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SOME NORMAL SUBGROUPS OF THE EXTENDED HECKE GROUPS $\overline{H}(\lambda_p)$

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ABSTRACT. We consider the extended Hecke groups $\overline{H}(\lambda_p)$ generated by $T(z) = -1/z$, $S(z) = -1/(z+\lambda_p)$ and $R(z) = 1/\bar{z}$ with $\lambda_p = 2\cos(\pi/p)$ for $p \geq 3$ prime number. In this article, we study the abstract group structure of the extended Hecke groups and the power subgroups $\overline{H}^m(\lambda_p)$ of $\overline{H}(\lambda_p)$. Then, we give the relations between commutator subgroups and power subgroups and also the information of interest about free normal subgroups of the extended Hecke groups.

1. Introduction. In [**7**], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.,

$$S(z) = -\frac{1}{z+\lambda}.$$

Hecke showed that H(λ) is discrete if and only if $\lambda = \lambda_q = 2\cos(\pi/q)$, $q \in \mathbf{N}$, $q \geq 3$, or $\lambda \geq 2$. We will focus on the discrete with $\lambda < 2$, i.e., those with $\lambda = \lambda_q$, $q \geq 3$. These groups have come to be known as the *Hecke groups*, and we will denote $H(\lambda_q)$ for $q \geq 3$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, and it has a presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q.$$

The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbf{Z})$, the modular group, $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H((1+\sqrt{5})/2)$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2, \mathbf{Z}[\lambda_q])$, for $q \geq 4$.

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The Hecke groups $H(\lambda_q)$ and their normal subgroups have been extensively studied for many aspects in the literature, see [**2, 5, 6**]. The Hecke group $H(\lambda_3)$, the modular group $PSL(2,\mathbf{Z})$, and its normal subgroups have especially been of great interest in many fields of mathematics, for example number theory, automorphic function theory and group theory, see [**12, 15, 16**].

The extended modular group, denoted by $\overline{H}(\lambda_3) = \prod = PGL(2,\mathbf{Z})$, has been defined in [**10, 11, 22**] by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group $H(\lambda_3)$. Then, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [**8, 17–19**] similar to the extended modular group by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the Hecke group $\overline{H}(\lambda_q)$. In [**17–19**], there were studied some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ (commutator subgroups, even subgroups, principal congruence subgroups, Fuchsian subgroups) and some relations between them. Also, in [**20, 21**], we investigated the power and free subgroups of the extended modular group $\overline{H}(\lambda_3)$ and the extended Hecke group $\overline{H}(\lambda_5)$ and the relations between power subgroups and commutator subgroups.

In this work, we continue our study to which properties of Hecke groups $H(\lambda_p)$, $p \geq 3$ prime number, hold for the extended Hecke groups $\overline{H}(\lambda_p)$. Firstly, we give a proof of the fact that the extended Hecke group $\overline{H}(\lambda_p)$ is isomorphic to the free product of two finite dihedral groups of orders 4 and $2p$ with amalgamation $\mathbf{Z}_2$. Secondly, we will study especially the power subgroups $\overline{H}^m(\lambda_p)$ of $\overline{H}(\lambda_p)$, and we will determine abstract group structure and generators of them. In fact, it is a well-known and important result that the only normal subgroups of the Hecke groups $H(\lambda_p)$ containing torsion are $H(\lambda_p)$, $H^2(\lambda_p)$ and $H^p(\lambda_p)$ of indices 1, 2, p respectively, see [**5**]. Here we show that this nicely generalizes to the extended Hecke groups $\overline{H}(\lambda_p)$ with $p \geq 3$ prime number. We specially discuss $\overline{H}^2(\lambda_p)$ and $\overline{H}^p(\lambda_p)$ as they are nicely related to $\overline{H}(\lambda_p)$ and its commutator subgroup $\overline{H}'(\lambda_p)$. Then, we give a classification theorem for these power subgroups. Also we discuss free normal subgroups of finite index in the extended Hecke groups $\overline{H}(\lambda_p)$, $p \geq 3$ prime number.

