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ON APPROXIMATION OF CONFORMAL MAPS WITH SLOWLY GROWING COMPLEX DILATATION BY THE BIEBERBACH POLYNOMIALS

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ABSTRACT. The uniform convergence of the Bieberbach polynomials to the conformal map with slowly growing complex dilatation is investigated. The rate of this convergence is estimated depending on the growth of the complex dilatation.

1. Introduction and new results. Let G be a finite simply connected domain in the complex plane C bounded by rectifiable Jordan curve L, and let $z_0 \in G$. By the Riemann mapping theorem, there exists a unique conformal mapping $w = \varphi_0(z)$ of G onto $D(0, r_0) := \{w : |w| < r_0\}$ with the normalization $\varphi_0(z_0) = 0$, $\varphi_0'(z_0) = 1$. The radius r_0 of this disk is called the conformal radius of G with respect to z_0. Let $\psi_0(w)$ be the inverse to $\varphi_0(z)$. Let also $G^- := \operatorname{ext} L$, $D := D(0,1) = \{w : |w| < 1\}$, $T := \partial D$, $D^- := \{w : |w| > 1\}$, and let φ_1 be the conformal mapping of G^- onto D^- normalized by

$$\varphi_1(\infty) = \infty, \quad \lim_{z\to\infty} \varphi_1(z)/z > 0.$$

The inverse mapping of φ_1 is denoted by ψ_1. For $u \in (0,1)$, we set

$$L_u := \{z : |\varphi_1(z)| = 1 + u\}, \quad \Omega_u := (\operatorname{int} L_u)\backslash\overline{G}, \quad G_{1+u} = \overline{G} \cup \Omega_u,$$

where by $\operatorname{int} L_u$ we denote the finite domain bounded by L_u.

For an arbitrary function f and $p > 0$ we also set

$$\|f\|_{\overline{G}} := \sup\{|f(z)|, z \in \overline{G}\}, \quad \|f\|^2_{L_2(G)} := \iint_G |f(z)|^2 \, d\sigma_z.$$

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The class of functions analytic in G with finite $L_2(G)$ norm is denoted by $A^2(G)$.

It is well known that the function $\varphi_0(z)$ minimizes the integral $\|f'\|^2_{L_2(G)}$ in the class of all functions analytic in G with the normalization $f(z_0) = 0$, $f'(z_0) = 1$. On the other hand, if Π_n is the class of all polynomials p_n of degree at most n satisfying the conditions $p_n(z_0) = 0$, $p'_n(z_0) = 1$, then the integral $\|p'_n\|^2_{L_2(G)}$ is minimized in Π_n by a unique polynomial π_n which is called the nth Bieberbach polynomial for the pair (G, z_0).

If G is a Caratheodory domain, then $\|\varphi'_0 - \pi'_n\|_{L_2(G)} \to 0$, $n \to \infty$, and from this it follows that $\pi_n(z) \to \varphi_0(z)$, $n \to \infty$, for $z \in G$ uniformly on compact subsets of G.

On the other hand, if φ_0 has a holomorphic extension to some domain $\widetilde{G} \supset \overline{G}$, then

$$\|\varphi_0 - \pi_n\|_{\overline{G}} = O(q^n)$$

for some $0 < q < 1$. That is, in this case take a geometric convergence on $\overline{G}$ of the polynomials π_n to φ_0. Naturally, the above cited estimation is true if φ_0 has a holomorphic extension to $\overline{G}$ or if ∂G is an analytic Jordan curve.

The deeper connection between the boundary properties of the domains G and the rate of the uniform convergence of π_n to φ_0 on $\overline{G}$ was first observed by Keldych [**13**]. He showed that if the boundary L of G is a smooth Jordan curve with bounded curvature, then there is an exponential convergence on $\overline{G}$; more exactly, for every $\varepsilon > 0$, there is a constant $c = c(\varepsilon)$ such that

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \frac{c}{n^{1-\varepsilon}}$$

for all natural numbers n. Thus, the uniform convergence of the sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\overline{G}$ and the estimate of the error $\|\varphi_0 - \pi_n\|_{\overline{G}}$ depend on the geometric properties of boundary L. If L has a certain degree of smoothness, this error tends to zero with a certain speed.

