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SEPARABILITY AND EFFICIENCY UNDER STANDARD WREATH PRODUCT IN TERMS OF CAYLEY GRAPHS

FIRAT ATEŞ AND A. SINAN ÇEVIK

ABSTRACT. In this paper we are mainly interested in separability and efficiency under the standard wreath product. To do that we will first obtain a presentation, say $\mathcal{P}_G$, for the standard wreath product in terms of Cayley graphs. Then we will prove our first main result of this paper, which can be thought of as an application of the result given in [**17**] (or the general result in [**6**]). Moreover, by considering the standard wreath product G of any finite groups B by A, we will define the relationship between B-separability and efficiency, on G, as another main result of this paper.

1. Introduction. Let G be a group, and let H be a subgroup of G. Then G is said to be *H-separable* if, for each $x \in G - H$, there exists $N \lhd G$ with finite index such that $x \notin NH$. Moreover, G is called *subgroup separable* if G is H-separable for all finitely generated subgroups H of G. The best known results about subgroup separability can be found, for instance, in [**2**, Section 3], [**11, 14**]. Furthermore, let S be a generating set for G. The *Cayley graph*, see, for example, [**3, 8, 12, 13**], of G, denoted by Γ_G, with respect to S has a vertex for every element of G, with an edge g to gs for all elements $g \in G$ and $s \in S$. Thus, the initial vertex of the edge is g and the terminal is gs.

Suppose that G is the *semi-direct product* of a group K by a group A, denoted by $K \rtimes_\theta A$ where $\theta : A \to \mathrm{Aut}\,(K)$, $a \to \theta_a$, $a \in A$, is a homomorphism. Suppose also that $\mathcal{P}_K = \langle \underline{\underline{y}}; \underline{\underline{s}} \rangle$ and $\mathcal{P}_A = \langle \underline{\underline{x}}; \underline{\underline{r}} \rangle$ are the presentations for the groups K and A, respectively, under the maps $y \to k_y$, $y \in \underline{\underline{y}}$, $x \to a_x$, $x \in \underline{\underline{x}}$. (We note that, throughout this paper, the notation "$\underline{\underline{\cdot}}$" used in group presentations denotes the finite set of

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generators and relators). Then, by [**9**], we have a presentation

$$\mathcal{P}_G = \langle \underline{\underline{y}}, \underline{\underline{x}} \,;\, \underline{\underline{s}}, \underline{\underline{r}}, \underline{\underline{t}} \rangle, \tag{1}$$

for $G = K \rtimes_\theta A$, where $\underline{\underline{t}} = \{yx\lambda_{yx}^{-1}x^{-1};\ y \in \underline{\underline{y}},\ x \in \underline{\underline{x}}\}$ and λ_{yx} is a word on $\underline{\underline{y}}$ representing the element $(k_y)\theta_{a_x}$ of K, $a \in A$, $k \in K$, $y \in \underline{\underline{y}}$, $x \in \underline{\underline{x}}$. (We recall that, by [**5**], the semi-direct product can also be defined as the *split extension*). Assume the group A is finite. Moreover, let B be any finite group, and let K be the direct product of $|A|$ copies of B. In [**6**] has been given the definition of the *standard wreath product* of B by A, denoted by $B \wr A$, which is actually the semi-direct product of K by A. (We should note that some authors, for instance Karpilovsky [**10**], use the notation $A \wr B$ instead of $B \wr A$. Here we use the notation as in [**15**]).