2. Extended Hecke groups $\overline{H}(\lambda_p)$ and their decomposition. Extended Hecke group $\overline{H}(\lambda_p)$ has a presentation

$$\overline{H}(\lambda_p) = \langle T, S, R \mid T^2 = S^p = R^2 = I,\ RT = TR,\ RS = S^{-1}R\rangle$$

or

$$\overline{H}(\lambda_p) = \langle T, S, R \mid T^2 = S^p = R^2 = (TR)^2 = (SR)^2 = I\rangle. \tag{2.1}$$

Hecke group $H(\lambda_p)$ is a subgroup of index 2 in $\overline{H}(\lambda_p)$, and it has a presentation

$$H(\lambda_p) = \langle T, S \mid T^2 = S^p = I\rangle \cong C_2 * C_p.$$

Now $H(\lambda_p)$ has trivial center, and its outer automorphism class group $\operatorname{Out} H(\lambda_p) = \operatorname{Aut} H(\lambda_p)/\operatorname{Inn} H(\lambda_p)$ is generated by the automorphism fixing T and inverting S, so the action of $\overline{H}(\lambda_p)$ on $H(\lambda_p)$ by conjugation induces as isomorphism $\overline{H}(\lambda_p) \cong \operatorname{Aut} H(\lambda_p)$, with R corresponding to the required outer automorphism.

The function

$$\alpha : T \longrightarrow RT, \quad S \to S, \quad R \to R$$

preserves the relations in (2.1), so it extends to an endomorphism of $\overline{H}(\lambda_p)$; since α^2 is the identity, α is an automorphism, which cannot be inner since $T \in H(\lambda_p) \trianglelefteq \overline{H}(\lambda_p)$ whereas $T\alpha = RT \notin H(\lambda_p)$. Therefore the outer automorphism class group $\operatorname{Out} \overline{H}(\lambda_p) = \operatorname{Aut} \overline{H}(\lambda_p)/\operatorname{Inn} \overline{H}(\lambda_p)$ has order 2, being generated by α.

In terms of (2.1) we have

$$\alpha : T \longrightarrow RT, \quad S \to S, \quad R \to R$$

so that

$$\alpha(H(\lambda_p)) = \langle RT, S \mid (RT)^2 = S^p = I\rangle.$$

The group $\alpha(H(\lambda_p))$ is a subgroup of index 2 in $\overline{H}(\lambda_p)$.

Now we give a theorem about the group structure of the extended Hecke group $\overline{H}(\lambda_p)$:

Theorem 2.1. *The extended Hecke group $\overline{H}(\lambda_p)$ is given directly as a free product of two groups G_1, G_2 with amalgamated subgroup $\mathbf{Z}_2$, where G_1 is the dihedral group D_2 and G_2 is the dihedral group D_p, that is, $\overline{H}(\lambda_p) \cong D_2 *_{\mathbf{Z}_2} D_p$.*

Proof. The result follows from a presentation of the extended Hecke group $\overline{H}(\lambda_p)$ given (2.1):

$$\overline{H}(\lambda_p) = \langle T, S, R \mid T^2 = S^p = R^2 = (TR)^2 = (SR)^2 = I \rangle.$$

Let $G_1 = \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong D_2$, and let $G_2 = \langle S, R \mid S^p = R^2 = (SR)^2 = I \rangle \cong D_p$. Then $\overline{H}(\lambda_p)$ is $G_1 * G_2$ with the identification $R = R$.

In G_1, the subgroup generated by R is $\mathbf{Z}_2$; this is also true in G_2. Therefore, the identification induces an isomorphism and $\overline{H}(\lambda_p)$ is a generalized free product with the subgroup $M \cong \mathbf{Z}_2$ amalgamated. $\square$

3. Power subgroups of $\overline{H}(\lambda_p)$. Let m be a positive integer. Let us define $\overline{H}^m(\lambda_p)$ to be the subgroup generated by the mth-powers of all elements of $\overline{H}(\lambda_p)$. The subgroup $\overline{H}^m(\lambda_p)$ is called the mth-*power subgroup* of $\overline{H}(\lambda_p)$. As fully invariant subgroups, they are normal in $\overline{H}(\lambda_p)$.

From the definition one can easily deduce that

$$\overline{H}^m(\lambda) > \overline{H}^{mk}(\lambda)$$

and that

$$(\overline{H}^m(\lambda))^k > \overline{H}^{mk}(\lambda).$$

Power subgroups of the Hecke groups were studied in [**3–5, 9, 15**]. In [**3, 5, 15**], they proved that

$$\begin{aligned}
&\left|H(\lambda_p) : H^2(\lambda_p)\right| = 2,\\
&|H(\lambda_p) : H^p(\lambda_p)| = p,\\
&|H(\lambda_p) : H'(\lambda_p)| = 2p,\\
&H'(\lambda_p) = H^2(\lambda_p) \cap H^p(\lambda_p),\\
&H^2(\lambda_p) = \langle S\rangle \star \langle TST\rangle,\\
&H^p(\lambda_p) = \langle T\rangle \star \langle STS^{p-1}\rangle \star \cdots \star \langle S^{p-1}TS\rangle\\
&H'(\lambda_p) = \langle TSTS^{p-1}\rangle \star \cdots \star \langle TS^{p-1}TS\rangle,
\end{aligned} \tag{3.1}$$

and $H^{2pm}(\lambda_p)$ are free groups. Also, Newman proved in [**15**] that $H''(\lambda_3) \subset H^6(\lambda_3) \subset H'(\lambda_3)$.