Later, this idea was demonstrated in other investigations, for example in the works [**2, 3, 6**–**8, 11, 12, 17**–**20**]. In all of these works, the rate of the convergence has an exponential or lower growth. In the better cases there were results such as

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \frac{\text{const}}{n^{\gamma}},$$

with a positive constant γ depending on the numbers characterizing the geometric properties of G.

In particular, if the boundary is a K-quasiconformal curve then, due to the results [**2, 11, 14**], we have that the latest inequality holds for every $\gamma \in (0,\, 1/2K^2)$.

To the best of the author's knowledge, the first result proved that there has been convergence; the rate substantially higher than exponential and nonexceeding geometric rate was observed by Andrievskii and Pritsker. In [**4**] they in particular proved that, if ∂G is piecewise quasianalytic, with x^p $(p > 1)$-type interior zero angles at the joint points, then there exist the constants $q = q(G)$ and $r = r(G)$, $0 < q$, $r < 1$, and $c = c(G) > 0$ such that

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq c q^{n^r}, \quad n \in N.$$

In this work we propose to consider the domains G admitting the same intermediate rate of convergence of the Bieberbach polynomials to the conformal mapping φ_0 on $\overline{G}$.

Note that a class of functions with an intermediate rate of polynomial approximation at the first time in fact was observed in the constructive theory by Belyi [**5**]. The direct theorems obtained in [**5**] were formulated in terms of quasiconformal extension. Later by Maimeskul [**16**] at the same terms the inverse problems were studied and were proved the inverse theorems reversing in the same sense the direct theorems given in [**5**].

It is well known that a Jordan curve L is quasiconformal if there exists a quasiconformal mapping of a domain $\widetilde{G} \supset L$ which carries L into a circle. Moreover, if L is a quasiconformal, then a conformal mapping φ of G onto unit disk D can be extended to a quasiconformal mapping of the whole plane with the complex dilatation $\mu(z) := \varphi_{\bar{z}}(z)/\varphi_z(z)$ satisfying the conditions

$$\sup_{z \in C} |\mu(z)| < 1 \quad \text{and} \quad \sup_{z \in G} |\mu(z)| = 0.$$

In particular, if L is a K-quasiconformal, then the mapping φ admits a K^2-quasiconformal extension from G to $\overline{C}$.

Let $\gamma(t)$ be a function given on $(1, \infty)$ with the following properties:

$1^{\diamond}$. $\gamma(t) > 0$,

$2^{\diamond}$. $\gamma(t)$ *is a function increasing and continuous on* $(1, \infty)$ *and* $\gamma(t) \to \infty$, $t \to \infty$,

$3^{\diamond}$. $\gamma(t)/t$ *is a decreasing function on* $(1, \infty)$ *and* $\gamma(t)/t \to 0$, $t \to \infty$,

$4^{\diamond}$. $\ln t = \tilde{o}(\gamma(t))$, *i.e.*, $(\ln t)/\gamma(t) \to 0$, $t \to \infty$.

In particular, the functions t^{α}, $0 < \alpha < 1$, $\ln^{\beta} t$, $1 < \beta < \infty$, $t^{\alpha}/\ln^{\beta} t$, $0 < \alpha \le 1$, $0 < \beta < \infty$, $t^{\alpha} \ln^{\beta} t$, $0 < \alpha < 1$, $0 < \beta < \infty$ and many other functions satisfy the above cited conditions.

Starting from the above mentioned idea, we introduce the subclasses of quasidisks G such that the conformal map of G onto the unit disk D admits a quasiconformal extension with a slowly growing dilatation (SGD) to a domain $\widetilde{G} \supset \overline{G}$.

We shall use $c, c_1, c_2, \ldots$ to denote constants depending only on numbers that are not important for the questions of interest.

