $f \in W^{\alpha}_{p(\cdot),\omega}$, *then for every* $n = 0, 1, 2, 3, \ldots$ *there exists a constant* $c > 0$ *independent of* n *such that*

$$E_n(f)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^{\alpha}} E_n(f^{(\alpha)})_{p(\cdot),\omega}$$

holds.

Corollary 1.3. *Under the conditions of Theorem* 1.2*,*

$$E_n(f)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^\alpha} \left\| f^{(\alpha)} \right\|_{p(\cdot),\omega}$$

with a constant $c > 0$ *independent of* n.

Theorem 1.4. *If* $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$,

$$\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}(\boldsymbol{T}) \quad \text{for some } \; p_0 \in (1, p_*(\boldsymbol{T}))$$

and $f \in L_\omega^{p(\cdot)}$*, then there exists a constant* $c > 0$ *dependent only on* r *and* p *such that*

$$E_n(f)_{p(\cdot),\omega} \leq c\ \Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot),\omega}$$

holds for $r \in \mathbb{R}^+$ *and* $n = 0, 1, 2, 3, \ldots$.

The following inverse theorem of trigonometric approximation holds.

Theorem 1.5. *Under the conditions of Theorem* 1.4*, the inequality*

$$\Omega_r(f, \frac{1}{n+1})_{p(\cdot),\omega} \leq \frac{c}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(f)_{p(\cdot),\omega}$$

holds for $r \in \mathbb{R}^+$ *and* $n = 0, 1, 2, 3, \ldots$*, where the constant* $c > 0$ *depends only on* r *and* p.

Corollary 1.6. *Under the conditions of Theorem* 1.4*, if the condition*

$$E_n(f)_{p(\cdot),\omega} = \mathcal{O}(n^{-\sigma}), \quad n = 1, 2, \ldots,$$

is satisfied for some $\sigma > 0$*, then*

$$\Omega_r(f, \delta)_{p(\cdot),\omega} = \begin{cases} \mathcal{O}(\delta^\sigma), & r > \sigma, \\ \mathcal{O}(\delta^\sigma \left|\log(1/\delta)\right|), & r = \sigma, \\ \mathcal{O}(\delta^r), & r < \sigma, \end{cases}$$

holds for $r \in \mathbb{R}^+$.

Definition 1.7. For $0 < \sigma < r$ we set

$$\operatorname{Lip}\sigma(r, p(\cdot), \omega) := \left\{ f \in L_\omega^{p(\cdot)} : \Omega_r(f, \delta)_{p(\cdot),\omega} = \mathcal{O}(\delta^\sigma), \delta > 0 \right\}.$$

Corollary 1.8. *Under the conditions of Theorem* 1.4*, if* $0 < \sigma < r$ *and*

$$E_n(f)_{p(\cdot),\omega} = \mathcal{O}(n^{-\sigma}) \quad \text{for } n = 1, 2, \ldots,$$

then $f \in \operatorname{Lip}\sigma(r, p(\cdot), \omega)$.

Corollary 1.9. *Under the conditions of Theorem* 1.4*, if* $0 < \sigma < r$*, then the following conditions are equivalent:*

(a) $f \in \operatorname{Lip}\sigma(r, p(\cdot), \omega)$.

(b) $E_n(f)_{p(\cdot),\omega} = \mathcal{O}(n^{-\sigma})$, $n = 1, 2, \ldots$.

Theorem 1.10. *Under the conditions of Theorem* 1.4*, if*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} < \infty$$

for some $\alpha \in (0, \infty)$*, then* $f \in W^{\alpha}_{p(\cdot),\omega}$ *and*

$$E_n(f^{(\alpha)})_{p(\cdot),\omega} \le c\left((n+1)^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega}\right)$$

hold, where the constant $c > 0$ *depends only on* α *and* p.

The latter theorem gives rise to

Corollary 1.11. *Under the conditions of Theorem* 1.4*, if* $r \in (0, \infty)$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} < \infty$$

for some $\alpha > 0$*, then there exists a constant* $c > 0$ *depending only on* α*,* r *and* p *such that*

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n+1}\right)_{p(\cdot),\omega} \le \frac{c}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+r-1} E_\nu(f)_{p(\cdot),\omega} + c \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega}$$

holds.

The following simultaneous approximation theorem is valid.

Theorem 1.12. *If* $p \in \mathcal{P}^{\log}_{\pm}(\boldsymbol{T})$*,*

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\boldsymbol{T}) \quad \text{for some } p_0 \in (1, p_*(\boldsymbol{T})),$$

$\alpha \in [0, \infty)$*, and* $f \in W^{\alpha}_{p(\cdot),\omega}$*, then there exist* $T \in \mathcal{T}_n$*,* $n = 1, 2, 3, \ldots$*, and a constant* $c > 0$ *depending only on* α *and* p *such that*

$$\|f^{(\alpha)} - T^{(\alpha)}\|_{p(\cdot),\omega} \le c E_n(f^{(\alpha)})_{p(\cdot),\omega}$$

holds.

Theorem 1.13. *If* $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\boldsymbol{T}) \quad \textit{for some} \;\; p_0 \in (1, p_*),$$

f belongs to the Hardy space $H^{p(\cdot)}$ with a variable exponent on the unit circumference $\mathbb{D}$ and $r \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that

$$\left\| f(z) - \sum_{k=0}^{n} \eta_k(f) z^k \right\|_{p(\cdot),\omega} \leq c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot),\omega}, \quad n = 0, 1, 2, \ldots,$$

where $\eta_k(f)$, $k = 0, 1, 2, 3, \ldots$, are the Taylor coefficients of f at the origin.

2 Some auxiliary results

We begin with

Lemma A ([27]). *For $\alpha \in \mathbb{R}^+$ we suppose that*

(i) $a_1 + a_2 + \cdots + a_n + \cdots$

and

(ii) $a_1 + 2^\alpha a_2 + \cdots + n^\alpha a_n + \cdots$

are two series in the Banach space $(B, \|\cdot\|)$. Let

$$R_n^{\langle\alpha\rangle} := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^\alpha\right) a_k$$

and

$$R_n^{\langle\alpha\rangle *} := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^\alpha\right) k^\alpha a_k$$

for $n = 1, 2, \ldots$. Then

$$\left\| R_n^{\langle\alpha\rangle *} \right\| \leq c, \quad n = 1, 2, \ldots,$$

for some $c > 0$ if and only if there exists $R \in B$ such that

$$\left\| R_n^{\langle\alpha\rangle} - R \right\| \leq \frac{C}{n^\alpha},$$

where c and C are constants depending only on each other.

Putting $A_k(x) := c_k(f)e^{ikx}$ in (1.7), we define

$$S_n(f) := S_n(x, f) := \sum_{k=0}^{n} (A_k(x) + A_{-k}(x))$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 0, 1, 2, \ldots,$$
$$R_n^{\langle\alpha\rangle}(f, x) := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right)(A_k(x) + A_{-k}(x))$$

and

$$\Theta_m^{\langle\alpha\rangle} := \frac{1}{1 - \left(\frac{m+1}{2m+1}\right)^{\alpha}} R_{2m}^{\langle\alpha\rangle} - \frac{1}{\left(\frac{2m+1}{m+1}\right)^{\alpha} - 1} R_m^{\langle\alpha\rangle} \quad \text{for } m = 1, 2, 3, \ldots. \tag{2.1}$$

Lemma 2.1. *Under the conditions of Theorem* 1.4*, there are constants* $c, C > 0$ *such that*

$$\|\tilde{f}\|_{p(\cdot),\omega} \leq c\,\|f\|_{p(\cdot),\omega} \tag{2.2}$$

and

$$\|S_n(\cdot, f)\|_{p(\cdot),\omega} \leq C\,\|f\|_{p(\cdot),\omega} \quad \textit{for } n = 1, 2, \ldots \tag{2.3}$$

hold.