Then, power subgroups of the extended modular group $\overline{H}(\lambda_3)$ were investigated in [**20**]. They showed that

$$\begin{aligned}
&\left|\overline{H}(\lambda_3) : \overline{H}^2(\lambda_3)\right| = 4, && \left|\overline{H}(\lambda_3) : \overline{H}^3(\lambda_3)\right| = 1,\\
&\overline{H}^2(\lambda_3) = \langle S\rangle \star \langle TST\rangle, && \overline{H}^3(\lambda_3) = \overline{H}(\lambda_3),\\
&\overline{H}'(\lambda_3) = \overline{H}^2(\lambda_3) = H^2(\lambda_3), && \overline{H}^6(\lambda_3) \subset \overline{H}''(\lambda_3) \subset \overline{H}'(\lambda_3),
\end{aligned}$$

and $\overline{H}^{6m}(\lambda_3)$ are free groups.

Here we discuss power subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$, p. prime. Firstly we find a presentation for the quotient $\overline{H}(\lambda_p)/\overline{H}^m(\lambda_p)$ by adding the relation $X^m = I$ to the presentation of $\overline{H}(\lambda_p)$ given by (2.1). The order of $\overline{H}(\lambda_p)/\ \overline{H}^m(\lambda_p)$ gives us the index. We have

$$\begin{aligned}
\overline{H}(\lambda_p)/\ \overline{H}^m(\lambda_p) \cong \langle T, S, R \mid\ & T^2 = S^p = R^2 = (TR)^2 = (SR)^2 = I,\\
& T^m = S^m = R^m = (TR)^m = (SR)^m = \cdots = I\rangle.
\end{aligned} \tag{3.2}$$

Then we use the Reidemeister-Schreier process to find the presentation of the power subgroups $\overline{H}^m(\lambda_p)$.

We begin with the case $m = 2$:

Theorem 3.1. *The normal subgroup* $\overline{H}^2(\lambda_p)$ *is isomorphic to the free product of two finite cyclic groups of order* p. *Also*

$$\overline{H}(\lambda_p)/\overline{H}^2(\lambda_p) \cong C_2 \times C_2,$$
$$\overline{H}(\lambda_p) = \overline{H}^2(\lambda_p) \cup T\overline{H}^2(\lambda_p) \cup R\overline{H}^2(\lambda_p) \cup TR\overline{H}^2(\lambda_p),$$

and

$$\overline{H}^2(\lambda_p) = \langle S \rangle \star \langle TST \rangle.$$

The elements of $\overline{H}^2(\lambda_p)$ *are characterized by the requirement that the sum of the exponents of* T *is even.*

Proof. By (3.2), we obtain $T^2 = R^2 = I$ and $S = I$ from the relations

$$S^p = S^2 = I.$$

Then we get

$$\overline{H}(\lambda_p)/\overline{H}^2(\lambda_p) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong C_2 \times C_2,$$

and therefore

$$\left|\overline{H}(\lambda_p) : \overline{H}^2(\lambda_p)\right| = 4.$$

Now we choose $\{I, T, R, TR\}$ as a Schreier transversal for $\overline{H}^2(\lambda_p)$. According to the Reidemeister-Schreier method, we can form all possible products:

$$\begin{array}{lll}
I.T.(T)^{-1} = I, & I.S.(I)^{-1} = S, & I.R.(R)^{-1} = I,\\
T.T.(I)^{-1} = I, & T.S.(T)^{-1} = TST, & T.R.(TR)^{-1} = I,\\
R.T.(TR)^{-1} = RTRT, & R.S.(R)^{-1} = RSR, & R.R.(I)^{-1} = I,\\
TR.T.(R)^{-1} = TRTR, & TR.S.(TR)^{-1} = TRSRT, & TR.R.(T)^{-1} = I.
\end{array}$$