Definition 1.1. Let G be a finite quasidisk and $\varphi_0(z)$ the conformal map of G onto $D(0, r_0)$ with the normalization $\varphi_0(z_0) = 0$, $\varphi_0'(z_0) = 1$. We say that $\varphi_0 \in \mathrm{SGD}(\gamma)$ if it admits a quasiconformal extension (possibly nonunivalently) to a domain $\widetilde{G} \supset \overline{G}$ being the solution of the Beltrami equation

$$\varphi_{0\bar{z}} = \mu(z)\,\varphi_{0z}$$

in $\widetilde{G}$ with a complex dilatation satisfying the conditions

$$(1) \qquad \mu(G_{1+\gamma(n)/n}) := \sup\{|\mu(z)|,\ z \in G_{1+\gamma(n)/n}\} \le ce^{-\gamma(n)}.$$

Note that the conditions imposing to the conformal map φ_0 in this definition are in fact imposed to a domain G. The main result is stated as

Theorem 1.2. *If* $\varphi_0 \in \mathrm{SGD}(\gamma)$, *then for every* $\varepsilon > 0$ *there is a positive constant* $c = c(\varepsilon)$ *such that*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le ce^{-(1-\varepsilon)\gamma(n)}.$$

From this theorem the different estimates may be obtained by varying the function $\gamma(n)$. For example, we have

Corollary 1.3. *If $\varphi_0 \in \mathrm{SGD}\,(t^\alpha)$, and $0 < \alpha < 1$, then*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le ce^{-(1-\varepsilon)n^\alpha}.$$

Corollary 1.4. *If $\varphi_0 \in \mathrm{SGD}\,(t^\alpha / \ln^\beta t)$ and $0 < \alpha \le 1$, $0 < \beta < \infty$, then*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le ce^{-(1-\varepsilon)n^\alpha / \ln^\beta n}.$$

Corollary 1.5. *If $\varphi_0 \in \mathrm{SGD}\,(t^\alpha \ln^\beta t)$, $0 < \alpha < 1$, $0 < \beta < \infty$, then*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le ce^{-(1-\varepsilon)n^\alpha / \ln^\beta n}.$$

Corollary 1.6. *If $\varphi_0 \in \mathrm{SGD}\,(\ln^\beta t)$, $1 < \beta < \infty$, then*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le cn^{-(1-\varepsilon)\ln^{\beta-1} n}.$$

2. Auxiliary results. It is well known that, if G is a quasidisk, then the conformal map φ_0 of G onto $D(0, r_0)$ admits a quasiconformal extension to a domain $\widetilde{G} \supset \overline{G}$.

Lemma 2.1. *Let G be a quasidisk, and let $\mu(z)$ be the complex dilatation of a quasiconformal extension for φ_0. Then*

$$\|\varphi_{0\overline{z}}\|_{L_2(\Omega_u)} \le c\mu(G_{1+u}),$$

for every $u \in (0,1)$.

Proof. Since $\varphi_{0\overline{z}}(z) = \mu(z)\varphi_{0z}(z)$, denoting $\mu(G_2) := \sup_{z\in G_2} |\mu(z)|$, we have

$$\begin{aligned}
\|\varphi_{0\overline{z}}\|^2_{L_2(\Omega_u)} &= \iint_{\Omega_u} |\mu(z)|^2 |\varphi_{0z}(z)|^2 \, d\sigma_z \\
&= \iint_{\Omega_u} |\mu(z)|^2 \left(1 - \frac{|\varphi_{0\overline{z}}(z)|^2}{|\varphi_{0z}(z)|^2}\right)^{-1} J_{\varphi_0}(z)\, d\sigma_z \\
&\le (1 - [\mu(G_2)]^2)^{-1} \iint_{\Omega_u} |\mu(z)|^2 J_{\varphi_0}(z)\, d\sigma_z \\
&\le c_1 [\mu(G_{1+u})]^2 \operatorname{mes} \varphi_0(\Omega_u).
\end{aligned}$$