Proof. Let $S_*(f) := S_*(f, x) := \sup_{k \geq 0} |S_k(f, x)|$. Then using Theorem 4.16 of [33] we obtain

$$\|S_n(\cdot, f)\|_{p(\cdot),\omega} \leq \|S_*(f)\|_{p(\cdot),\omega} \leq C\,\|f\|_{p(\cdot),\omega}\,.$$

For (2.2) we use extrapolation Theorem 3.2 of [33]. For any $p > 1$ we have ([18])

$$\|\tilde{f}\|_{p,\omega} \leq c\,\|f\|_{p,\omega}$$

and [33, Theorem 3.2, (3.3)] is satisfied for $p = p_0 = q_0$ and $q(x) = p(x)$. Therefore (2.2) holds,

$$\|\tilde{f}\|_{p(\cdot),\omega} \leq c\,\|f\|_{p(\cdot),\omega}\,. \qquad \square$$

Remark 2.2. Under the conditions of Theorem 1.4, it can be easily seen from (2.2) and (2.3) that there exists a constant $c > 0$ such that

$$\|f - S_n(\cdot, f)\|_{p(\cdot),\omega} \leq c E_n(f)_{p(\cdot),\omega} \asymp E_n(\tilde{f})_{p(\cdot),\omega}.$$

Under the conditions of Theorem 1.4, using (2.3) and the Abel transform, we get

$$\|R_n^{\langle\alpha\rangle}(f,x)\|_{p(\cdot),\omega} \leq c\,\|f\|_{p(\cdot),\omega} \quad \text{for } n=1,2,3,\ldots,\ x\in \boldsymbol{T},\ f\in L_\omega^{p(\cdot)} \tag{2.4}$$

and therefore (2.1) and (2.4) imply

$$\|\Theta_m^{\langle\alpha\rangle}(f,x)\|_{p(\cdot),\omega} \leq c\,\|f\|_{p(\cdot),\omega} \quad \text{for } m=1,2,3,\ldots,\ x\in \boldsymbol{T},\ f\in L_\omega^{p(\cdot)}.$$

From the property

$$\Theta_m^{\langle\alpha\rangle}(f)(x) = \frac{1}{\sum_{k=m+1}^{2m}\left[(k+1)^\alpha - k^\alpha\right]}\sum_{k=m+1}^{2m}\left[(k+1)^\alpha - k^\alpha\right]S_k(x,f) \quad \text{for } x\in\boldsymbol{T},\ f\in L^1$$

it follows that

$$\Theta_m^{\langle\alpha\rangle}(T_m) = T_m, \tag{2.5}$$

where $T_m \in \mathcal{T}_m$ for $m = 1,2,3,\ldots$.

Lemma 2.3. *Under the conditions of Theorem* 1.4*, if $T_n \in \mathcal{T}_n$ and $\alpha \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that*

$$\|T_n^{(\alpha)}\|_{p(\cdot),\omega} \leq cn^\alpha\,\|T_n\|_{p(\cdot),\omega}$$

holds.

Proof. Without loss of generality one can assume that $\|T_n\|_{p(\cdot),\omega} = 1$. Since

$$T_n = \sum_{k=0}^{n}(A_k(x) + A_{-k}(x)), \qquad \frac{\tilde{T}_n}{n^\alpha} = \sum_{k=1}^{n}\left[(A_k(x) - A_{-k}(x))/n^\alpha\right]$$

and

$$\frac{T_n^{(\alpha)}}{(in)^\alpha} = \sum_{k=1}^{n} k^\alpha\left[(A_k(x) - A_{-k}(x))/n^\alpha\right],$$

we have by (2.4) and (2.2) that

$$\left\|R_m^{\langle\alpha\rangle}\left(\frac{\tilde{T}_n}{n^\alpha}\right)\right\|_{p(\cdot),\omega} \leq \frac{c}{n^\alpha}\|\tilde{T}_n\|_{p(\cdot),\omega} \leq \frac{c}{n^\alpha}\,\|T_n\|_{p(\cdot),\omega} = \frac{c}{n^\alpha}$$

and by Lemma A

$$\left\|R_m^{\langle\alpha\rangle}\left(\frac{T_n^{(\alpha)}}{(in)^\alpha}\right)\right\|_{p(\cdot),\omega} \leq c.$$

Hence by (2.5)

$$\|T_n^{(\alpha)}\|_{p(\cdot),\omega} = n^{\alpha} \left\| \Theta_m^{\langle\alpha\rangle} \left(\frac{T_n^{(\alpha)}}{(in)^{\alpha}} \right) \right\|_{p(\cdot),\omega} \leq cn^{\alpha} \|T_n\|_{p(\cdot),\omega} .$$

A general case follows immediately from this. □

Lemma 2.4. *If $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$ and $f \in W^2_{p(\cdot),\omega}$, then there exists a constant $c > 0$ such that for $r = 1, 2, 3, \ldots$ and $\delta \geq 0$*

$$\Omega_r(f, \delta)_{p(\cdot),\omega} \leq c\delta^2 \Omega_{r-1}(f'', \delta)_{p(\cdot),\omega}$$

holds.

Proof. Putting

$$g(x) := \prod_{i=2}^{r} (I - \mathcal{A}_{h_i}) f(x),$$

we have

$$(I - \mathcal{A}_{h_1}) g(x) = \prod_{i=1}^{r} (I - \mathcal{A}_{h_i}) f(x)$$

and

$$\begin{aligned} \prod_{i=1}^{r} (I - \mathcal{A}_{h_i}) f(x) &= \frac{1}{h_1} \int_{-h_1/2}^{h_1/2} (g(x) - g(x+t)) dt \\ &= -\frac{1}{2h_1} \int_0^{h_1/2} \int_0^{2t} \int_{-u/2}^{u/2} g''(x+s) ds\, du\, dt. \end{aligned}$$

Therefore from (1.3)

$$\begin{aligned} &\left\| \prod_{i=1}^{r} (I - \mathcal{A}_{h_i}) f(x) \right\|_{p(\cdot),\omega} \\ &\quad \leq \frac{c}{2h_1} \sup_{g_0 \in L_{2\pi}^{p'(\cdot)}} \left\{ \int_{\boldsymbol{T}} \left| \int_0^{h_1/2} \int_0^{2t} \int_{-u/2}^{u/2} g''(x+s) ds du dt \right| \omega(x) |g_0(x)| \, dx : \right. \\ &\qquad\qquad \left. \int_{\boldsymbol{T}} |g_0(x)|^{p'(x)} dx \leq 1 \right\} \\ &\quad \leq \frac{c}{2h_1} \int_0^{h_1/2} \int_0^{2t} u \left\| \frac{1}{u} \int_{-u/2}^{u/2} g''(x+s) ds \right\|_{p(\cdot),\omega} du\, dt \\ &\quad \leq \frac{c}{2h_1} \int_0^{h_1/2} \int_0^{2t} u \, \|g''\|_{p(\cdot),\omega} \, du dt = C h_1^2 \, \|g''\|_{p(\cdot),\omega} . \end{aligned}$$

Since

$$g''(x) = \prod_{i=2}^{r} (I - \mathcal{A}_{h_i}) f''(x),$$

we obtain

$$\begin{aligned}
\Omega_r(f,\delta)_{p(\cdot),\omega} &\leq C \sup_{\substack{0<h_i\leq\delta \\ i=1,2,\ldots,r}} h_1^2 \left\| g'' \right\|_{p(\cdot),\omega} \\
&= c\delta^2 \sup_{\substack{0<h_i\leq\delta \\ i=2,3,\ldots,r}} \left\| \prod_{i=2}^{r} (I - \mathcal{A}_{h_i}) f''(x) \right\|_{p(\cdot),\omega} \\
&= c\delta^2 \sup_{\substack{0<h_j\leq\delta \\ j=1,2,\ldots,r-1}} \left\| \prod_{j=1}^{r-1} (I - \mathcal{A}_{h_j}) f''(x) \right\|_{p(\cdot),\omega} \\
&= C\delta^2 \Omega_{r-1}(f'',\delta)_{p(\cdot),\omega}
\end{aligned}$$

and Lemma 2.4 is proved. □

Corollary 2.5. *If* $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, $r = 1, 2, 3, \ldots,$ *and* $f \in W_{p(\cdot),\omega}^{2r}$, *then there exists a constant* $c > 0$ *depending only on* r *and* p *such that*

$$\Omega_r(f,\delta)_{p(\cdot),\omega} \leq c\delta^{2r} \| f^{(2r)} \|_{p(\cdot),\omega}$$

holds for $\delta \geq 0$.