Since $RTRT = I$, $TRTR = I$, $RSR = S^{-1}$ and $TRSRT = TS^{-1}T = (TST)^{-1}$, the generators are S and TST. Thus we have

$$\overline{H}^2(\lambda_p) = \langle S, TST \mid S^p = (TST)^p = I \rangle \cong C_p * C_p,$$

and

$$\overline{H}(\lambda_p) = \overline{H}^2(\lambda_p) \cup T\overline{H}^2(\lambda_p) \cup R\overline{H}^2(\lambda_p) \cup TR\overline{H}^2(\lambda_p). \qquad \square$$

Theorem 3.2. *Let $p \geq 3$ be a prime number. Then $\overline{H}^p(\lambda_p) = \overline{H}(\lambda_p)$.*

Proof. By (3.2), we find $S = T = R = I$ from the relations

$$R^2 = R^p = I, \quad S^p = (SR)^2 = (SR)^p = I, \quad T^2 = T^p = I.$$

Thus we have

$$\left|\overline{H}(\lambda_p) : \overline{H}^p(\lambda_p)\right| = 1,$$

that is,

$$\overline{H}^p(\lambda_p) = \overline{H}(\lambda_p). \qquad \square$$

We can now obtain a classification of these subgroups:

Theorem 3.3. *Let m be a positive integer, and let $p \geq 3$ be a prime number.*

i) $\overline{H}^m(\lambda_p) = \overline{H}(\lambda_p)$ *if* $2 \nmid m$,

ii) $\overline{H}^m(\lambda_p) = \overline{H}^2(\lambda_p)$ *if* $2 \mid m$ *but* $2p \nmid m$.

Proof. i) If $2 \nmid m$, then by (3.2), we find $S = T = R = I$ from the relations

$$R^2 = R^m = I, \quad S^p = S^m = (SR)^2 = (SR)^m = I, \quad T^2 = T^m = I.$$

Thus $\overline{H}(\lambda_p)/\overline{H}^p(\lambda_p)$ is trivial and hence $\overline{H}^m(\lambda_p) = \overline{H}(\lambda_p)$.

ii) If $2 \mid m$ but $2p \nmid m$, then $(2, m) = 2$. By (3.2), we obtain $S = T^2 = R^2 = I$ from the relations

$$R^2 = R^m = I, \quad S^p = S^m = I, \quad T^2 = T^m = I.$$

These show that

$$\overline{H}(\lambda_p)/\overline{H}^p(\lambda_p) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I\rangle \cong D_2,$$

and

$$\left|\overline{H}(\lambda_p) : \overline{H}^p(\lambda_p)\right| = 4.$$

Since $\overline{H}^2(\lambda_p)$ is the only normal subgroup of index 4, we have $\overline{H}^m(\lambda_p) = \overline{H}^2(\lambda_p)$. $\square$

Therefore we have only the subgroups $\overline{H}^{2pm}(\lambda_p)$ left to consider. In this case the above techniques do not say much about $\overline{H}^{2pm}(\lambda_p)$.

To discuss $\overline{H}^{2pm}(\lambda_p)$, we first need to consider the commutator subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$:

Theorem 3.4. (i) $\overline{H}(\lambda_p)/\ \overline{H}'(\lambda_p) \cong V_4 \cong C_2 \times C_2$;

(ii) $\overline{H}'(\lambda_p) = \langle S, TST \mid S^p = (TST)^p = I\rangle \cong C_p \star C_p$;

(iii) $\overline{H}'(\lambda_p)/\overline{H}''(\lambda_p) \cong V_{p^2}$, *here* V_{p^2} *denotes an elementary abelian group of order* p^2;

(iv) $\overline{H}''(\lambda_p)$ *is a free group with basis* $[S, TST]$, $[S, TS^2T], \dots,$ $[S, TS^{p-1}T]$, $[S^2, TST]$, $[S^2, TS^2T], \dots, [S^2, TS^{p-1}T], \dots, [S^{p-1}, TST]$, $[S^{p-1}, TS^2T], \dots, [S^{p-1}, TS^{p-1}T]$.

(v) *For* $n > 2$, $\left|\overline{H}(\lambda_p) : \overline{H}^{(n)}(\lambda_p)\right| = \infty$.

Proof. (i)–(iv). Please refer to [**18, 19**].

(v) Taking relations and abelianizing we find that the resulting quotient is infinite. It follows that $\overline{H}'''(\lambda_p)$ has infinite index in $\overline{H}''(\lambda_p)$. Further, since this has infinite index it follows that the derived series from this point on has infinite index. $\square$

Notice that $\overline{H}'(\lambda_p)$ is a subgroup of index 2 of the extended Hecke groups $H(\lambda_p)$, consisting of the words in T and S for which T has even exponent-sum.