That is,

$$\|\varphi_{0\overline{z}}\|_{L_2(\Omega_u)} \leq c\mu(G_{1+u}),$$

as required. $\square$

Now, for $0 < u < 1$, we set

$$B_u := \widetilde{G}\backslash\overline{\operatorname{int} L_u}, \qquad I(z) := -\frac{1}{\pi}\iint_{\Omega_u} \frac{\varphi_{0\overline{\zeta}}(\zeta)}{(\zeta - z)^2}\, d\sigma_\zeta,$$
$$J(z) := \frac{1}{2\pi i}\int_{\partial\widetilde{G}} \frac{\varphi_0(\zeta)}{(\zeta - z)^2}\, d\zeta - \frac{1}{\pi}\iint_{B_u} \frac{\varphi_{0\overline{\zeta}}(\zeta)}{(\zeta - z)^2}\, d\sigma,$$

and let

$$E_n(f, G)_2 := \inf_{p_n} \|f - p_n\|_{A_2(G)},$$

where inf is taken over all polynomials p_n of degree at most n, be the best approximation of order n of the function f in the Bergman space $A^2(G)$.

Lemma 2.2. *Let G be a quasidisk and $u \in (0, 1)$. Then*

$$\|\varphi_0' - \pi_n'\|_{L_2(G)} \leq c\left[\frac{E_n(J, \operatorname{int} L_u)_2}{(1+u)^n \sqrt{u}} + \mu(G_{1+u})\right].$$

Proof. Since the quasiconformal mapping φ_0 has the generalized L^2-derivatives $\varphi_{0\bar{z}}$ and φ_{0z} in $\widetilde{G}$, the following integral representations hold (see, for example, [**15**, p. 154])

$$\varphi_0(z) = \frac{1}{2\pi i}\int_{\partial\widetilde{G}} \frac{\varphi_0(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\iint_{\widetilde{G}\backslash\overline{G}} \frac{\varphi_{0\overline{\zeta}}(\zeta)}{\zeta - z}\, d\sigma_\zeta, \quad z \in G,$$

and

$$\varphi_0'(z) = \frac{1}{2\pi i}\int_{\partial\widetilde{G}} \frac{\varphi_0(\zeta)}{(\zeta - z)^2}\, d\zeta - \frac{1}{\pi}\iint_{\widetilde{G}\backslash\overline{G}} \frac{\varphi_{0\overline{\zeta}}(\zeta)}{(\zeta - z)^2}\, d\sigma_\zeta, \quad z \in G.$$

Then, under the above-mentioned notations for $J(z)$ and $I(z)$,

$$\varphi_0'(z) = J(z) + I(z), \tag{2}$$

for every $z \in G$.

The first component of $J(z)$ is analytic in $\overline{\operatorname{int} L_u}$. The second component of $J(z)$ is analytic in $\operatorname{int} L_u$ and, moreover, according to the Calderon-Zigmund inequality (see, for example, [**1**, p. 89]) about the boundedness of the Hilbert transformation from L_2 into itself, it belongs to the Bergman space $A^2(\operatorname{int} L_u)$. Therefore, the function $J(z)$ also belongs to the Bergman space $A^2(\operatorname{int} L_u)$. Then, by virtue of the result [**10**, Theorem 3], there exists a polynomial $q_{n-1}(z)$ of degree $n-1$ such that

$$\|J(z) - q_{n-1}(z)\|_{L_2(G)} \le c_3 \frac{E_n(J, \operatorname{int} L_u)_2}{(1+u)^n \sqrt{u}}. \tag{3}$$

On the other hand, according to the Calderon-Zigmund inequality and by virtue of Lemma 2.1, we have

$$\|I(z)\|_{L_2(G)} \le c_4 \|\varphi_{0\overline{z}}\|_{L_2(\Omega_u)} \le c_5 \mu(G_{1+u}). \tag{4}$$

