Turk J Math
31 (2007) , 171 – 179.

On P-Sasakian Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor

Cihan Özgür and Mukut Mani Tripathi

Abstract

We classify P-Sasakian manifolds, which satisfy the conditions $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot S = 0$ and $Z(\xi, X) \cdot C = 0$.

Key Words: P-Sasakian manifold, concircular curvature tensor, Weyl conformal curvature tensor.

1. Introduction

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is Levi-Civita connection of the Riemannian metric. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0, \qquad X, Y \in TM,$$

where $R(X, Y)$ acts on R as a derivation.

Locally symmetric and semisymmetric P-Sasakian manifolds are studied in [2] and [5]. After the curvature tensor, the Weyl conformal curvature tensor C and the concircular curvature tensor Z are the next most important tensors. In this paper, we study several derivation conditions on P-Sasakian manifolds. The paper is organized as follows. In

2000 *Mathematics Subject Classification:* 53C25

section 2, we give a brief account of P-Sasakian manifolds, the Weyl conformal curvature tensor and the concircular curvature tensor. In section 3, we find necessary and sufficient conditions for P-Sasakian manifolds satisfying the conditions like $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot S = 0$ and $Z(\xi, X) \cdot C = 0$. In Section 4, we prove that for an n-dimensional P-Sasakian manifold M the following three statements are equivalent: (a) M is locally symmetric, (b) M is concircularly symmetric and (c) M is locally isometric to the Hyperbolic space $H^n(-1)$.

2. P-Sasakian Manifolds

An n-dimensional differentiable manifold M is called an *almost paracontact manifold* if it admits an almost paracontact structure (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{2.2}$$

or equivalently,

$$g(X, \varphi Y) = g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in TM$. Then, M becomes an *almost paracontact Riemannian manifold* equipped with an almost paracontact Riemannian structure (φ, ξ, η, g).

An almost paracontact Riemannian manifold is called a *P-Sasakian manifold* if it satisfies

$$(\nabla_X \varphi) Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \qquad X, Y \in TM, \tag{2.4}$$

where ∇ is Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla\xi = \varphi, \tag{2.5}$$

$$(\nabla_X \eta) Y = g(X, \varphi Y) = (\nabla_Y \eta) X, \qquad X \in TM. \tag{2.6}$$

In an n-dimensional P-Sasakian manifold M, the curvature tensor R, the Ricci tensor S, and the Ricci operator Q satisfy

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$S(X,\xi) = -(n-1)\,\eta(X), \tag{2.10}$$

$$Q\xi = -(n-1)\,\xi, \tag{2.11}$$

$$\eta(R(X,Y)U) = g(X,U)\eta(Y) - g(Y,U)\eta(X), \tag{2.12}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.13}$$

$$\eta(R(\xi,X)Y) = \eta(X)\eta(Y) - g(X,Y). \tag{2.14}$$

An almost paracontact Riemannian manifold M is said to be *η-Einstein* [2] if the Ricci operator Q satisfies

$$Q = a\,Id + b\,\eta \otimes \xi, \tag{2.15}$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold. For more details about almost paracontact Riemannian manifolds we refer to [2], [6] and [7].

Let (M,g) be an n-dimensional Riemannian manifold. Then the *concircular curvature tensor* Z and the *Weyl conformal curvature tensor* C are defined by [9]

$$Z\,(X,Y)\,U = R\,(X,Y)\,U - \frac{r}{n(n-1)}\,(g(Y,U)X - g(X,U)Y)\,, \tag{2.16}$$

$$\begin{aligned} C(X,Y)U \quad = \quad & R(X,Y)U - \frac{1}{n-2}\,\{S(Y,U)X - S(X,U)Y \\ & + g(Y,U)QX - g(X,U)QY\} \\ & + \frac{r}{(n-1)(n-2)}\,\{g(Y,U)X - g(X,U)Y\} \end{aligned} \tag{2.17}$$

for all $X,Y,U \in TM$, respectively, where r is the scalar curvature of M.

3. Main Results

In this section, we obtain necessary and sufficient conditions for P-Sasakian manifolds satisfying the derivation conditions $Z(\xi,X)\cdot Z=0$, $Z(\xi,X)\cdot R=0$, $R(\xi,X)\cdot Z=0$, $Z(\xi,X)\cdot S=0$ and $Z(\xi,X)\cdot C=0$.

Theorem 3.1 *An n-dimensional P-Sasakian manifold M satisfies*

$$Z(\xi,X)\cdot Z=0$$

if and only if either the scalar curvature r of M is $r=n(1-n)$ or M is locally isometric to the Hyperbolic space $H^n(-1)$.