We can give the following results using the Theorem 3.1 and Theorem 3.4:

Corollary 3.5. (i) $\overline{H}'(\lambda_p) = H(\lambda_p) \cap \alpha(H(\lambda_p))$,

(ii) $H'(\lambda_p)$ *is a subgroup of index* p *in* $\overline{H}'(\lambda_p)$,

(iii) $\overline{H}''(\lambda_p)$ *is a subgroup of index* p *in* $H'(\lambda_p)$.

Proof. (i) Both $H(\lambda_p)$ and $\alpha(H(\lambda_p))$ have index 2, so $H(\lambda_p) \cap \alpha(H(\lambda_p))$ has index 4, and hence we find $\overline{H}'(\lambda_p) = H(\lambda_p) \cap \alpha(H(\lambda_p))$.

(ii)–(iii) It is easily seen from Theorem 3.4 and by (3.1). □

Theorem 3.6. *The commutator subgroup* $\overline{H}'(\lambda_p)$ *of* $\overline{H}(\lambda_p)$ *satisfies*

$$\overline{H}'(\lambda_p) = \overline{H}^2(\lambda_p).$$

Theorem 3.7. *Let* $p \geq 3$ *be a prime number.*

i) $H^2(\lambda_p) = \overline{H}^2(\lambda_p) = \overline{H}^2(\lambda_p) \cap \overline{H}^p(\lambda_p)$;

ii) $(\overline{H}'(\lambda_p))^p \subset \overline{H}''(\lambda_p)$.

Theorem 3.8. *Let* m *be a positive integer. The groups* $\overline{H}^{2pm}(\lambda_p)$ *are the subgroups of the second commutator subgroup* $\overline{H}''(\lambda_p)$.

Proof. i) Since $\overline{H}^{2p}(\lambda_p) \subset (\overline{H}^2(\lambda_p))^p \subset \overline{H}^2(\lambda_p)$ and $\overline{H}'(\lambda_p) = \overline{H}^2(\lambda_p)$ implies that $\overline{H}^{2p}(\lambda_p) \subset (\overline{H}'(\lambda_p))^p \subset \overline{H}'(\lambda_p)$ and $\overline{H}^{2pm}(\lambda_p) \subset \overline{H}^{2p}(\lambda_p) \subset \overline{H}''(\lambda_p)$. Since $\overline{H}'(\lambda_p)$ does not contain any reflection, $\overline{H}^{2pm}(\lambda_p)$ does not contain any reflection. Also we know that $H^{2pm}(\lambda_p) \subset \overline{H}^{2pm}(\lambda_p)$. Thus, we get

$$\overline{H}^{2pm}(\lambda_p) = H^{2pm}(\lambda_p) \subset \overline{H}''(\lambda_p). \qquad \square$$

By means of these results, we are going to be able to investigate the subgroups $\overline{H}^{2pm}(\lambda_p)$. We have by Schreier's theorem the following theorem:

Theorem 3.9. *The subgroups $\overline{H}^{2pm}(\lambda_p)$ are free.*

Therefore,

$$\begin{aligned}\left|\overline{H}(\lambda_p) : \overline{H}^{2pm}(\lambda_p)\right| &= \left|\overline{H}(\lambda_p) : H^{2pm}(\lambda_p)\right| \\ &= \left|\overline{H}(\lambda_p) : H(\lambda_p)\right| . \left|H(\lambda_p) : H^{2pm}(\lambda_p)\right| \\ &= 2\left|H(\lambda_p) : H^{2pm}(\lambda_p)\right|\end{aligned}$$

since $\left|\overline{H}(\lambda_p) : H(\lambda_p)\right| = 2$.

Now we find the subgroups of low index of the extended Hecke groups $\overline{H}(\lambda_p)$. This will be done using the commutator subgroup $\overline{H}^{'}(\lambda_p)$ of $\overline{H}(\lambda_p)$.