Let us set

$$p_n(z) := \int_{z_0}^{z} q_{n-1}(\zeta)\, d\zeta, \quad \widehat{p}_n(z) := p_n(z) + [1 - p_n'(z_0)]\,(z - z_0).$$

Then $\hat{p}_n(z_0) = 0$, $\hat{p}_n'(z_0) = 1$, and according to (2), (3) and (4), we have

$$\begin{aligned} \|\varphi_0'(z) - \hat{p}_n'(z)\|_{L_2(G)} \le c_3 \frac{E_n(J \operatorname{int} L_u)_2}{(1+u)^n \sqrt{u}} + c_5 \mu(G_{1+u}) \\ + \|1 - q_{n-1}(z_0)\|_{L_2(G)}. \end{aligned} \tag{5}$$

For the second time applying the same relations (2), (3), and (4), we obtain

$$\begin{aligned} \|1 - q_{n-1}(z_0)\|_{L_2(G)} &= \|I(z_0) + J(z_0) - q_{n-1}(z_0)\|_{L_2(G)} \\ &\le \|I(z_0)\|_{L_2(G)} + \|J(z_0) - q_{n-1}(z_0)\|_{L_2(G)} \\ &\le c_6 \mu(G_{1+u}) + c_7 \frac{E_n(J, \operatorname{int} L_u)_2}{(1+u)^n \sqrt{u}}. \end{aligned} \tag{6}$$

Hence, from the relations (5) and (6), we conclude that

$$\|\varphi_0'(z) - \hat{p}_n'(z)\|_{L_2(G)} \leq c\left[\frac{E_n(J, \operatorname{int} L_u)_2}{(1+u)^n\sqrt{u}} + \mu(G_{1+u})\right].$$

Finally, in view of the extremal property of the polynomials π_n, the proof of this lemma is completed. □

Corollary 2.3. *If $\varphi_0 \in \mathrm{SGD}\,(\gamma)$ then, for every $\varepsilon > 0$, there is a positive constant $c = c(\varepsilon)$ such that*

$$\|\varphi_0'(z) - \pi_n'(z)\|_{L_2(G)} \leq ce^{-(1-\varepsilon)\gamma(n)}.$$

Proof. Let $u := \gamma(n)/n$. Then, by inequality (1),

$$\mu(G_{1+\gamma(n)/n}) := \sup\{|\mu(z)|, z \in G_{1+\gamma(n)/n}\} \leq ce^{-\gamma(n)},$$

and, taking the boundedness of the quantity $E_n(J, \operatorname{int} L_u)_2$ into account from Lemma 2.2, we get

$$\|\varphi_0'(z) - \pi_n'(z)\|_{L_2(G)} \leq c_8\left[\left(\frac{n+\gamma(n)}{n}\right)^{n/\gamma(n)}\right]^{-\gamma(n)}\left[n/\gamma\,(n)\right]^{1/2} + c_9e^{-\gamma(n)}.$$

On the other hand, for every $\varepsilon > 0$ there is a natural number $n_0 = n_0(\varepsilon)$ such that

$$\left[\left(\frac{n+\gamma(n)}{n}\right)^{n/\gamma(n)}\right]^{-1} \leq e^{-(1-\varepsilon)}$$

as soon as $n \geq n_0\,(\varepsilon)$. Hence, for every $n \geq n_0\,(\varepsilon)$, we obtain that

$$\begin{aligned}\|\varphi_0'(z) - \pi_n'(z)\|_{L_2(G)} &\leq c_8e^{-(1-\varepsilon)\gamma(n)}\left[n/\gamma\,(n)\right]^{1/2} + c_9e^{-\gamma(n)} \\ &\leq c_{10}e^{-(1-\varepsilon)\gamma(n)}\left[n/\gamma\,(n)\right]^{1/2}.\end{aligned}$$