Furthermore, let us suppose that G is a finitely presented group with a finite presentation $\mathcal{P} = \langle \underline{\underline{x}}; \underline{\underline{r}} \rangle$. Then the *Euler characteristic* of $\mathcal{P}$ is defined by $\chi(\mathcal{P}) = 1 - |\underline{\underline{x}}| + |\underline{\underline{r}}|$, where $|\,.\,|$ denotes the number of elements in the set. Also there exists an upper bound $\delta(G) = 1 - \mathrm{rk}_{\mathbf{Z}}(H_1(G)) + d(H_2(G))$, where $\mathrm{rk}_{\mathbf{Z}}(\,.\,)$ denotes the $\mathbf{Z}$-rank of the torsion-free part and $d(\,.\,)$ denotes the minimal number of generators. In fact, by [**6**], it is always true that $\chi(\mathcal{P}) \geq \delta(G)$. We then define $\chi(G) = \min\{\chi(\mathcal{P}) : \mathcal{P}$ is a finite presentation for $G\}$. Hence, a presentation, say $\mathcal{P}_0$, for G is called *minimal* if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$, for all presentations $\mathcal{P}$ of G. Also, the presentation $\mathcal{P}_0$ is called *efficient* if $\chi(\mathcal{P}_0) = \delta(G)$. In addition, G is called *efficient* if $\chi(G) = \delta(G)$. We note that examples of efficient and inefficient groups have been referenced in detail in [**6**].

In this paper we will investigate, mainly, the relationship between the efficiency and separability under standard wreath products. First of all, we will obtain a presentation for this product by defining a construction based on a Cayley graph. (We note that the use of Cayley graphs in presentations is well known in semigroups [**1**]). In [**15**] has been given the efficiency for p-groups under general wreath products, and in [**6**] has been defined the sufficient conditions for finite groups to be efficient under standard wreath product. After these two results, as a first main result of this paper, we will give sufficient conditions for a standard wreath product presentation of the collection of p-groups to be efficient on the minimal number of generators. (Actually the importance of this result is that it was obtained using Cayley graphs). Further, for a standard wreath product, say G, of any finite groups B by A, we will prove, by the meaning of the efficiency on the minimal number of generators, that G is B-separable. Moreover, we will give an example and some applications for these two results.

2. Main theorems. Let A be a nontrivial finite abelian group. Then, clearly, A can be uniquely written as

$$A = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \cdots \oplus \mathbf{Z}_{n_r}, \quad n_1 \mid n_2 \mid \cdots \mid n_r.$$

We define the first torsion number of A, called $t(A)$, to be n_1. If A is a trivial finite abelian group, then $t(A) = 0$.

Suppose that the following three conditions hold for finite groups A and B.

(i) A and B have efficient presentations $\mathcal{P}_A = \langle \underline{\underline{x}}\,;\, \underline{\underline{r}} \rangle$ and $\mathcal{P}_B = \langle \underline{\underline{y}}\,;\, \underline{\underline{s}} \rangle$, respectively, on g, n, $g, n \in \mathbf{N}$, generators where $n = d(B)$,

(ii) $d(B) = d(H_1(B))$,

(iii) *either* the orders of A and $H_1(B)$ are even and also $t(H_2(A))$, $t(H_2(B))$ and $t(H_1(B))$ are even *or* the order of A is odd and there exists an odd prime p dividing $t(H_2(A)), t(H_2(B))$ and $t(H_1(B))$, where $t(\,.\,)$ is the first torsion number of the abelian group as defined above.

The proof of the following result can be found in [**6**, Theorem 1.1] which is the generalization of the result in [**17**].

Theorem 2.1. *Let $G = B \wr A$, and suppose that* (i), (ii) *and* (iii) *hold. Then G has an efficient presentation on $g + n$ generators.*

Remark 2.2. As depicted in [**6**, Remark 1.2], there is interest not just in finding efficient presentations for a finite group G, but in finding presentations that are efficient on the minimal number of generators, see [**18**].

Now suppose that both A and B are p-groups, for some prime p, and let the above condition (i) hold. In fact we do not need conditions (ii), by Proposition 3.5 below, and (iii) since A and B are p-groups.

Thus, we can give the following result as a consequence of Theorem 2.1.

Corollary 2.3. *Let $G = B \wr A$, where A and B are p-groups, satisfying condition* (i). *Then G has an efficient presentation on $g + n$ generators.*

By extending the meaning of Corollary 2.3, we obtain the following theorem as a first main result of this paper.