Lemma 3.10. *There are exactly 3 normal subgroups of index 2 in $\overline{H}(\lambda_p)$. Explicitly these are $H(\lambda_p) = \langle T, S \mid T^2 = S^p = I\rangle \cong C_2 * C_p$, $\overline{H}_0(\lambda_p) = \langle R, S, TST \mid R^2 = S^p = (TST)^p = (RS)^2 = (RTST)^2 = I\rangle \cong D_p *_{\mathbf{Z}_2} D_p$ and $\alpha(H(\lambda_p)) = \langle TR, S \mid (TR)^2 = S^p = I\rangle \cong C_2 * C_p$.*

Proof. Let $N \triangleleft \overline{H}(\lambda_p)$ with $\left|\overline{H}(\lambda_p) : N\right| = 2$. Since $\overline{H}(\lambda_p)/N$ is abelian, we have $\overline{H}(\lambda_p) \supset N \supset \overline{H}^{'}(\lambda_p)$.

Now $\overline{H}(\lambda_p)/\overline{H}^{'}(\lambda_p) = C_2 \times C_2 = D_2$, a Klein 4−group. This has exactly 3 normal subgroups of index 2. Therefore these pull back to exactly 3 normal subgroups of index 2 in $\overline{H}(\lambda_p)$ containing $\overline{H}^{'}(\lambda_p)$. Since N contains $\overline{H}^{'}(\lambda_p)$, N must be one of these. □

Notice that these results coincide with the ones given in [**1**, p. 343, Proposition 3.5] for M*-groups when $q = 3$, i.e., $G_1 = \overline{H}_0(\lambda_3)$, $G_2 = H(\lambda_3)$, $G_3 = \alpha(H(\lambda_3))$ and $G_4 = \overline{H}^{'}(\lambda_3)$.

Lemma 3.11. *$\overline{H}(\lambda_p)$ has no normal subgroups of index 3.*

Proof. Suppose $N \lhd \overline{H}(\lambda_p)$ with $\left|\overline{H}(\lambda_p) : N\right| = 3$. Let $A = \overline{H}(\lambda_p)/N$ and so $|A| = 3$ and thus A is abelian. Therefore $N \supset \overline{H}'(\lambda_p)$ which is impossible since $\left|\overline{H}(\lambda_p) : \overline{H}'(\lambda_p)\right| = 4$. $\square$

Notice that Lemmas 3.10 and 3.11 also follow from (2.1) by looking at all possible homomorphisms onto cyclic groups of orders 2 and 3.

Lemma 3.12. *There are exactly* 2 *normal subgroups of index* $2p$ *in* $\overline{H}(\lambda_p)$. *Explicitly these are*

$$H^p(\lambda_p) = \langle T\rangle \star \langle STS^{p-1}\rangle \star \cdots \star \langle S^{p-1}TS\rangle$$

and

$$\overline{H}_2(\lambda_p) = \langle TR\rangle \star \langle RSTS\rangle \star \cdots \star \langle RS^{p-1}TS^{p-1}\rangle.$$

Notice that $\overline{H}_2(\lambda_p)$ is the principal congruence subgroup of the extended Hecke groups $\overline{H}(\lambda_p)$ and $H_2(\lambda_p)$ is the principal congruence subgroup of Hecke groups $H(\lambda_p)$, see [**13, 18**]. Also $\alpha(\overline{H}_2(\lambda_p)) = H^p(\lambda_p)$.

Therefore, we can form the following diagram which the relation between power subgroups and commutator subgroups of the extended Hecke group $\overline{H}(\lambda_p)$.

4. Free normal subgroups of $\overline{H}(\lambda_p)$. A well-known theorem of Karrass and Solitar, see [**14**], states that if N is a subgroup of G, a free product of two groups A, B with amalgamated subgroup of U, then N can be obtained by two constructions from the intersection of N and certain conjugates of A, B, and U. The constructions are those of a tree product (a special kind of generalized free product) and of a Higman-Neumann-Neumann extension. In the case G is an extended Hecke group, A and B are dihedral groups of order 4 and $2p$ and U is cyclic of order 2. Thus extended Hecke group $\overline{H}(\lambda_p)$ has two kinds of normal subgroups: Free ones and free products of some infinite cyclic groups, some cyclic groups of order 2 and order p, some dihedral groups D_2 and D_p with some dihedral groups D_{m_1} and D_{m_2} with amalgamation $\mathbf{Z}_2$ where m_i divides 2 or p. Therefore, the study of free normal subgroups