Lemma 2.6. *If* $p \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, $n = 0, 1, 2, \ldots,$ $T_n \in \mathcal{T}_n$ *and* $r \in \mathbb{R}^+$, *then there exists a constant* $c > 0$ *depending only on* r *and* p *such that*

$$\Omega_r\left(T_n, \frac{1}{n+1}\right)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^r} \| T_n^{(r)} \|_{p(\cdot),\omega}$$

holds.

Proof. First we prove that if $0 < \alpha < \beta$, $\alpha, \beta \in \mathbb{R}^+$, then

$$\Omega_\beta(f,\cdot)_{p(\cdot),\omega} \leq c\Omega_\alpha(f,\cdot)_{p(\cdot),\omega}. \tag{2.6}$$

It is easily seen that if $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{Z}^+$, then

$$\Omega_\beta(f,\cdot)_{p(\cdot),\omega} \leq c(\alpha,\beta,p)\Omega_\alpha(f,\cdot)_{p(\cdot),\omega}. \tag{2.7}$$

Now, we assume that $0 < \alpha < \beta < 1$. In that case, putting

$$\Phi(x) := \sigma_h^{\alpha} f(x)$$

we have

$$\begin{aligned}
&\sigma_h^{\beta-\alpha} \Phi(x) \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} \Phi(x+u_1+\cdots+u_j) du_1 \cdots du_j \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} \\
&\qquad\quad \times \Bigg[\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f(x+u_1+\cdots+u_j \\
&\qquad\qquad\qquad + u_{j+1}+\cdots+u_{j+k}) du_1 \cdots du_j du_{j+1} \cdots du_{j+k} \Bigg] \\
&\quad = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-\alpha}{j} \binom{\alpha}{k} \\
&\qquad\quad \times \left[\frac{1}{h^{j+k}} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f(x+u_1+\cdots+u_{j+k}) du_1 \cdots du_{j+k} \right] \\
&\quad = \sum_{\upsilon=0}^{\infty} (-1)^{\upsilon} \binom{\beta}{\upsilon} \frac{1}{h^{\upsilon}} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f(x+u_1+\cdots+u_{\upsilon}) du_1 \cdots du_{\upsilon} \\
&\quad = \sigma_h^{\beta} f(x) \quad \text{a.e.}
\end{aligned}$$

Then by (1.6)

$$\left\| \sigma_h^{\beta} f(\cdot) \right\|_{p(\cdot),\omega} = \left\| \sigma_h^{\beta-\alpha} \Phi(\cdot) \right\|_{p(\cdot),\omega} \le c \left\| \sigma_h^{\alpha} f(\cdot) \right\|_{p(\cdot),\omega}$$

and

$$\Omega_{\beta}(f,\cdot)_{p(\cdot),\omega} \le c\,\Omega_{\alpha}(f,\cdot)_{p(\cdot),\omega}. \tag{2.8}$$

We note that if $r_1, r_2 \in \mathbb{Z}^+$, $\alpha_1, \beta_1 \in (0,1)$, taking $\alpha := r_1 + \alpha_1$, $\beta := r_2 + \beta_1$ for the remaining cases $r_1 = r_2$, $\alpha_1 < \beta_1$ or $r_1 < r_2$, $\alpha_1 = \beta_1$ or $r_1 < r_2$, $\alpha_1 < \beta_1$, it can be easily obtained from (2.7) and (2.8) that the required inequality (2.6) holds.

Using (2.6), Corollary 2.5 and Lemma 2.3, we get

$$\begin{aligned}\Omega_r\left(T_n, \frac{1}{n+1}\right)_{p(\cdot),\omega} &\leq c\Omega_{[r]}\left(T_n, \frac{1}{n+1}\right)_{p(\cdot),\omega} \\ &\leq c\left(\frac{1}{n+1}\right)^{2[r]} \left\| T_n^{(2[r])} \right\|_{p(\cdot),\omega} \\ &\leq \frac{c}{(n+1)^{2[r]}}(n+1)^{[r]-(r-[r])} \left\| T_n^{(r)} \right\|_{p(\cdot),\omega} \\ &= \frac{c}{(n+1)^r} \left\| T_n^{(r)} \right\|_{p(\cdot),\omega}\end{aligned}$$

which is the required result. □

Definition 2.7. For $p \in \mathcal{P}(\boldsymbol{T})$, $f \in L_\omega^{p(\cdot)}$, $\delta > 0$ and $r = 1, 2, 3, \ldots$ the *Peetre K-functional* is defined as

$$K(\delta, f; L_\omega^{p(\cdot)}, W_{p(\cdot),\omega}^r) := \inf_{g \in W_{p(\cdot),\omega}^r} \left\{ \left\| f - g \right\|_{p(\cdot),\omega} + \delta \left\| g^{(r)} \right\|_{p(\cdot),\omega} \right\}. \quad (2.9)$$

Theorem 2.8. *If $p \in \mathcal{P}_\pm^{\log}(\boldsymbol{T})$, $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, $r = 1, 2, 3, \ldots$, and $f \in L_\omega^{p(\cdot)}$, then $K(\delta^{2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot),\omega}^{2r})$in* (2.9) *and the modulus $\Omega_r(f, \delta)_{p(\cdot),\omega}$ are equivalent.*

Proof. If $h \in W_{p(\cdot),\omega}^{2r}$, then we have by Corollary 2.5 and (2.9) that

$$\begin{aligned}\Omega_r(f, \delta)_{p(\cdot),\omega} &\leq c \left\| f - h \right\|_{p(\cdot),\omega} + c\delta^{2r} \left\| h^{(2r)} \right\|_{p(\cdot),\omega} \\ &\leq cK(\delta^{2r}, f; L_\omega^{p(\cdot)}, W_{p(\cdot),\omega}^{2r}).\end{aligned}$$

Putting

$$(L_\delta f)(x) := 3\delta^{-3} \int_0^{\delta/2} \int_0^{2t} \int_{-u/2}^{u/2} g''(x+s) ds\, du\, dt \quad \text{for } x \in \boldsymbol{T},$$

we have

$$\frac{d^2}{dx^2} L_\delta f = \frac{c}{\delta^2}(I - \mathcal{A}_\delta) f$$

and hence

$$\frac{d^{2r}}{dx^{2r}} L_\delta^r f = \frac{c}{\delta^{2r}}(I - \mathcal{A}_\delta)^r \quad \text{for } r = 1, 2, 3, \ldots.$$

On the other hand, we find

$$\left\| L_\delta f \right\|_{p(\cdot),\omega} \leq 3\delta^{-3} \int_0^\delta \int_0^{2t} u \left\| \mathcal{A}_u f \right\|_{p(\cdot),\omega} du\, dt \leq c \left\| f \right\|_{p(\cdot),\omega}.$$

Now, let $F_\delta^r := I - (I - L_\delta^r)^r$. Then $F_\delta^r f \in W^{2r}_{p(\cdot),\omega}$ and

$$\left\| \frac{d^{2r}}{dx^{2r}} F_\delta^r f \right\|_{p(\cdot),\omega} \leq c \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p(\cdot),\omega} = \frac{c}{\delta^{2r}} \left\| (I - \mathcal{A}_\delta)^r f \right\|_{p(\cdot),\omega}$$
$$\leq \frac{c}{\delta^{2r}} \Omega_r(f,\delta)_{p(\cdot),\omega}.$$