Theorem 2.4. *Let $A_1, A_2, \dots, A_r$ and B be finite p-groups. Also, let*

$$G_0 = B, \quad G_1 = G_0 \wr A_1, \quad G_2 = G_1 \wr A_2, \quad \dots, \quad G_r = G_{r-1} \wr A_r.$$

If B has an efficient presentation on $d(B)$ generators, then G_r has an efficient presentation on $d(G_r)$ generators.

Suppose that the above three conditions hold for any finite groups A and B, and also suppose that $G = B \wr A$. Then, as another main result of this paper, we have the following theorem which gives the relationship between efficiency and separability.

Theorem 2.5. *Let us suppose that G has an efficient presentation on $d(G)$ generators, that is, with a minimal number of generators. Then G is B-separable.*

3. Preliminaries. We should note that some of the following material in this section can also be found in [**6**].

Proposition 3.1 [**16**]. *Let B be a finite group. Then*

(i) *$H_2(B)$ is a finite group, whose elements have order dividing the order of B;*

(ii) *$H_2(B) = 1$ if B is cyclic.*

Let A be an abelian group. Then we denote by $A\#A$ the subgroup of the factor group of $A \otimes A$ generated by the elements of the form $a \otimes b + b \otimes a$ $(a, b \in A)$. Also, for any group K, an element of order 2 is called an *involution*.

Theorem 3.2 [**4**]. *Let m denote the number of involutions in the group A. Then*

$$H_2(B \wr A) = H_2(B) \oplus H_2(A) \oplus (H_1(B) \otimes H_1(B))^{(|A|-m-1)/2} \oplus (H_1(B)\#H_1(B))^m.$$

In the rest of the paper $\mathbf{Z}_n$ will denote the cyclic group of order n.

Lemma 3.3. *Let B be a finite group, let*

$$H_1(B) \cong \bigoplus_{i=1}^{t} \mathbf{Z}_{n_i},$$

and let s be the number of even n_i, $1 \leq i \leq t$. Then

$$H_1(B)\#H_1(B) \cong \bigoplus_{1 \leq i < j \leq t} \mathbf{Z}_{(n_i, n_j)} \oplus \mathbf{Z}_2^{(s)},$$

where $\mathbf{Z}_2^{(s)}$ is a direct product of s copies of $\mathbf{Z}_2$.

The proofs of the above proposition, theorem and lemma can be found in [**10**]. Also, the proof of the following lemma can be seen easily using a simple calculation.

Lemma 3.4. *Let A and B be finite p-groups for some prime p. Then*

$$d(A \oplus B) = d(A) + d(B).$$

It is clear that Lemma 3.4 can be extended for more than two groups.

The following proposition will also be needed for our proofs, and the proof of it can be obtained by a standard way.

Proposition 3.5. *Let B be an arbitrary finite p-group. Then* $d(B) = d(H_1(B))$.

4. Proof of main theorems.

4.1. Proof of Corollary 2.3 and Theorem 2.4. In this section we assume that $G = B \wr A$, where A and B are p-groups, satisfying condition (i). We recall that the other two conditions have already held for p-groups.

We will first prove Corollary 2.3, which has been shown in [**6**] by a different technique, and then, by iterating this progress, we will obtain the proof of Theorem 2.4. This section will be divided in two parts, namely "*calculation of* $\delta(G)$" (which is the homological part) and "to obtain an efficient presentation for $B \wr A$" (which is the geometric part).

1) *Calculation of* $\delta(G)$. We note that since G is a finite group, $\text{rk}_{\mathbf{Z}}(H_1(G)) = 0$. Hence, $\delta(G) = 1 + d(H_2(G))$.

