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POWER SUBGROUPS OF SOME HECKE GROUPS II

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Abstract. Let $q \geq 3$ be an odd integer and let $H(\lambda_q)$ be the Hecke group associated to q. Let m be a positive integer and $H^m(\lambda_q)$ be the m-th power subgroup of $H(\lambda_q)$. In this work, the power subgroups $H^m(\lambda_q)$ are discussed. The Reidemeister-Schreier method and the permutation method are used to obtain the abstract group structure and generators of $H^m(\lambda_q)$; their signatures are then also determined. A similar result on the Hecke groups $H(\lambda_q)$, q prime, which says that $H'(\lambda_q) \cong H^2(\lambda_q) \cap H^q(\lambda_q)$, is generalized to Hecke groups $H(\lambda_q)$ with $q \geq 3$ odd integer.

1. Introduction

In [6], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

$PSL(2,\mathbb{R})$ denotes the group of orientation preserving isometries of the upper half plane. A Fuchsian group is a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$. It is well known that every Fuchsian group has a presentation of the following type:

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Generators	$a_1, b_1, ..., a_g, b_g$	(hyperbolic)
	$x_1, ..., x_r$	(elliptic)
	$p_1, ..., p_t$	(parabolic)
	$h_1, ..., h_u$	(hyperbolic boundary)
Relations	$x_j^{m_j} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{t} p_k \prod_{l=1}^{u} h_l = 1$	

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator of a_i and b_i. We then say that the group has signature $(g; m_{1,...,} m_r; t; u)$. Here g is the genus of the Riemann surface corresponding to the group and m_i are the integers greater than 1, called the periods of the group. Most Fuchsian groups including Hecke groups have no hyperbolic boundary elements, therefore we take u = 0, and omit it in the signatures.

E. Hecke showed that H(λ) is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, for $q = 3, 4, 5, \dots$, or $\lambda \geq 2$. We are going to be interested in the former case. These groups have come to be known as the *Hecke groups*, and we will denote them by $H(\lambda_q)$, for $q \geq 3$. Then the Hecke group $H(\lambda_q)$ is the discrete group generated by T and S, and it is isomorphic to the free product of two finite cyclic groups of orders 2 and q. $H(\lambda_q)$ has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 \star C_q, \ [3]. \tag{1.1}$$

Note that the Hecke groups $H(\lambda_q)$ can be thought of as triangle groups having an infinity as one of the entries. Coxeter and Moser [4] have shown that the triangle group $(g; k, l, m)$ is finite when $(1/k + 1/l + 1/m) > 1$ and infinite when $(1/k + 1/l + 1/m) \leq 1$. Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is they are infinite triangle groups. The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$.

Hecke groups $H(\lambda_q)$ and their normal subgroups have been extensively studied for many aspects in the literature, [1], [7], [11]. The Hecke group $H(\lambda_3)$, the modular group $PSL(2, \mathbb{Z})$, and its normal subgroups have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory and group theory [8], [10].

Let m be a positive integer. Let us define $H^m(\lambda_q)$ to be the subgroup generated by the m^{th} powers of all elements of $H(\lambda_q)$. The subgroup $H^m(\lambda_q)$ is called the *m-th power subgroup* of $H(\lambda_q)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

From the definition one can easily deduce that

$$H^m(\lambda_q) > H^{mk}(\lambda_q) \tag{1.2}$$

and that

$$(H^m(\lambda_q))^k > H^{mk}(\lambda_q).$$

Using the last two inequalities imply that

$$H^m(\lambda_q).H^k(\lambda_q) = H^{(m,k)}(\lambda_q)$$

where (m,k) denotes the greatest common divisor of m and k.