Since the condition $\ln n = \tilde{o}\,(\gamma\,(n))$ implies the relation $\left[n/\gamma\,(n)\right]^{1/2} = \widetilde{o}(e^{\varepsilon\gamma(n)})$, from the last inequality we conclude that the relation

$$\|\varphi_0'(z) - \pi_n'(z)\|_{L_2(G)} \leq ce^{-(1-\varepsilon)\gamma(n)},$$

holds for every $\varepsilon > 0$ and for some constant $c = c(\varepsilon)$. $\square$

3. Proofs of the new results.

Proof of Theorem 1.2. For any natural number $n \geq 2$ with $2^k \leq n \leq 2^{k+1}$, by Corollary 2.3, we have

$$\|\pi'_{2^{k+1}}(z) - \pi'_{n}(z)\|_{L_2(G)} \leq c_{11} e^{-(1-\varepsilon)\gamma(n)},$$

and later using Andrievskii's [2] polynomial lemma, see also Gaier [**9**],

$$\|p_{n}(z)\|_{\overline{G}} \leq c\,(\ln n)^{1/2}\,\|p'_{n}(z)\|_{L_2(G)},$$

which holds for every polynomial $p_{n}(z)$ of degree $\leq n$ with $p_n\,(z_0) = 0$, we get

$$\|\pi_{2^{k+1}}(z) - \pi_{n}(z)\|_{\overline{G}} \leq c_{12}\sqrt{n}\, e^{-(1-\varepsilon)\gamma(n)},$$

and analogously,

$$\|\pi_{2^{k+1}}(z) - \pi_{2^k}(z)\|_{\overline{G}} \leq c_{13}\sqrt{k}\, e^{-(1-\varepsilon)\gamma(2^k)}.$$

Then, from relation

$$\varphi_0\,(z) - \pi_n(z) = \pi_{2^{k+1}}(z) - \pi_{n}(z) + \sum_{j>k}\left[\pi_{2^{j+1}}(z) - \pi_{2^j}(z)\right], \quad z \in G,$$

we conclude that

(7)

$$\begin{aligned}\|\varphi_0\,(z) - \pi_n(z)\|_{\overline{G}} &\leq \|\pi_{2^{k+1}}(z) - \pi_n(z)\|_{\overline{G}} + \sum_{j>k}\|\pi_{2^{j+1}}(z) - \pi_{2^j}(z)\|_{\overline{G}} \\ &\leq c_{12}\sqrt{n}\, e^{-(1-\varepsilon)\gamma(n)} + c_{13}\sum_{j>k}\sqrt{j}\, e^{-(1-\varepsilon)\gamma(2^j)}.\end{aligned}$$

Since the condition $\ln t = \tilde{o}(\gamma\,(t)$ implies that, for every fixed $\varepsilon > 0$ the relation $\sqrt{n} = \tilde{o}(e^{\varepsilon\gamma(n)})$ holds, from (7) we get

$$\|\varphi_0\,(z) - \pi_n(z)\|_{\overline{G}} \leq c_{14} e^{-(1-\varepsilon)\gamma(n)} + c_{15}\sum_{j>k} e^{-(1-\varepsilon)\gamma(2^j)}.$$

The same condition $\ln t = \tilde{o}(\gamma(t))$ implies also the inequality

$$\gamma(2^{j+1}) - \gamma(2^j) \geq c \ln 2,$$

for some constant $c = c(\gamma) > 0$. Now denoting $q := 1/2^{c(1-\varepsilon)}$, we finally obtain

$$\begin{aligned}\|\varphi_0(z) - \pi_n(z)\|_{\overline{G}} &\leq c_{14} e^{-(1-\varepsilon)\gamma(n)} + c_{16} e^{-(1-\varepsilon)\gamma(2^{k+1})} \sum_{m \geq 0} q^m \\ &\leq c e^{-(1-\varepsilon)\gamma(n)}.\end{aligned}$$

Hence, the result. □

4. Sharpness of the estimate of Theorem 1.2. We now discuss the precision of Theorem 1.2 by using the inverse theorem of polynomial approximation with intermediate rate due to Maimeskul. If $\gamma(t)$ is a function satisfying the condition $1^\diamond - 4^\diamond$ from Section 1 and

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq c e^{-\gamma(n)}, \tag{8}$$

then the inverse theorem of Maimeskul [**16**] states that

$$\varphi_0(z) + cz \in \mathrm{SGD}(\gamma^*),$$

for some constant c, where

$$\gamma^* := \frac{\nu}{5} \gamma\left(\frac{n}{\nu}\right)$$

and $\nu \in (0,1)$.