Proof. In a P-Sasakian manifold M, we have

$$Z(X,Y)\xi=\left(1-\frac{r}{n(n-1)}\right)(\eta(Y)X-\eta(X)Y)\,, \tag{3.18}$$

$$Z(\xi,X)Y=\left(1-\frac{r}{n(n-1)}\right)(g(X,Y)\xi-\eta(Y)X)\,. \tag{3.19}$$

The condition $Z(\xi,U)\cdot Z=0$ implies that

$$0=[Z(\xi,U),Z(X,Y)]\,\xi-Z(Z(\xi,U)X,Y)\xi-Z(X,Z(\xi,U)Y)\xi,$$

which in view of (3.19) gives

$$\begin{aligned}0 \;=\; & \left(1+\frac{r}{n(n-1)}\right)\{-g(U,Z(X,Y)\xi)\xi+\ g(U,X)Z(\xi,Y)\xi\\ & -\eta(X)Z(U,Y)\xi+g(U,Y)Z(X,\xi)\xi\\ & -\ \eta(Y)Z(X,U)\xi+\eta(U)Z(X,Y)\xi-Z(X,Y)U\}\,.\end{aligned}$$

Equation (3.18) then gives

$$\left(1+\frac{r}{n(n-1)}\right)\left(Z(X,Y)U-\left(1+\frac{r}{n(n-1)}\right)(g(Y,U)X-g(X,U)Y)\right)=0.$$

Therefore either the scalar curvature $r=n(1-n)$ or

$$Z(X,Y)U-\left(1-\frac{r}{n(n-1)}\right)(g(Y,U)X-g(X,U)Y)=0$$

which in view of (2.16) gives

$$R\,(X,Y)\,U = g(U,X)Y - g(U,Y)X.$$

The above equation implies that M is of constant curvature -1 and consequently it is locally isometric to the Hyperbolic space $H^n(-1)$.

Conversely, if M has scalar curvature $r = n(1-n)$ then from (3.19) it follows that $Z(\xi, X) = 0$. Similarly, in the second case, since M is of constant curvature $r = n\,(1-n)$, therefore we again get $Z(\xi, X) = 0$. □

Using the fact that $Z(\xi, X) \cdot R$ denotes $Z(\xi, X)$ acting on R as a derivation, we have the following Theorem as a corollary of Theorem 3.1.

Theorem 3.2 *An n-dimensional P-Sasakian manifold M satisfies*

$$Z(\xi, X) \cdot R = 0$$

if and only if either M is locally isometric to the Hyperbolic space $H^n(-1)$ or M has constant scalar curvature $r = n(1-n)$.

Proposition 3.3 *Let (M, g) be an n-dimensional Riemannian manifold. Then $R \cdot Z = R \cdot R$.*

Proof. Let $X, Y, U, V, W \in TM$. Then

$$\begin{aligned}(R(X,Y)\cdot Z)(U,V,W) &= R(X,Y)Z(U,V)W - Z(R(X,Y)U,V)W \\ &\quad - Z(U,R(X,Y)V)W - Z(U,V)R(X,Y)W.\end{aligned}$$

So from (2.16) and the symmetry properties of the curvature tensor R we have

$$\begin{aligned}(R(X,Y)\cdot Z)(U,V,W) &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ &= (R(X,Y)\cdot R)(U,V,W),\end{aligned}$$

which proves the proposition. □

Now, in view of Theorem 2.1 of [2] and Proposition 3.3 we have the following theorem:

Theorem 3.4 *An n-dimensional P-Sasakian manifold M satisfies*

$$R(\xi, X) \cdot Z = 0$$

if and only if M is locally isometric to the Hyperbolic space $H^n(-1)$.

Next, we prove the following

Theorem 3.5 *An n-dimensional P-Sasakian manifold M satisfies*

$$Z(\xi, X) \cdot S = 0$$

if and only if either M has scalar curvature $r = n(1-n)$ or M is an Einstein manifold with the scalar curvature $r = n(1-n)$.

Proof. The condition $Z(\xi, X) \cdot S = 0$ implies that

$$S(Z(\xi, X)Y, \xi) + S(Y, Z(\xi, X)\xi) = 0,$$

which in view of (3.19) gives

$$0 = \left(1 + \frac{r}{n(n-1)}\right)\left(-g(X,Y)S(\xi,\xi) + \eta(Y)S(X,\xi) - \eta(X)S(Y,\xi) + S(X,Y)\right).$$

So by the use of (2.10) we have

$$\left(1 + \frac{r}{n\,(n-1)}\right)(S - (1-n)g) = 0.$$

Therefore either the scalar curvature r of M is $r = n(1-n)$ which is of constant or $S = (1-n)g$ which implies that M is an Einstein manifold with the scalar curvature $r = n(1-n)$. The converse statement is trivial. □

Theorem 3.6 *An n-dimensional P-Sasakian manifold M satisfies*

$$Z(\xi, X) \cdot C = 0$$

if and only if either M has scalar curvature $r = n(1-n)$ or M is conformally flat, in which case M is a SP-Sasakian manifold.