Since

$$I - L_\delta^r = (I - L_\delta) \sum_{j=0}^{r-1} L_\delta^j,$$

we get

$$\begin{aligned}\left\| (I - L_\delta^r) g \right\|_{p(\cdot),\omega} &\leq c \left\| (I - L_\delta) g \right\|_{p(\cdot),\omega} \\ &\leq 3c\delta^{-3} \int_0^\delta \int_0^{2t} u \left\| (I - \mathcal{A}_u) g \right\|_{p(\cdot),\omega} du\, dt \\ &\leq c \sup_{0<u\leq\delta} \left\| (I - \mathcal{A}_u) g \right\|_{p(\cdot),\omega}.\end{aligned}$$

Taking into account

$$\left\| f - F_\delta^r f \right\|_{p(\cdot),\omega} = \left\| (I - L_\delta^r)^r f \right\|_{p(\cdot),\omega},$$

by a recursive procedure we obtain

$$\begin{aligned}\left\| f - F_\delta^r f \right\|_{p(\cdot),\omega} &\leq c \sup_{0<t_1\leq\delta} \left\| (I - \mathcal{A}_{t_1})(I - L_\delta^r)^{r-1} f \right\|_{p(\cdot),\omega} \\ &\leq c \sup_{0<t_1\leq\delta} \sup_{0<t_2\leq\delta} \left\| (I - \mathcal{A}_{t_1})(I - \mathcal{A}_{t_2})(I - L_\delta^r)^{r-2} f \right\|_{p(\cdot),\omega} \\ &\;\;\vdots \\ &\leq C \sup_{\substack{0<t_i\leq\delta \\ i=1,2,\ldots,r}} \left\| \prod_{i=1}^{r} (I - \mathcal{A}_{t_i}) f(\cdot) \right\|_{p(\cdot),\omega} = C\Omega_r(f,\delta)_{p(\cdot),\omega}\end{aligned}$$

and the proof is completed. □

3 Proofs of the main results

Proof of Theorem 1.2. First of all we note that by (1.5) and Theorem 3.2 of [33], the condition

$$\text{“}\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\boldsymbol{T}) \textit{ for some } p_0 \in (1, p_*(\boldsymbol{T}))\text{”}$$

implies that $\omega \in A_{p(\cdot)}(\boldsymbol{T})$. We set $A_k(x, f) := a_k \cos kx + b_k \sin kx$. Since the set of trigonometric polynomials is dense in $L_\omega^{p(\cdot)}$, for given $f \in L_\omega^{p(\cdot)}$ we have

$$E_n(f)_{p(\cdot),\omega} \to 0 \quad \text{as } n \to \infty.$$

By the first inequality in Remark 2.2 we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x, f)$$

in $\|\cdot\|_{p(\cdot),\omega}$ norm. For $k = 1, 2, 3, \ldots$ we know that

$$\begin{aligned} A_k(x, f) &= a_k \cos k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) + b_k \sin k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) \\ &= A_k\left(x + \frac{\alpha\pi}{2k}, f\right) \cos \frac{\alpha\pi}{2} + A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right) \sin \frac{\alpha\pi}{2} \end{aligned}$$

and

$$A_k\left(x, f^{(\alpha)}\right) = k^\alpha A_k\left(x + \frac{\alpha\pi}{2k}, f\right).$$

Therefore

$$\begin{aligned} \sum_{k=0}^{\infty} A_k(x, f) = A_0(x, f) &+ \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, f\right) \\ &+ \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right) \\ = A_0(x, f) &+ \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{-\alpha} A_k(x, f^{(\alpha)}) \\ &+ \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{-\alpha} A_k(x, \tilde{f}^{(\alpha)}) \end{aligned}$$

and hence

$$\begin{aligned} f(x) - S_n(x, f) &= \cos \frac{\alpha\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} A_k(x, f^{(\alpha)}) \\ &\quad + \sin \frac{\alpha\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} A_k(x, \tilde{f}^{(\alpha)}). \end{aligned}$$

Since

$$\begin{aligned}\sum_{k=n+1}^{\infty} & k^{-\alpha} A_k(x, f^{(\alpha)}) \\ &= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[(S_k(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot)) - (S_{k-1}(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot))\right] \\ &= \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha})(S_k(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot)) \\ &\qquad - (n+1)^{-\alpha}(S_n(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot))\end{aligned}$$

and

$$\begin{aligned}\sum_{k=n+1}^{\infty} k^{-\alpha} A_k(x, \tilde{f}^{(\alpha)}) = &\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha})(S_k(\cdot, \tilde{f}^{(\alpha)}) - \tilde{f}^{(\alpha)}(\cdot)) \\ &- (n+1)^{-\alpha}(S_n(\cdot, \tilde{f}^{(\alpha)}) - \tilde{f}^{(\alpha)}(\cdot)),\end{aligned}$$

we obtain

$$\begin{aligned}\| f(\cdot) - & S_n(\cdot, f)\|_{p(\cdot),\omega} \\ \le & \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\| S_k(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot)\right\|_{p(\cdot),\omega} \\ & + (n+1)^{-\alpha} \left\| S_n(\cdot, f^{(\alpha)}) - f^{(\alpha)}(\cdot)\right\|_{p(\cdot),\omega} \\ & + \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\| S_k(\cdot, \tilde{f}^{(\alpha)}) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p(\cdot),\omega} \\ & + (n+1)^{-\alpha} \left\| S_n(\cdot, \tilde{f}^{(\alpha)}) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p(\cdot),\omega} \\ \le & \, c\Bigg[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) E_k(f^{(\alpha)})_{p(\cdot),\omega} \\ & \qquad + (n+1)^{-\alpha} E_n(f^{(\alpha)})_{p(\cdot),\omega}\Bigg] \\ & + c\Bigg[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) E_k(\tilde{f}^{(\alpha)})_{p(\cdot),\omega} \\ & \qquad + (n+1)^{-\alpha} E_n(\tilde{f}^{(\alpha)})_{p(\cdot),\omega}\Bigg].\end{aligned}$$

Consequently, from the equivalence in Remark 2.2 we have

$$\begin{aligned}
&\|f(\cdot)-S_n(\cdot,f)\|_{p(\cdot),\omega}\\
&\quad\le c\left[\sum_{k=n+1}^{\infty}(k^{-\alpha}-(k+1)^{-\alpha})+(n+1)^{-\alpha}\right]\\
&\qquad\times\left\{E_k(f^{(\alpha)})_{p(\cdot),\omega}+E_n(\tilde{f}^{(\alpha)})_{p(\cdot),\omega}\right\}\\
&\quad\le cE_n(f^{(\alpha)})_{p(\cdot),\omega}\left[\sum_{k=n+1}^{\infty}(k^{-\alpha}-(k+1)^{-\alpha})+(n+1)^{-\alpha}\right]\\
&\quad\le \frac{c}{(n+1)^{\alpha}}E_n(f^{(\alpha)})_{p(\cdot),\omega}
\end{aligned}$$

and Theorem 1.2 is proved. □

Proof of Theorem 1.4. First we give the proof for $r\in\mathbb{Z}^+$. In case $g\in W^{2r}_{p(\cdot),\omega}$ we have by Corollary 2.5, (2.9) and Theorem 2.8 that