Conversely, applying Theorem 1.2 in case $\gamma := \gamma^*$, we get the estimate

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq c e^{-((1-\varepsilon)\nu/5)\gamma(n/\nu)}. \tag{9}$$

Since, for a fixed $\varepsilon \in (0,1)$ and $\nu \in 0,1)$,

$$\gamma^* = \frac{(1-\varepsilon)\nu}{5} \gamma\left(\frac{n}{\nu}\right) \geq c\gamma(n)$$

for some constant $c > 0$, we conclude that the rates of convergence in the estimations (8) and (9) are equal with respect to the order. In this sense Theorem 1.2 is sharp.

REFERENCES

1. L. Ahlfors, *Lectures on quasi-conformal mappings*, Van Nostrand, Princeton, New Jersey, 1966.

2. V. V. Andrievskii, *Convergence of Bieberbach polynomials in domains with quasiconformal boundary*, Ukrainian Math. J. **35** (1983), 233–236.

3. ———, *Uniform convergence of Bieberbach polynomials in domains with piecewise-quasiconformal boundary*, in *Theory of mappings and approximation of functions*, Naukova Dumka, Kiev, 1983 (in Russian).

4. V.V. Andrievskii and I.E. Pritsker, *Convergence of Bieberbach polynomials in domains with interior cusps*, J. Analyse Math. **82** (2000), 315–332.

5. V.I. Belyi, *Quasiconformal mappings and approximation of functions*, in Proc. Internat. Conf. on Approximation of Functions (Sofia, 1983), pp. 28–33.

6. D. Gaier, *On the convergence of the Bieberbach polynomials in regions with corners*, Constr. Approx. **4** (1988), 289–305.

7. ———, *On the convergence of the Bieberbach polynomials in regions with piecewise analytic boundary*, Arch. Math. **58** (1992), 289–305.

8. ———, *Polynomial approximation of conformal maps*, Constr. Approx. **14** (1998), 27–40.

9. ———, *On a polynomial lemma of Andrievskii*, Arch. Math. **49** (1987), 119–123.

10. D.M. Israfilov, *Approximation by generalized Faber series in weighted Bergman spaces on finite domains with quasi-conformal boundary*, East J. Approx. **4** (1998), 1–13.

11. ———, *Uniform convergence of some extremal polynomials in domains with quasi-conformal boundary*, East J. Approx. **4** (1998), 527–539.

12. ———, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G)$ and the Bieberbach polynomials*, Constr. Approx. **17** (2001), 335–351.

13. M.V. Keldych, *Sur l'approximation en moyenne quadratique des fonctions analytiques*, Math. Sb. **5** (1939), 391–401.

14. M. Leclerc, *A note on a theorem of V.V. Andrievskii*, Arch. Math. **46** (1986), 159–161.

15. O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, New York, 1973.

16. V.V. Maimeskul, *On approximation of functions with quasiconformal extension*, Akad. Nauk Ukrain. SSR Inst. (1985), 14–20.

17. S.N. Mergelyan, *Certain questions of the constructive theory of functions*, Trudy Math. Inst. Steklov **37** (1951), 1–91 (in Russian).

18. I.B. Simonenko, *On the convergence of Bieberbach polynomials in the case of a Lipschitz domain*, Math. USSR-Izv. **13** (1978), 166–174.

19. P.K. Suetin, *Polynomials ortogonal over a region and Bieberbach polynomials*, Proc. Steklov Inst. Math. **100** (1974).

20. Wu Xue-Mou, *On Bieberbach polynomials*, Acta Math. Sinica **13** (1963), 145–151.

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