The power subgroups of the modular group $H(\lambda_3)$ have been studied and classified in [8], [9] by Newman. In [8], Newman showed that

$$H'(\lambda_3) = H^2(\lambda_3) \cap H^3(\lambda_3)$$

where $H'(\lambda_3)$ is called *the commutator subgroup* of the modular group $H(\lambda_3)$. In fact, it is a well-known [9] and important result that the only normal subgroups of $H(\lambda_3)$ with torsion are $H(\lambda_3)$, $H^2(\lambda_3)$ and $H^3(\lambda_3)$ of indices 1, 2, 3 respectively. These results have been generalized to Hecke groups $H(\lambda_q)$, q prime, by Cangül and Singerman in [3]. They proved that

$$H'(\lambda_q) = H^2(\lambda_q) \cap H^q(\lambda_q) \text{ (also see [2])}$$

and if q is prime then the only normal subgroups of $H(\lambda_q)$ with torsion are $H(\lambda_q)$, $H^2(\lambda_q)$ and $H^q(\lambda_q)$ of indices 1, 2, q, respectively.

The power subgroups of the Hecke groups $H(\lambda_q)$, $q \geq 4$ even integer, were investigated by İkikardes, Koruoğlu and Sahin in [5]. Also in [11], [12], [13] and [14], Schmidt and Sheingorn used the results related to the power subgroups of some Hecke groups $H(\lambda_q)$.

In this work we compute the group structure of certain Fuchsian groups, the power subgroups of the odd-indexed subfamily of the Hecke triangle groups. We achieve this by applying standard techniques of combinatorial group theory (The Reidemeister-Schreier method and the permutation method). Also we give the signatures of $H^m(\lambda_q)$, of finite index, as all of them are not necessarily of finite index, and we proved that for $q \geq 3$ odd integer,

$$H'(\lambda_q) = H^2(\lambda_q) \cap H^q(\lambda_q).$$

Finally we have made corrections to the findings by Sheingorn in [14].

2. Structure of Power Subgroups of $H(\lambda_q)$

Now we consider the presentation of the Hecke group $H(\lambda_q)$ given in (1.1):

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I > .$$

Firstly we find a presentation for the quotient $H(\lambda_q)/H^m(\lambda_q)$ by adding the relation $X^m = I$ for all $X \in H(\lambda_q)$ to the presentation of $H(\lambda_q)$. The order of $H(\lambda_q)/H^m(\lambda_q)$ gives us the index which is finite by our choice. We have

$$H(\lambda_q)/H^m(\lambda_q) \cong< T, S \mid T^2 = S^q = T^m = S^m = (TS)^m = ... = I > . \quad (2.1)$$

Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups $H^m(\lambda_q)$, $q \geq 3$ odd integer. The idea is as follows: We first choose (not uniquely) a Schreier transversal Σ for $H^m(\lambda_q)$. (This method, in general, applies to all normal subgroups of finite index). Σ consists of certain words in T and S. Then we take all possible products in the following order:

$$\text{(An element of } \Sigma) \times \text{(A generator of } H(\lambda_q))$$
$$\times \text{(coset representative of the preceding product)}^{-1}$$

We now discuss the group theoretical structure of these subgroups. First we begin with the case $m = 2$:

Theorem 2.1. *Let $q \geq 3$ be an odd integer. The normal subgroup $H^2(\lambda_q)$ is the free product of two finite cyclic groups of order q. Also*

$$H(\lambda_q)/H^2(\lambda_q) \cong C_2,$$
$$H(\lambda_q) = H^2(\lambda_q) \cup T\ H^2(\lambda_q)$$

and

$$H^2(\lambda_q) =< S > \star < TST > .$$

The elements of $H^2(\lambda_q)$ can be characterized by the requirement that the sum of the exponents of T are even.

Proof. By (2.1), we have

$$H(\lambda_q)/H^2(\lambda_q) \cong< T, S \mid T^2 = S^q = T^2 = S^2 = (TS)^2 = ... = I > .$$

Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups $H^2(\lambda_q)$. We have

$$H(\lambda_q)/H^2(\lambda_q) \cong< T \mid T^2 = I >,$$

since $S^2 = S^q = I$ and $(m, q) = 1$. Thus we get

$$H(\lambda_q)/H^2(\lambda_q) \cong C_2,$$

and

$$\left|H(\lambda_q) : H^2(\lambda_q)\right| = 2.$$

Now we choose I, T. Hence, all possible products are

$$\begin{array}{ll} I.T.(T)^{-1} = I & I.S.(I)^{-1} = S \\ T.T.(I)^{-1} = I & T.S.(T)^{-1} = TST^{-1} \end{array}$$