$$\begin{aligned}
E_n(f)_{p(\cdot),\omega}&\le E_n(f-g)_{p(\cdot),\omega}+E_n(g)_{p(\cdot),\omega}\\
&\le c\left[\|f-g\|_{p(\cdot),\omega}+(n+1)^{-2r}\|g^{(2r)}\|_{p(\cdot),\omega}\right]\\
&\le cK((n+1)^{-2r},f;L^{p(\cdot)}_{\omega},W^{2r}_{p(\cdot),\omega})\\
&\le c\Omega_r\left(f,\frac{1}{n+1}\right)_{p(\cdot),\omega}
\end{aligned}$$

as required for any $r\in\mathbb{Z}^+$. Therefore by the last inequality and (2.6) we get

$$\begin{aligned}
E_n(f)_{p(\cdot),\omega}&\le c\Omega_{[r]+1}\left(f,\frac{1}{n+1}\right)_{p(\cdot),\omega}\\
&\le c\Omega_r\left(f,\frac{1}{n+1}\right)_{p(\cdot),\omega},\quad n=0,1,2,3,\ldots,
\end{aligned}$$

and the assertion follows for general $r>0$. □

Proof of Theorem 1.5. Let $T_n\in\mathcal{T}_n$ be the best approximating polynomial of the function $f\in L^{p(\cdot)}_{\omega}$ and let $m\in\mathbb{Z}^+$. Then by Remark 1.1 (ii)

$$\begin{aligned}
\Omega_r\left(f,\frac{1}{n+1}\right)_{p(\cdot),\omega}&\le\Omega_r\left(f-T_{2^m},\frac{1}{n+1}\right)_{p(\cdot),\omega}+\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(\cdot),\omega}\\
&\le cE_{2^m}(f)_{p(\cdot),\omega}+\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(\cdot),\omega}.
\end{aligned}$$

By Lemma 2.6 we have

$$\Omega_r\left(T_{2^m}, \frac{1}{n+1}\right)_{p(\cdot),\omega} \le c\left(\frac{1}{n+1}\right)^r \left\|T_{2^m}^{(r)}\right\|_{p(\cdot),\omega}.$$

Since

$$T_{2^m}^{(r)}(\cdot) = T_1^{(r)}(\cdot) + \sum_{\nu=0}^{m-1}\left\{T_{2^{\nu+1}}^{(r)}(\cdot) - T_{2^\nu}^{(r)}(\cdot)\right\},$$

we get

$$\Omega_r\left(T_{2^m}, \frac{1}{n+1}\right)_{p(\cdot),\omega} \le \frac{c}{(n+1)^r}\left\{\left\|T_1^{(r)}\right\|_{p(\cdot),\omega} + \sum_{\nu=0}^{m-1}\left\|T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right\|_{p(\cdot),\omega}\right\}.$$

Lemma 2.3 gives

$$\begin{aligned}\left\|T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right\|_{p(\cdot),\omega} &\le c2^{\nu r}\left\|T_{2^{\nu+1}} - T_{2^\nu}\right\|_{p(\cdot),\omega} \\ &\le c2^{\nu r+1}E_{2^\nu}(f)_{p(\cdot),\omega}\end{aligned}$$

and

$$\left\|T_1^{(r)}\right\|_{p(\cdot),\omega} = \left\|T_1^{(r)} - T_0^{(r)}\right\|_{p(\cdot),\omega} \le cE_0(f)_{p(\cdot),\omega}.$$

Hence

$$\Omega_r\left(T_{2^m}, \frac{1}{n+1}\right)_{p(\cdot),\omega} \le \frac{c}{(n+1)^r}\left\{E_0(f)_{p(\cdot),\omega} + \sum_{\nu=0}^{m-1} 2^{(\nu+1)r}E_{2^\nu}(f)_{p(\cdot),\omega}\right\}.$$

It is easy to see that

$$2^{(\nu+1)r}E_{2^\nu}(f)_{p(\cdot),\omega} \le c^* \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{r-1}E_\mu(f)_{p(\cdot),\omega}, \quad \nu = 1, 2, 3, \dots,$$

where

$$c^* = \begin{cases} 2^{r+1}, & 0 < r < 1, \\ 2^{2r}, & r \ge 1. \end{cases}$$

Therefore

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^r}\left\{E_0(f)_{p(\cdot),\omega}+2^r E_1(f)_{p(\cdot),\omega} + c\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{r-1}E_\mu(f)_{p(\cdot),\omega}\right\}$$
$$\leq \frac{c}{(n+1)^r}\left\{E_0(f)_{p(\cdot),\omega}+\sum_{\mu=1}^{2^m}\mu^{r-1}E_\mu(f)_{p(\cdot),\omega}\right\}$$
$$\leq \frac{c}{(n+1)^r}\sum_{\nu=0}^{2^m-1}(\nu+1)^{r-1}E_\nu(f)_{p(\cdot),\omega}.$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{p(\cdot),\omega},$$
$$E_{2^m}(f)_{p(\cdot),\omega} \leq E_{2^{m-1}}(f)_{p(\cdot),\omega} \leq \frac{c}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{p(\cdot),\omega}.$$

the last two inequalities complete the proof. □

Proof of Theorem 1.10. For the polynomial T_n of the best trigonometric approximation for $f \in L_\omega^{p(\cdot)}$ we have

$$\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(\cdot),\omega} \leq 2E_{2^i}(f)_{p(\cdot),\omega}$$

and from Lemma 2.3 it follows that

$$\left\|T_{2^{i+1}}^{(\alpha)}-T_{2^i}^{(\alpha)}\right\|_{p(\cdot),\omega} \leq c2^{(i+1)\alpha}E_{2^i}(f)_{p(\cdot),\omega}.$$

Hence

$$\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{W^\alpha_{p(\cdot),\omega}} = \sum_{i=1}^{\infty}\left\|T_{2^{i+1}}^{(\alpha)}-T_{2^i}^{(\alpha)}\right\|_{p(\cdot),\omega}+\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(\cdot),\omega}$$
$$\leq c\sum_{m=2}^{\infty}m^{\alpha-1}E_m(f)_{p(\cdot),\omega} < \infty.$$

Therefore

$$\left\| T_{2^{i+1}} - T_{2^i} \right\|_{W^{\alpha}_{p(\cdot),\omega}} \to 0 \quad \text{as } i \to \infty.$$

This means that $\{T_{2^i}\}$ is a Cauchy sequence in $L^{p(\cdot)}_{\omega}$. Since $T_{2^i} \to f$ in $L^{p(\cdot)}_{\omega}$ and $W^{\alpha}_{p(\cdot),\omega}$ is a Banach space, we obtain $f \in W^{\alpha}_{p(\cdot),\omega}$.

On the other hand, since

$$\begin{aligned}\left\| f^{(\alpha)} - S_n(f^{(\alpha)}) \right\|_{p(\cdot),\omega} \le \left\| S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)}) \right\|_{p(\cdot),\omega} \\ + \sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}(f^{(\alpha)}) - S_{2^k}(f^{(\alpha)}) \right\|_{p(\cdot),\omega},\end{aligned}$$

we have for $2^m < n < 2^{m+1}$

$$\begin{aligned}\left\| S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)}) \right\|_{p(\cdot),\omega} &\le c 2^{(m+2)\alpha} E_n(f)_{p(\cdot),\omega} \\ &\le c(n+1)^{\alpha} E_n(f)_{p(\cdot),\omega}.\end{aligned}$$

Thus, we find

$$\begin{aligned}&\sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}(f^{(\alpha)}) - S_{2^k}(f^{(\alpha)}) \right\|_{p(\cdot),\omega} \\ &\quad\le c \sum_{k=m+2}^{\infty} 2^{(k+1)\alpha} E_{2^k}(f)_{p(\cdot),\omega} \\ &\quad\le c \sum_{k=m+2}^{\infty} \sum_{\mu=2^{k-1}+1}^{2^k} \mu^{\alpha-1} E_{\mu}(f)_{p(\cdot),\omega} \\ &\quad= c \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{p(\cdot),\omega} \\ &\quad\le c \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{p(\cdot),\omega}\end{aligned}$$

and Theorem 1.10 is proved. □

Proof of Theorem 1.12. In the case of $\alpha = 0$ the result follows from Remark 2.2 and the property $S_n(f) \in \mathcal{T}_n$:

$$\| f - S_n(f) \|_{p(\cdot),\omega} \le c E_n(f)_{p(\cdot),\omega}.$$

For $\alpha > 0$ we set

$$W_n(f) := W_n(x, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x, f) \quad \text{for } n = 0, 1, 2, \ldots.$$

Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f),$$

we have

$$\begin{aligned} \left\| f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot, f) \right\|_{p(\cdot),\omega} \leq \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{p(\cdot),\omega} & \\ + \left\| S_n^{(\alpha)}(\cdot, W_n(f)) - S_n^{(\alpha)}(\cdot, f) \right\|_{p(\cdot),\omega} & \\ + \left\| W_n^{(\alpha)}(\cdot, f) - S_n^{(\alpha)}(\cdot, W_n(f)) \right\|_{p(\cdot),\omega} & \\ =: I_1 + I_2 + I_3. & \end{aligned}$$

In this case, from the boundedness of the operator S_n in $L_\omega^{p(\cdot)}$ we obtain the boundedness of the operator W_n in $L_\omega^{p(\cdot)}$ and there holds

$$\begin{aligned} I_1 &\leq \left\| f^{(\alpha)}(\cdot) - S_n(\cdot, f^{(\alpha)}) \right\|_{p(\cdot),\omega} + \left\| S_n(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)}) \right\|_{p(\cdot),\omega} \\ &\leq c E_n(f^{(\alpha)})_{p(\cdot),\omega} + \left\| W_n(\cdot, S_n(f^{(\alpha)}) - f^{(\alpha)}) \right\|_{p(\cdot),\omega} \\ &\leq c E_n(f^{(\alpha)})_{p(\cdot),\omega}. \end{aligned}$$

From Lemma 2.3 we get

$$I_2 \leq c n^\alpha \left\| S_n(\cdot, W_n(f)) - S_n(\cdot, f) \right\|_{p(\cdot),\omega}$$

and

$$I_3 \leq c(2n)^\alpha \left\| W_n(\cdot, f) - S_n(\cdot, W_n(f)) \right\|_{p(\cdot),\omega} \leq c(2n)^\alpha E_n(W_n(f))_{p(\cdot),\omega}.$$

Now we have

$$\begin{aligned} \left\| S_n(\cdot, W_n(f)) - S_n(\cdot, f) \right\|_{p(\cdot),\omega} &\leq \left\| S_n(\cdot, W_n(f)) - W_n(\cdot, f) \right\|_{p(\cdot),\omega} \\ &\quad + \left\| W_n(\cdot, f) - f(\cdot) \right\|_{p(\cdot),\omega} \\ &\quad + \left\| f(\cdot) - S_n(\cdot, f) \right\|_{p(\cdot),\omega} \\ &\leq c E_n(W_n(f))_{p(\cdot),\omega} + c E_n(f)_{p(\cdot),\omega} \\ &\quad + c E_n(f)_{p(\cdot),\omega}. \end{aligned}$$

Since

$$E_n(W_n(f))_{p(\cdot),\omega} \le cE_n(f)_{p(\cdot),\omega},$$

we get

$$\begin{aligned}\|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot, f)\|_{p(\cdot),\omega} &\le cE_n(f^{(\alpha)})_{p(\cdot),\omega} + cn^\alpha E_n(W_n(f))_{p(\cdot),\omega} \\ &\quad + cn^\alpha E_n(f)_{p(\cdot),\omega} \\ &\quad + c(2n)^\alpha E_n(W_n(f))_{p(\cdot),\omega} \\ &\le cE_n(f^{(\alpha)})_{p(\cdot),\omega} + Cn^\alpha E_n(f)_{p(\cdot),\omega}.\end{aligned}$$

Since by Theorem 1.2

$$E_n(f)_{p(\cdot),\omega} \le \frac{c}{(n+1)^\alpha} E_n(f^{(\alpha)})_{p(\cdot),\omega},$$

we obtain

$$\|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot, f)\|_{p(\cdot),\omega} \le cE_n(f^{(\alpha)})_{p(\cdot),\omega}$$

and the proof is completed. □

Proof of Theorem 1.13. Let $\sum_{k=-\infty}^{\infty} c_k(g)e^{ik\theta}$ be the Fourier series of the boundary function g of $f \in H^{p(\cdot)}(\mathbb{D})$, and $S_n(g,\theta) := \sum_{k=-n}^{n} c_k(g)e^{ik\theta}$ be its nth partial sum. Since $g \in H^1(\mathbb{D})$, we have ([13, p. 38])

$$c_k(g) = \begin{cases} 0 & \text{for } k < 0, \\ \eta_k(f) & \text{for } k \ge 0. \end{cases}$$

Therefore

$$\left\| f(z) - \sum_{k=0}^{n} \eta_k(f) z^k \right\|_{p(\cdot),\omega} = \|g - S_n(g,\cdot)\|_{p(\cdot),\omega} \tag{3.1}$$

If t_n^* is the best approximating trigonometric polynomial for g in $L_\omega^{p(\cdot)}$, then from (2.3), (3.1) and Theorem 1.4 we get

$$\begin{aligned}\left\| f(z) - \sum_{k=0}^{n} \eta_k(f) z^k \right\|_{p(\cdot),\omega} &\le \|g - t_n^*\|_{p(\cdot),\omega} + \|S_n(g - t_n^*, \cdot)\|_{p(\cdot),\omega} \\ &\le cE_n(g)_{p(\cdot),\omega} = cE_n(f)_{p(\cdot),\omega} \\ &\le c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot),\omega}\end{aligned}$$

and the proof of Theorem 1.13 is completed. □

Some of the above results can be extended to the complex case.

4 Polynomial approximation in $E_{\omega}^{p(\cdot)}(G_0)$

Let G_0 and G_∞ be, respectively, the bounded and the unbounded components of a closed rectifiable curve Γ of the complex plane $\mathbb{C}$. Without loss of generality we may assume that $0 \in G_0$. Let $w = \varphi(z)$ and $w = \varphi_1(z)$ be the conformal mappings of G_∞ and G_0 onto the complement $\mathbb{D}_\infty$ of $\mathbb{D}$, normalized by the conditions

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \varphi(z)/z > 0$$

and

$$\varphi_1(0) = \infty, \qquad \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively. We denote by ψ and ψ_1, the inverse mappings of φ and φ_1, respectively.

Denote by $\mathcal{P}(\Gamma)$ the class of Lebesgue measurable functions $p = p(z) : \Gamma \to (1, \infty)$ with $1 < p_*(\Gamma) := \operatorname{ess\,inf}_{z\in\Gamma} p(z) \le p^*(\Gamma) := \operatorname{ess\,sup}_{z\in\Gamma} p(z) < \infty$.

Let $p \in \mathcal{P}(\Gamma)$ be a bounded measurable function and let $\omega : \Gamma \to [0, \infty]$ be a weight with

$$|\{t \in \Gamma : \omega(t) = 0\}| = 0.$$

For these p and ω we denote by $L_{\omega}^{p(\cdot)}(\Gamma)$ the class of functions $f : \Gamma \to \mathbb{C}$ for which

$$\int_\Gamma |f(z)\omega(z)|^{p(z)} \, |dz| < \infty.$$

The space $L_{\omega}^{p(\cdot)}(\Gamma)$ is a Banach space with the norm

$$\|f\|_{\Gamma,p(\cdot),\omega} := \inf_{\alpha>0} \left\{ \int_\Gamma \left| \frac{f(z)\omega(z)}{\alpha} \right|^{p(z)} |dz| \le 1 \right\}.$$

If p and ω are as above, the set of bounded rational functions defined on Γ is dense in $L_{\omega}^{p(\cdot)}(\Gamma)$ (cf. [31]). If $1 < p_*(\Gamma) \le p(z) \le p^*(\Gamma) < \infty$ for $z \in \Gamma$ and $\omega \equiv 1$, then the space $L_{\omega}^{p(\cdot)}(\Gamma)$ coincides with

$$\left\{ f : \left| \int_\Gamma f(z)g(z)dz \right| < \infty \text{ for all } g \in L_{\omega}^{p'(\cdot)}(\Gamma) \right\},$$

where $p'(z) := p(z)/(p(z) - 1)$ is the conjugate exponent of $p(z)$.