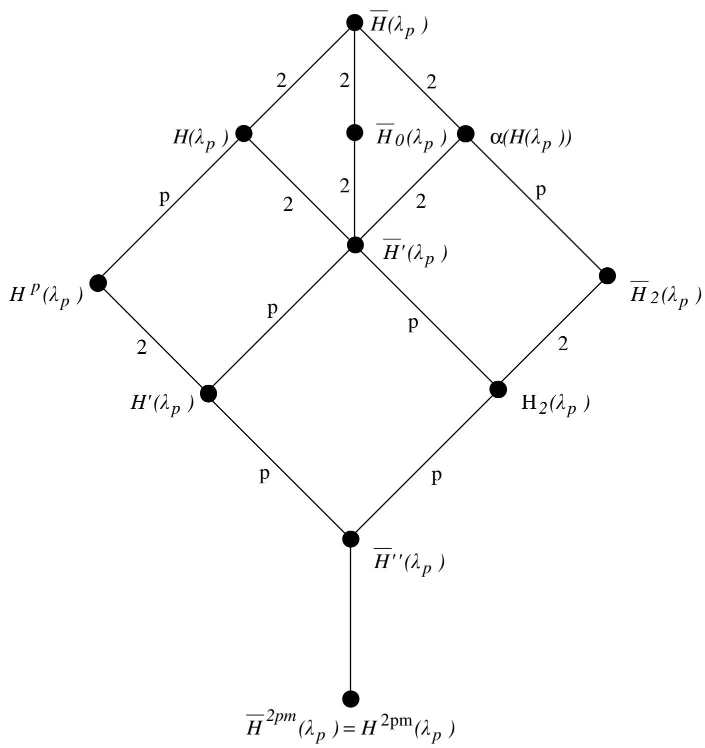

FIGURE 1. Some subgroups of $\overline{H}(\lambda_p)$.

and their group theoretical structures will be important to us. Here we discuss them for prime p. These have been done for modular group by Newman in [**16**] and for Hecke groups $H(\lambda_q)$, q prime, by Cangül in [**2**] and for extended modular group $\overline{H}(\lambda_3)$ by Sahin, Ikikardes and Koruoğlu in [**20**]. Their results can be generalized to the extended Hecke groups $\overline{H}(\lambda_p)$ with p prime number. When q is a composite odd number, it is possible to obtain similar results, however, it looks difficult to find all normal free subgroups in this case.

Before giving the main theorem we need the following lemmas.

Lemma 4.1. *Let N be a nontrivial normal subgroup of finite index in $\overline{H}(\lambda_p)$. Then N is free if and only if it contains no elements of finite order.*

Proof. By (2.1), $\overline{H}(\lambda_p)$ is isomorphic to a free product of D_2 and D_p each amalgamated over $\mathbf{Z}_2$. A subgroup of finite index in $\overline{H}(\lambda_p)$ is isomorphic to a free product of the groups F, C_r and $D_{m_1} *_{\mathbf{Z}_2} D_{m_2}$, where r and each m_i divide 2 or p. Thus, if N is a subgroup of finite index in $\overline{H}(\lambda_p)$, it follows that

$$N = F * \prod\nolimits_* C_r * \prod\nolimits_* (D_{m_1} *_{\mathbf{Z}_2} D_{m_2}) \tag{4.1}$$

where F is either free or $\{I\}$ and each C_r is conjugate to $\{T\}$ or to $\{S\}$ or to $\{R\}$ and each D_{m_i} is conjugate to $\{T,R\}$ or to $\{S,R\}$. As N contains no elements of finite order the free product $\prod_* C_r * \prod_* (D_{m_1} *_{\mathbf{Z}_2} D_{m_2})$ is vacuous, and also as N is nontrivial, N must be free.

Conversely, if N is free, then by definition, it contains no elements of finite order. $\square$

Notice that this lemma is true for all noncocompact NEC groups.

Lemma 4.2. *The only normal subgroups of finite index in $\overline{H}(\lambda_p)$ containing elements of finite order are*

$$\overline{H}(\lambda_p),\ H(\lambda_p),\ \overline{H}_0(\lambda_p),\ \alpha(H(\lambda_p)),\ H^2(\lambda_p), H^p(\lambda_p) \text{ and } \overline{H}_2(\lambda_p).$$

Proof. Let N be a normal subgroup of finite index in $\overline{H}(\lambda_p)$ containing an element of finite order. Then N contains an element of order 2 or an element of order p or two elements of order 2 or two elements of order 2 and p or three elements so that two elements of order 2 and an element of order p. From [**22**], we know that an element of order 2 in $\overline{\Gamma}$ is conjugate to T or to R or to TR and an element of order p in $\overline{\Gamma}$ is conjugate to S or to $S^2, \dots,$ or to $S^{(p-1)/2}$. Therefore if a normal subgroup N contains an element of finite order, then it contains T or R or TR or $S, S^2, \dots, S^{(p-1)/2}$. Therefore there are nine cases:

(i) If N contains T, R and one of $S, S^2, \dots, S^{(p-1)/2}$, then $N = \overline{H}(\lambda_p)$.