Since $T^{-1} = T$, the generators of $H^2(\lambda_q)$ are S, TST. Thus $H^2(\lambda_q)$ has a presentation

$$H^2(\lambda_q) = < S > \star < TST >$$

and we get

$$H(\lambda_q) = H^2(\lambda_q) \cup TH^2(\lambda_q).$$

Let us now we use the permutation method (see [15]) to find the signature of $H^2(\lambda_q)$. We consider the homomorphism

$$H(\lambda_q) \to H(\lambda_q)/H^2(\lambda_q) \cong C_2.$$

Here T is mapped to an element of order two and S is mapped to the identity. Hence TS is mapped to an element of order two. Then they have the following permutation representations :

$$\begin{array}{rcl} T & \to & (1\ 2), \\ S & \to & (1)\ (2), \\ TS & \to & (1\ 2). \end{array}$$

Therefore the signature of $H^2(\lambda_q)$ is $(g;\, q,\, q,\, \infty,) = (g;\, q^{(2)},\, \infty)$. Now by the Riemann-Hurwitz formula, $g = 0$. Thus we obtain $H^2(\lambda_q) = (0;\, q^{(2)},\, \infty)$. □

Notice that this result coincides with the group $H^2(\lambda_q)$ given in [14] for the Hecke groups $H(\lambda_q)$. The formula for the signature of $H^2(\lambda_q)$ in [14] is not correct, in general, because the signature of $H^2(\lambda_q)$ is $(0;\, (q/2)^{(2)},\, \infty^{(2)})$ only when q even (see [5]).

Now we have generally the following result:

Corollary 1. *Let $q \geq 3$ an odd integer and let m be a positive integer such that $(m, 2) = 2$ and $(m, q) = 1$. The normal subgroup $H^m(\lambda_q)$ is isomorphic to the normal subgroup $H^2(\lambda_q)$, i.e.,*

$$H^m(\lambda_q) \cong H^2(\lambda_q).$$

Theorem 2.2. *Let $q \geq 3$ an odd integer. The normal subgroup $H^q(\lambda_q)$ is the free product of q finite cyclic groups of order 2. Also*

$$H(\lambda_q)/H^q(\lambda_q) \cong C_q,$$
$$H(\lambda_q) = H^q(\lambda_q) \cup S\ H^q(\lambda_q) \cup S^2\ H^q(\lambda_q) \cup \cdots \cup S^{q-1}\ H^q(\lambda_q),$$

and

$$H^q(\lambda_q) =< T > \star < STS^{-1} > \star < S^2TS^{-2} > \star \cdots \star < S^{q-1}TS > .$$

The elements of $H^q(\lambda_q)$ can be characterized by the requirement that the sum of the exponents of S are even.

Proof. By (2.1), we obtain

$$H(\lambda_q)/H^q(\lambda_q) \cong < S \mid S^q = I > \cong C_q,$$

from the relations $T^2 = T^q = I$ and as $(2, q) = 1$. Thus

$$|H(\lambda_q) : H^q(\lambda_q)| = q.$$

Therefore we choose $\{I,\ S,\ S^2, ..., S^{q-1}\}$ as a Schreier transversal for $H^q(\lambda_q)$. According to the Reidemeister-Schreier method, we can form all possible products:

$$\begin{array}{ll} I.T.(I)^{-1} = T, & I.S.(S)^{-1} = I, \\ S.T.(S)^{-1} = STS^{-1}, & S.S.(S^2)^{-1} = I, \\ S^2.T.(S^2)^{-1} = S^2TS^{-2}, & S^2.S.(S^3)^{-1} = I, \\ \vdots & \vdots \\ S^{q-1}.T.(S^{q-1})^{-1} = S^{q-1}TS, & S^{q-1}.S.(S^q)^{-1} = I. \end{array}$$

The generators are T, STS^{-1}, S^2TS^{-2}, ..., $S^{q-1}TS$. Thus $H^q(\lambda_q)$ has a presentation

$$H^q(\lambda_q) =< T > \star < STS^{-1} > \star < S^2TS^{-2} > \star \cdots \star < S^{q-1}TS > .$$