We define for $p \in \mathcal{P}(\Gamma)$ and a weight ω

$$E_{\omega}^{p(\cdot)}(G_0) := \left\{ f \in E^1(G_0) : f \in L_{\omega}^{p(\cdot)}(\Gamma) \right\},$$

$$E_{\omega}^{p(\cdot)}(G_\infty) := \left\{ f \in E^1(G_\infty) : f \in L_{\omega}^{p(\cdot)}(\Gamma) \right\}$$

and

$$\tilde{E}_{\omega}^{p(\cdot)}(G_\infty) := \left\{ f \in E_{\omega}^{p(\cdot)}(G_\infty) : f(\infty) = 0 \right\},$$

where $E^p(X)$, $1 \leq p < \infty$, is a Smirnov space of analytic functions defined on a simply connected domain $X \subset \mathbb{C}$. If $p(z) = p$ is constant, then $E_{\omega}^{p(\cdot)}(X)$ coincides with a usual weighted Smirnov space on X.

Basic approximation problems in the spaces $E^p(G_0)$ were proposed by several mathematicians. Walsh and Russel [46] gave the results in $E^p(G_0)$, $1 < p < \infty$, for polynomial approximation orders in the case of an analytic boundary. Al'per [6] proved direct and inverse approximation theorems by algebraic polynomials in the spaces $E^p(G_0)$, $1 < p < \infty$, for a Dini smooth boundary. Kokilashvili [28] improved Al'per's direct and inverse results for algebraic polynomial approximation and, assuming that the Cauchy singular integral operator is bounded (corners permitted), he obtained the improved direct and inverse approximation theorems in the Smirnov spaces $E^p(G_0)$, $1 < p < \infty$ ([30]). Andersson [7] proved that Kokilashvili's results also hold in $E^1(G_0)$. When the boundary is a Carleson curve, the approximation of functions of $E^p(G_0)$, $1 < p < \infty$, by the partial sum of Faber series was investigated by Israfilov in [19] and [9]. These results are generalized to the Muckenhoupt weighted case in [20] and [21]. The approximation properties of Faber series in so-called weighted and nonweighted Smirnov–Orlicz spaces are investigated in [29], [15], [26], [2], [3], [22], [4] and [23]. Most of the above results use the partial sum of Faber series as approximation tool.

In this section we prove the main theorems of approximation, respectively, by algebraic polynomials and rational functions in the weighted variable Smirnov spaces $E_{\omega}^{p(\cdot)}(G_0)$ and $\tilde{E}_{\omega}^{p(\cdot)}(G_\infty)$.

A smooth Jordan curve Γ will be called *Dini-smooth* ([37]) if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arc length s, has the modulus of continuity $\Omega(\theta, s)$ satisfying the Dini condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s} ds < \infty, \quad \delta > 0.$$

If Γ is Dini-smooth, then ([47])

$$0 < c < \left|\psi'(w)\right| < C < \infty, \quad |w| \geq 1, \tag{4.1}$$

with some constants c and C. Similar inequalities hold also for ψ_1' and φ_1' in the case of $|w| = 1$ and $z \in \Gamma$, respectively.

Let $\mathcal{P}_{\pm}^{\log}(\Gamma) := \{p \in \mathcal{P}(\Gamma) : p$ satisfies (1.4) with the replacements $x_1 \to z_1$, $x_2 \to z_2$ and $\boldsymbol{T} \to \Gamma\}$.

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For given $p \in \mathcal{P}(\Gamma)$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(z)}\right\|_{A_{p(\cdot)}(\Gamma)} := \sup_{B \in \mathcal{B}(\Gamma)} \frac{1}{|B|^{p_B}} \left\|\omega^{p(z)}\right\|_{L^1(B)} \left\|\frac{1}{\omega^{p(z)}}\right\|_{B,(p'(\cdot)/p(\cdot))} < \infty$$

is denoted by $A_{p(\cdot)}(\Gamma)$. Here $p_B := (\frac{1}{|B|}\int_B \frac{1}{p(z)}\,|dz|)^{-1}$ and

$$\mathcal{B}(\Gamma) := \{B(z,r) \cap \Gamma : B(z,r) \text{ is a ball in } \mathbb{C} \text{ of radius } r \text{ with } z \in \Gamma\}.$$

For given $f \in L_\omega^{p(\cdot)}(\Gamma)$ we define

$$f_0(e^\theta) := f(\psi(e^\theta)), \quad f_1(e^\theta) := f(\psi_1(e^\theta)) \quad \text{for } \theta \in \boldsymbol{T}$$

and

$$\omega_0(e^\theta) := \omega(\psi(e^\theta)), \quad \omega_1(e^\theta) := \omega(\psi_1(e^\theta)) \quad \text{for } \theta \in \boldsymbol{T}.$$

Remark 4.1. If Γ is Dini-smooth and $f \in L_\omega^{p(\cdot)}(\Gamma)$, then

(i) the functions $f_0(e^\theta)$and $f_1(e^\theta)$ belong to $L_\omega^{p(\cdot)}$,

(ii) the conditions $\omega \in A_{p(\cdot)}(\Gamma)$ and $\omega_0 \in A_{p(\cdot)}(\boldsymbol{T}) \ni \omega_1$ are equivalent,

(iii) the conditions $p \circ \psi, p \circ \psi_1 \in \mathcal{P}_{\pm}^{\log}(\boldsymbol{T})$ and $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$ are equivalent.

If $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$ and $\omega \in A_{p(\cdot)}(\Gamma)$, we define a degree $r > 0$ of the moduli of smoothness of $f \in L_\omega^{p(\cdot)}(\Gamma)$ as

$$\Omega_r(f,\delta)_{\Gamma,p(\cdot),\omega} := \Omega_r(f_0^+,\delta)_{\boldsymbol{T},p(\cdot),\omega_0}, \quad \delta > 0,$$
$$\tilde{\Omega}_r(f,\delta)_{\Gamma,p(\cdot),\omega} := \Omega_r(f_1^+,\delta)_{\boldsymbol{T},p(\cdot),\omega_1},$$

where the functions $f_0^+(t)$ and $f_1^+(t)$ are the nontangential boundary values of the functions

$$f_0^+(w) := \frac{1}{2\pi i}\int_{\boldsymbol{T}} \frac{f_0(t)}{t-w}dt, \quad f_1^+(w) := \frac{1}{2\pi i}\int_{\boldsymbol{T}} \frac{f_1(t)}{t-w}dt, \quad t \in \mathbb{D}.$$

We set

$$E_n(f)_{\mathbb{D},p(\cdot),\omega} := \inf_{P \in \mathcal{P}_n} \|f - P\|_{\boldsymbol{T},p(\cdot),\omega},$$
$$\tilde{E}_n(g)_{\Gamma,p(\cdot),\omega} := \inf_{R \in \mathcal{R}_n} \|g - R\|_{\Gamma,p(\cdot),\omega},$$

where $f \in E_\omega^{p(\cdot)}(\mathbb{D})$, $g \in E_\omega^{p(\cdot)}(G_\infty)$, and $\mathcal{R}_n$ is the set of rational functions of the form $\sum_{k=0}^n a_k z^{-k}$.