(ii) If N contains T and one of $S, S^2, \dots, S^{(p-1)/2}$, but not R and TR, then $\overline{H}(\lambda_p)/N \cong \langle r \mid r^2 = I\rangle \cong C_2$, as we have the relations $t^2 = s^p = r^2 = (rs)^2 = (tr)^2 = s^i = t = I$ where $1 \le i \le (p-1)/2$. Then by the Reidemeister-Schreier method we have $N = H(\lambda_p)$.

(iii) If N contains R and one of $S, S^2, \dots, S^{(p-1)/2}$, but not T and TR, then $\overline{H}(\lambda_p)/N \cong \langle t \mid t^2 = I\rangle \cong C_2$, as we have the relations $t^2 = s^p = r^2 = (rs)^2 = (tr)^2 = s^i = r = I$ where $1 \le i \le (p-1)/2$. Then by the Reidemeister-Schreier method we have $N = \overline{H}_0(\lambda_p)$.

(iv) If N contains TR and one of $S, S^2, \dots, S^{(p-1)/2}$, but not T and R, then $\overline{H}(\lambda_p)/N \cong \langle t \mid t^2 = I\rangle \cong C_2$, as, this time, we have the relations $t^2 = s^p = r^2 = (rs)^2 = (tr)^2 = s^i = (tr) = I$ where $1 \le i \le (p-1)/2$. Similarly $N = H(\lambda_p)\alpha$.

(v) If N contains T and R but not $S, S^2, \dots, S^{(p-1)/2}$, then $\overline{H}(\lambda_p)/N \cong C_1$. Therefore we get $N = \overline{H}(\lambda_p)$. Thus this case is impossible.

(vi) If N contains one of $S, S^2, \dots, S^{(p-1)/2}$ but not T and R, then $\overline{H}(\lambda_p)/N \cong \langle t, r \mid t^2 = r^2 = (tr)^2 = I\rangle \cong D_2$. Then by the Reidemeister-Schreier method we have $N = H^2(\lambda_p)$.

(vii) If N contains T but not R, $S, S^2, \dots, S^{(p-1)/2}$, then $\overline{H}(\lambda_p)/N \cong \langle s, r \mid s^p = r^2 = (sr)^2 = I\rangle \cong D_p$ as we have the relations $t^2 = s^p = r^2 = (rs)^2 = (tr)^2 = t = I$. Then by the Reidemeister-Schreier method we have $N = H^p(\lambda_p)$.

(viii) If N contains TR but not T, R and $S, S^2, \dots, S^{(p-1)/2}$, then $\overline{H}(\lambda_p)/N \cong \langle t, s \mid t^2 = s^p = (ts)^2 = I\rangle \cong D_p$ as we have the relations $t^2 = s^p = r^2 = (rs)^2 = (tr)^2 = (tr) = I$. Then by the Reidemeister-Schreier method we have $N = \overline{H}_2(\lambda_p)$.

(ix) If N contains R but not T, $S, S^2, \dots, S^{(p-1)/2}$, then $\overline{H}(\lambda_p)/N \cong \langle t \mid t^2 = I\rangle \cong C_2$. Therefore we have $N = \overline{H}_0(\lambda_p)$. Thus this case is impossible. □

Theorem 4.3. *Let N be a nontrivial normal subgroup of finite index in $\overline{H}(\lambda_p)$ different from $\overline{H}(\lambda_p)$, $H(\lambda_p)$, $\overline{H}_0(\lambda_p)$, $\alpha(H(\lambda_p))$, $H^2(\lambda_p)$, $H^p(\lambda_p)$ and $\overline{H}_2(\lambda_p)$. Then N is a free group.*

Proof. It is clear by Lemma 4.1 and Lemma 4.2. □

Theorem 4.4. *Let N be a normal subgroup of finite index in $\overline{H}(\lambda_p)$ different from $\overline{H}(\lambda_p)$, $H(\lambda_p)$, $\overline{H}_0(\lambda_p)$, $\alpha(H(\lambda_p))$, $H^2(\lambda_p)$, $H^p(\lambda_p)$ and $\overline{H}_2(\lambda_p)$ such that $\left|\overline{H}(\lambda_p) : N\right| = \mu < \infty$. Then μ is divisible by $4p$.*

Proof. The quotient group contains subgroups of orders 2, 4 and $2p$, so its order is divisible by $4p$. □

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