Now consider the homomorphism

$$H(\lambda_q) \to H(\lambda_q)/H^q(\lambda_q) \cong C_q.$$

Here T is mapped to the identity and S is mapped to an element of order q. Hence TS is mapped to an element of order q as well. Then they have the following permutation representations :

$$\begin{array}{rcl} T & \to & (1)\ (2)\ ...\ (q) \\ S & \to & (1\ 2\ ...\ q) \\ TS & \to & (1\ 2\ ...\ q) \end{array}$$

Therefore $H^q(\lambda_q)$ has the signature $(0; 2^{(q)}, \infty)$ similarly to the previous cases. □

Notice that this result coincides with the group Γ_q given in [12] for the Hecke groups $H(\lambda_q)$. In [12], q must be odd integer ≥ 3, otherwise Γ_q has not the signature $(0; 2^{(q)}, \infty)$. Therefore Γ_q is not the analog of Γ^3.

Theorem 2.3. *Let $q \geq 3$ an odd integer and let m be a positive integer such that $(m, 2) = 1$ and $(m, q) = d$. The normal subgroup $H^q(\lambda_q)$ is the free product of d finite cyclic groups of order two and the finite cyclic group of order q/d. Also*

$$H(\lambda_q)/H^m(\lambda_q) \cong C_d,$$
$$H(\lambda_q) = H^m(\lambda_q) \cup SH^m(\lambda_q) \cup S^2 H^m(\lambda_q) \cup ... \cup S^{d-1} H^m(\lambda_q),$$

and

$$H^m(\lambda_q) =< T > \star < STS^{q-1} > \star < S^2TS^{q-2} > \star ... \star < S^{d-1}TS^{q-d+1} > \star < S^d > .$$

Proof. If $(m, 2) = 1$ and $(m, q) = d$, then by (2.1), we find

$$H(\lambda_q)/H^m(\lambda_q) \cong < S \mid S^d = I > \cong C_d$$

from the relations $T^2 = T^m = I$ and $S^q = S^m = I$. Thus

$$|H(\lambda_q) : H^m(\lambda_q)| = d.$$

Therefore we choose $\{I,\ S,\ S^2, ...,\ S^{d-1}\}$ as a Schreier transversal for $H^m(\lambda_q)$. According to the Reidemeister-Schreier method, we can form all possible products:

$$\begin{array}{ll} I.T.(I)^{-1} = T, & I.S.(S)^{-1} = I, \\ S.T.(S)^{-1} = STS^{q-1}, & S.S.(S^2)^{-1} = I, \\ S^2.T.(S^2)^{-1} = S^2TS^{q-2}, & S^2.S.(S^3)^{-1} = I, \\ \vdots & \vdots \\ S^{d-1}.T.(S^{d-1})^{-1} = S^{d-1}TS^{q-d+1}, & S^{d-1}.S.(I)^{-1} = S^d. \end{array}$$

The generators are T, S^d, STS^{q-1}, S^2TS^{q-2}, . . . , $S^{d-1}TS^{q-d+1}$. Thus $H^m(\lambda_q)$ has a presentation

$$H^m(\lambda_q) =< T > \star < STS^{q-1} > \star < S^2TS^{q-2} > \star ... \star < S^{d-1}TS^{q-d+1} > \star < S^d >$$

and we get

$$H(\lambda_q) = H^m(\lambda_q) \cup SH^m(\lambda_q) \cup S^2 H^m(\lambda_q) \cup ... \cup S^{d-1} H^m(\lambda_q).$$

Now we consider the homomorphism

$$H(\lambda_q) \to H(\lambda_q)/H^m(\lambda_q) \cong C_d.$$

Here T is mapped to the identity and S is mapped to an element of order d. Hence TS is mapped to an element of order d as well. Then they have the following permutation representations :

$$\begin{aligned} T &\rightarrow (1)\ (2)\ ...\ (d) \\ S &\rightarrow (1\ 2\ ...\ d) \\ TS &\rightarrow (1\ 2\ ...\ d) \end{aligned}$$

Therefore the group $H^m(\lambda_q)$ has the signature $(0; 2^{(d)}, q/d, \infty)$. □