Proof. $Z(\xi, U) \cdot C = 0$ implies that

$$0 = [Z(\xi, U), C(X, Y)]W - C(Z(\xi, U)X, Y)W - C(X, Z(\xi, U)Y)W,$$

which in view of (3.19) we have

$$\begin{aligned} 0 &= (1 + \frac{r}{n(n-1)})[\eta(C(X,Y)W)U - C(X,Y,W,U)\xi - \eta(X)C(U,Y)W \\ &+ g(U,X)C(\xi,Y)W - \eta(Y)C(X,U)W + g(U,Y)C(X,\xi)W \\ &- \eta(W)C(X,Y)U + g(U,W)C(X,Y)\xi]. \end{aligned}$$

So either the scalar curvature of M is $r = n(1-n)$ or the equation

$$\begin{aligned} 0 &= \eta(C(X,Y)W)U - C(X,Y,W,U)\xi - \eta(X)C(U,Y)W \\ &+ g(U,X)C(\xi,Y)W - \eta(Y)C(X,U)W + g(U,Y)C(X,\xi)W \\ &- \eta(W)C(X,Y)U + g(U,W)C(X,Y)\xi \end{aligned}$$

holds on M. Taking the inner product of the last equation with ξ we get

$$\begin{aligned} 0 &= \eta(C(X,Y)W)\eta(U) - C(X,Y,W,U) \\ &\quad -\eta(X)\eta(C(U,Y)W) + g(U,X)\eta(C(\xi,Y)W) - \eta(Y)\eta(C(X,U)W) \\ &\quad +g(U,Y)\eta(C(X,\xi)W) - \eta(W)\eta(C(X,Y)U). \end{aligned} \tag{3.20}$$

Hence using (2.10), (2.12) and (2.17) the equation (3.20) turns the form

$$\begin{aligned} 0 &= g(U,Y)g(X,W) - g(U,X)g(Y,W) \\ &\quad +\frac{1-n}{n-2}\{-g(Y,W)g(X,U) + g(X,W)g(U,Y) \\ &\quad +g(X,U)\eta(Y)\eta(W) - g(U,Y)\eta(X)\eta(W)\} \\ &\quad +\frac{1}{n-2}\{S(Y,U)\eta(X)\eta(W) - S(X,U)\eta(Y)\eta(W) \\ &\quad +g(Y,W)S(X,U) - g(X,W)S(Y,U)\} - R(X,Y,W,U). \end{aligned} \tag{3.21}$$

Hence by a suitable contraction of (3.21) we have

$$S(Y,W) = (1 + \frac{r}{n-1})g(Y,W) + (-n + \frac{r}{1-n})\eta(Y)\eta(W), \tag{3.22}$$

which implies that M is an η-Einstein manifold. So using (3.22) in (3.20) we obtain $C = 0$ on M. Thus using the fact from [1] that a conformally flat P-Sasakian manifold is an SP-Sasakian, M becomes an SP-Sasakian manifold. The converse statement is trivial.

□

4. An application

A Riemannian manifold is said to be *concircularly symmetric* if the concircular curvature tensor Z is parallel, that is, $\nabla Z = 0$. Now, we prove the following theorem.

Theorem 4.1 *In a P-Sasakian manifold M the following conditions are equivalent:*

(a) *M is locally symmetric,*

(b) *M is concircularly symmetric,*

(c) *M is locally isometric to the Hyperbolic space $H^n(-1)$.*

Proof. It is obvious that the condition $\nabla T = 0$, $T \in \{R, Z\}$, implies the condition $R \cdot T = 0$. From Theorem 2.1 of [2] and Theorem 3.4, it follows that M satisfies the condition $R(\xi, X) \cdot T = 0$, $T \in \{R, Z\}$ if and only if M is locally isometric to the Hyperbolic space $H^n(-1)$. □

References

[1] Adati, T. and Miyazawa, T.: Some properties of P-Sasakian manifolds , TRU Math. 13, 25-32 (1977).

[2] Adati, T. and Miyazawa, T.: On P-Sasakian manifolds satisfying certain conditions, Tensor (N.S.) 33, 173-178 (1979).

[3] Blair, D. E., Kim, J. S and Tripathi, M. M.: On the concircular curvature tensor of a contact metric manifold, J. Korean Math. Soc. 42, no. 5, 883-892 (2005).

[4] De, U. C. and Pathak, G.: On P-Sasakian manifolds satisfying certain conditions, J. Indian Acad. Math. 16, 72-77 (1994).

[5] De, U. C. and Tarafdar, D.: On a type of P-Sasakian manifold, Math. Balkanica (N.S.) 7, 211-215 (1993).

[6] Satō, I.: On a structure similar to the almost contact structure, Tensor (N.S.) 30, no. 3, 219-224 (1976).

[7] Satō, I. and Matsumoto, K.: On P-Sasakian manifolds satisfying certain conditions, Tensor (N.S.) 33, 173-178 (1979).

[8] Tripathi, M. M. and Kim, J. S.: On the concircular curvature tensor of a (κ, μ)-manifold, Balkan J. Geom. Appl. 9, no.1, 104-114 (2004).

[9] Yano, K. and Kon, M.: Structures on Manifolds, Series in Pure Math., Vol 3, World Sci., 1984.

Cihan ÖZGÜR
Department of Mathematics
Balıkesir University
10145, Çağış, Balıkesir-TURKEY
e-mail: cozgur@balikesir.edu.tr

Received 06.10.2005

Mukut Mani TRIPATHI
Department of Mathematics and Astronomy
Lucknow University
Lucknow-226 007, INDIA
e-mail: mmtripathi66@yahoo.com