Theorem 4.2. *Let* Γ *be a Dini-smooth curve,* $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \textit{for some} \;\; p_0 \in (1, p_*(\Gamma)),$$

$r > 0$ *and* $f \in L_{\omega}^{p(\cdot)}(\Gamma)$. *Then there is a constant* $c > 0$ *such that for any natural number* n

$$\|f - R_n(\cdot, f)\|_{\Gamma,p(\cdot),\omega} \leq c \left\{ \Omega_r \left(f, \frac{1}{n+1} \right)_{\Gamma,p(\cdot),\omega} + \tilde{\Omega}_r \left(f, \frac{1}{n+1} \right)_{\Gamma,p(\cdot),\omega} \right\},$$

where $R_n(\cdot, f)$ *is the* n*th partial sum of the Faber–Laurent series of* f.

Corollary 4.3. *Let* Γ *be a Dini-smooth curve,* $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \textit{for some} \;\; p_0 \in (1, p_*(\Gamma)),$$

$r > 0$ *and* $f \in E_{\omega}^{p(\cdot)}(G_0)$. *Then there is a constant* $c > 0$ *such that for any natural number* n

$$\|f - P_n(\cdot, f)\|_{\Gamma,p(\cdot),\omega} \leq c \; \Omega_r \left(f, \frac{1}{n+1} \right)_{\Gamma,p(\cdot),\omega},$$

where $P_n(\cdot, f)$ *is the* n*th partial sum of the Faber series of* f.

Corollary 4.4. *Let* Γ *be a Dini-smooth curve,* $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \textit{for some} \;\; p_0 \in (1, p_*(\Gamma)),$$

$r > 0$ *and* $f \in \tilde{E}_{\omega}^{p(\cdot)}(G_\infty)$. *Then there is a constant* $c > 0$ *such that for any natural number* n

$$\|f - R_n(\cdot, f)\|_{\Gamma,p(\cdot),\omega} \leq c \; \tilde{\Omega}_r \left(f, \frac{1}{n+1} \right)_{\Gamma,p(\cdot),\omega},$$

where $R_n(\cdot, f)$ *is as in Theorem* 4.2.

Theorem 4.5. *Under the conditions of Corollary* 4.3, *the inequality*

$$\Omega_r \left(f, \frac{1}{n} \right)_{\Gamma,p(\cdot),\omega} \leq \frac{c}{n^r} \left\{ E_0(f)_{\Gamma,p(\cdot),\omega} + \sum_{k=1}^{n} k^{r-1} E_k(f)_{\Gamma,p(\cdot),\omega} \right\}$$

holds with a constant $c > 0$.

Corollary 4.6. *Under the conditions of Corollary* 4.3*, if*

$$E_n(f)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0, \quad n = 1,2,3,\ldots,$$

then

$$\Omega_r(f,\delta)_{\Gamma,p(\cdot),\omega} = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha, \\ \mathcal{O}(\delta^\alpha \left|\log \frac{1}{\delta}\right|), & r = \alpha, \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases}$$

Definition 4.7. Let $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \text{for some } \ p_0 \in (1, p_*(\Gamma))$$

and $r \in \mathbb{R}^+$. If $f \in E_\omega^{p(\cdot)}(G_0)$, then for $0 < \sigma < r$ we set

$$\operatorname{Lip}\sigma(r,\Gamma,p(\cdot),\omega) := \left\{f \in E_\omega^{p(\cdot)}(G_0) : \Omega_r(f,\delta)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(\delta^\sigma), \delta > 0\right\}$$

and

$$\widetilde{\operatorname{Lip}}\,\sigma(r,\Gamma,p(\cdot),\omega) := \left\{f \in \tilde{E}_\omega^{p(\cdot)}(G_\infty) : \tilde{\Omega}_r(f,\delta)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(\delta^\sigma)\right\}.$$

Corollary 4.8. *Let* $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \textit{for some } \ p_0 \in (1, p_*(\Gamma))$$

and $r \in \mathbb{R}^+$. *If* $f \in E_\omega^{p(\cdot)}(G_0)$, $0 < \sigma < r$ *and* $E_n(f)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(n^{-\sigma})$ *for* $n = 1,2,\ldots,$ *then* $f \in \operatorname{Lip}\sigma(r,\Gamma,p(\cdot),\omega)$.

By Corollary 4.3 and Corollary 4.6 we have the constructive characterization of the classes $\operatorname{Lip}\sigma(\alpha,\Gamma,p(\cdot),\omega)$.

Corollary 4.9. *Let* $p \in \mathcal{P}_{\pm}^{\log}(\Gamma)$,

$$\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma) \quad \textit{for some } \ p_0 \in (1, p_*(\Gamma)),$$

$0 < \sigma < r$ *and* $f \in E_\omega^{p(\cdot)}(G_0)$. *Then the following conditions are equivalent:*

(a) $f \in \operatorname{Lip}\sigma(r,\Gamma,p(\cdot),\omega)$.

(b) $E_n(f)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(n^{-\sigma})$, $n = 1,2,\ldots$.

The inverse theorem for unbounded domains is formulated as follows.

Theorem 4.10. *Under the conditions of Corollary* 4.4*, there is a constant* $c > 0$ *such that for every natural number n*

$$\tilde{\Omega}_r\left(f, \frac{1}{n}\right)_{\Gamma,p(\cdot),\omega} \leq \frac{c}{n^r}\left\{\tilde{E}_0(f)_{\Gamma,p(\cdot),\omega} + \sum_{k=1}^{n} k^{r-1}\tilde{E}_k(f)_{\Gamma,p(\cdot),\omega}\right\}$$

holds.

In a similar way as for $E_\omega^{p(\cdot)}(G_0)$ we obtain the following corollaries.

Corollary 4.11. *Under the conditions of Corollary* 4.4*, if*

$$\tilde{E}_n(f)_{M,\Gamma,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0, \quad n = 1, 2, 3, \dots,$$

then

$$\tilde{\Omega}_r(f, \delta)_{\Gamma,p(\cdot),\omega} = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha, \\ \mathcal{O}(\delta^\alpha \left|\log \frac{1}{\delta}\right|), & r = \alpha, \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases}$$

Using Corollary 4.11 and Definition 2.7 we get

Corollary 4.12. *Under the conditions of Corollary* 4.4*, if*

$$\tilde{E}_n(f)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(n^{-\sigma}), \quad \sigma > 0, \quad n = 1, 2, 3, \dots,$$

then $f \in \widetilde{\mathrm{Lip}}\,\sigma(r, \Gamma, p(\cdot), \omega)$.

By Corollary 4.11 and 4.12 we have

Corollary 4.13. *Let* $0 < \sigma < r$ *and the conditions of Corollary* 4.4 *be fulfilled. Then the following conditions are equivalent.*

(a) $f \in \widetilde{\mathrm{Lip}}\,\sigma(r, \Gamma, p(\cdot), \omega)$,

(b) $\tilde{E}_n(f)_{\Gamma,p(\cdot),\omega} = \mathcal{O}(n^{-\sigma})$, $n = 1, 2, 3, \dots$.

Remark 4.14. We note that the proof methods of these results are similar to those of given in [23] and [1], and for the proofs we use the following facts:

(i) ([33]) If Γ is a Dini-smooth curve, $p \in \mathcal{P}_\pm^{\log}(\Gamma)$, $\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}(\Gamma)$ for some $p_0 \in (1, p_*(\Gamma))$ and $f \in L_\omega^{p(\cdot)}(\Gamma)$, then

$$\|S_\Gamma f\|_{p(\cdot),\omega} \leq c\,\|f\|_{p(\cdot),\omega}\,,$$

where S_Γ is the Cauchy singular integral operator on Γ.

(ii) If $p \in \mathcal{P}_\pm^{\log}(\boldsymbol{T})$ and $\omega \in A_{p(\cdot)}(\boldsymbol{T})$, then the class of continuous functions is dense in the space $L_\omega^{p(\cdot)}$.

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