Corollary 2. *Let $q \geq 3$ be an odd integer and let m be a positive odd integer such that $(m, q) = 1$. Then*

$$H^m(\lambda_q) = H(\lambda_q).$$

Now we are only left to consider the case where $(m, 2) = 2$ and $(m, q) = d > 2$. Then in $H(\lambda_q)/H^m(\lambda_q)$ we have the relations $t^2 = s^d = (ts)^m$, where t, s and ts are the images of T, S, and TS, respectively, under the homomorphism of $H(\lambda_q)$ to $H(\lambda_q)/H^m(\lambda_q)$. Then the order of the factor group is unknown. Therefore the above techniques do not say much about $H^m(\lambda_q)$ in this case apart from the fact they are all normal subgroups with torsion.

We also require the structure of the commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$. This is well known (see [2], [11]), and we have

Lemma 1. *The commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$ is isomorphic to a free group of rank $q - 1$. Also*

$$\begin{aligned} |H(\lambda_q) : H'(\lambda_q)| &= 2q, \\ H(\lambda_q) &= H'(\lambda_q) \cup T\ H'(\lambda_q) \cup S\ H'(\lambda_q) \cup ... \\ &\quad \cup S^{q-1}\ H'(\lambda_q) \cup TS\ H'(\lambda_q) \cup ... \cup TS^{q-1}\ H'(\lambda_q) \end{aligned}$$

and $H'(\lambda_q) =< STS^{q-1}T > \star < S^2TS^{q-2}T > \star \ldots \star < S^{q-1}TST >$.

Let

$$a_1 = STS^{q-1}T,\ a_2 = S^2TS^{q-2}T, \ldots, a_{q-1} = S^{q-1}TST.$$

Note that since q is odd the quotient groups $H(\lambda_q)/H^2(\lambda_q)$ and $H(\lambda_q)/H^q(\lambda_q)$ are cyclic and therefore abelian so that

$$H^2(\lambda_q) > H'(\lambda_q),\ H^q(\lambda_q) > H'(\lambda_q).$$

Hence

$$H^2(\lambda_q) \cap H^q(\lambda_q) > H'(\lambda_q).$$

Since $H^2(\lambda_q)$ and $H^q(\lambda_q)$ are normal subgroups of $H(\lambda_q)$, we have, by one of the isomorphism theorems, that

$$H^2(\lambda_q).H^q(\lambda_q)/H^q(\lambda_q) \cong H^2(\lambda_q)/(H^2(\lambda_q) \cap H^q(\lambda_q)).$$

As $H^2(\lambda_q).H^q(\lambda_q) \cong H(\lambda_q)$, we have

$$\left|H^2(\lambda_q) : \left(H^2(\lambda_q) \cap H^q(\lambda_q)\right)\right| = q.$$

Then

$$\left|H(\lambda_q) : \left(H^2(\lambda_q) \cap H^q(\lambda_q)\right)\right| = 2q.$$

Now we have

$$H(\lambda_q) > H^2(\lambda_q) \cap H^q(\lambda_q) > H'(\lambda_q)$$

and

$$|H(\lambda_q) : H'(\lambda_q)| = \left|H(\lambda_q) : \left(H^2(\lambda_q) \cap H^q(\lambda_q)\right)\right| = 2q.$$

These together imply the following result:

Theorem 2.4. *The commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$ satisfies*

$$H'(\lambda_q) = H^2(\lambda_q) \cap H^q(\lambda_q). \tag{2.2}$$

By means of this result, we are going to be able to investigate the subgroups $H^{2qm}(\lambda_q)$. As $H^2(\lambda_q) > H^{2q}(\lambda_q)$ and $H^q(\lambda_q) > H^{2q}(\lambda_q)$, (2.2) implies that

$$H'(\lambda_q) > H^{2q}(\lambda_q).$$

As $H'(\lambda_q)$ is a free group, we can conclude that $H^{2q}(\lambda_q)$ is also a free group. Moreover (1.2) implies that

$$H^{2q}(\lambda_q) > H^{2qm}(\lambda_q)$$

for $m \in \mathbb{N}$. Therefore we have

Theorem 2.5. *The subgroups $H^{2qm}(\lambda_q)$ are free.*

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