Case 1. *p is odd.* Since $m = 0$, by Theorem 3.2, we have

$$H_2(B \wr A) = H_2(B) \oplus H_2(A) \oplus (H_1(B) \otimes H_1(B))^{(|A|-1)/2}.$$

Then, by Lemma 3.4,

$$\begin{aligned} 1 + d(H_2(G)) = 1 + d(H_2(B)) + 1 + d(H_2(A)) - 1 \\ + d((H_1(B) \otimes H_1(B))^{(|A|-1)/2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \delta(G) &= \delta(A) + \delta(B) - 1 + d((H_1(B) \otimes H_1(B))^{(|A|-1)/2}) \\ &= \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + d((H_1(B) \otimes H_1(B))^{(|A|-1)/2}) \\ &\quad \text{since } A \text{ and } B \text{ have efficient presentations on } d(A) \\ &\quad \text{and } d(B) \text{ generators.} \end{aligned}$$

Now let us calculate $d((H_1(B) \otimes H_1(B))^{(|A|-1)/2})$. We know that B is a p-group and so the abelianization group $H_1(B)$ of B is also a

p-group. Moreover, in condition (i), we assumed that $|y| = d(B)$, that is, B has an efficient presentation $\langle \underline{\underline{y}}; \underline{\underline{s}} \rangle$ on $d(B)$ generators. Then, by Proposition 3.5, $d(B) = n = d(H_1(B))$. So we can write

$$H_1(B) = \mathbf{Z}_{p^{k_1}} \times \mathbf{Z}_{p^{k_2}} \times \cdots \times \mathbf{Z}_{p^{k_n}}.$$

Then,

$$\begin{aligned} H_1(B) \otimes H_1(B) &= (\mathbf{Z}_{p^{k_1}} \times \mathbf{Z}_{p^{k_2}} \times \cdots \times \mathbf{Z}_{p^{k_n}}) \\ &\otimes (\mathbf{Z}_{p^{k_1}} \times \mathbf{Z}_{p^{k_2}} \times \cdots \times \mathbf{Z}_{p^{k_n}}) \\ &= (\mathbf{Z}_{p^{\min(k_1,k_1)}}) \oplus (\mathbf{Z}_{p^{\min(k_1,k_2)}}) \oplus \cdots \oplus (\mathbf{Z}_{p^{\min(k_1,k_n)}}) \\ &\oplus (\mathbf{Z}_{p^{\min(k_2,k_1)}}) \oplus (\mathbf{Z}_{p^{\min(k_2,k_2)}}) \oplus \cdots \oplus (\mathbf{Z}_{p^{\min(k_2,k_n)}}) \oplus \\ &\qquad \vdots \qquad\qquad \vdots \\ &(\mathbf{Z}_{p^{\min(k_n,k_1)}}) \oplus (\mathbf{Z}_{p^{\min(k_n,k_2)}}) \oplus \cdots \oplus (\mathbf{Z}_{p^{\min(k_n,k_n)}}). \end{aligned}$$

Thus, $d(H_1(B) \otimes H_1(B)) = n^2$ and so

$$d(H_1(B) \otimes H_1(B))^{(|A|-1)/2} = \frac{1}{2}(|A| - 1)n^2.$$

Hence,

$$\delta(G) = \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + \frac{1}{2}(|A| - 1)n^2. \tag{2}$$

Case 2. *p is even.* Due to the fact that the number of involutions is uncertain in 2-group, we can just keep it as m in our calculations. By Theorem 3.2, we have

$$\begin{aligned} H_2(G) = H_2(B) \oplus H_2(A) \oplus (H_1(B) \otimes H_1(B))^{(|A|-m-1)/2} \\ \oplus (H_1(B) \# H_1(B))^m. \end{aligned}$$

Then, by using Lemma 3.4, as in the previous case, we get

$$\begin{aligned} 1 + d(H_2(G)) = 1 + d(H_2(B)) + 1 + d(H_2(A)) - 1 \\ + d((H_1(B) \otimes H_1(B)))^{(|A|-m-1)/2} \\ + d(H_1(B) \# H_1(B))^m. \end{aligned}$$

Since A and B have efficient presentation on $d(A)$ and $d(B)$ generators, respectively, we have

$$\delta(G) = \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + d((H_1(B) \otimes H_1(B))^{(|A|-m+1)/2}) + d(H_1(B)\#H_1(B))^m.$$

Hence, we can write $H_1(B) = \mathbf{Z}_{2^{k_1}} \times \mathbf{Z}_{2^{k_2}} \times \cdots \times \mathbf{Z}_{2^{k_n}}$, and we then get $d(H_1(B) \otimes H_1(B)) = n^2$, where $d(B) = n = d(H_1(B))$ by Proposition 3.5, so that

$$d(H_1(B) \otimes H_1(B))^{(|A|-m+1)/2} = \frac{1}{2}(|A| - m + 1)n^2.$$

Also, by Lemma 3.3, we get

$$\begin{aligned} H_1(B)\#H_1(B) = {} & (\mathbf{Z}_{2^{\min(k_1,k_2)}}) \oplus (\mathbf{Z}_{2^{\min(k_1,k_3)}}) \oplus \cdots \oplus (\mathbf{Z}_{2^{\min(k_1,k_n)}}) \\ & \oplus (\mathbf{Z}_{2^{\min(k_2,k_3)}}) \oplus (\mathbf{Z}_{2^{\min(k_2,k_4)}}) \oplus \cdots \oplus (\mathbf{Z}_{2^{\min(k_2,k_n)}}) \\ & \qquad \vdots \qquad\qquad \vdots \\ & \oplus (\mathbf{Z}_{2^{\min(k_{n-2},k_{n-1})}}) \oplus (\mathbf{Z}_{2^{\min(k_{n-2},k_n)}}) \oplus (\mathbf{Z}_{2^{\min(k_{n-1},k_n)}}) \\ & \oplus \mathbf{Z}_2^{(n)} \end{aligned}$$

since $H_1(B)$ is a 2-group, we take $s = n$.

Now, by Lemma 3.4, $d(H_1(B)\#H_1(B)) = (n-1) + (n-2) + \cdots + 2 + 1 + n = (n^2+n)/2$. Therefore,

$$d(H_1(B)\#H_1(B))^m = m\left(\frac{n^2+n}{2}\right).$$

After that we have

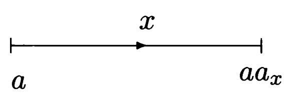
$$\delta(G) = \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + \frac{1}{2}n^2\left(|A| + \frac{m}{n} - 1\right). \tag{3}$$

2) *To obtain an efficient presentation for G.* In fact, we first need to obtain a standard presentation for $G = B \wr A$ where A and B are finite p-groups, for any prime p, by the following construction. (This

construction contains some geometric steps and Tietze transformations [**12**]). The following process can be followed for the construction.

Let $\{a_x : x \in \underline{\underline{x}}\}$ be a generating set for A corresponding to the presentation $\mathcal{P}_A = \langle \underline{\underline{x}}; \underline{\underline{r}} \rangle$, and let $\{b_y : y \in \underline{\underline{y}}\}$ be a generating set for B corresponding to the presentation $\mathcal{P}_B = \langle \underline{\underline{y}}; \underline{\underline{s}} \rangle$. We let $A = \{a_1, a_2, \dots, a_n\}$.

- Choose an ordering $a_1 < a_2 < \cdots < a_n$ where $a_1 = 1$.
- Draw a Cayley graph Γ_A of A on its elements.
- For each vertex $a \in A$, take a copy $\langle \underline{\underline{y}}^{(a)}; \underline{\underline{s}}^{(a)} \rangle$ of $\mathcal{P}_B$.
- For each pair of vertices a, a', where $a \neq a'$, write down relations
$$y^{(a)} z^{(a')} = z^{(a')} y^{(a)}, \quad y, z \in \underline{\underline{y}}.$$
- For each positive edge



in the Cayley graph, write down the relations
$$x^{-1} y^{(a)} x = y^{(aa_x)}.$$

After these steps, we can get the following lemma which can be proved directly by the meaning of standard wreath product and by considering the presentation $\mathcal{P}_G$, as in (1).

Lemma 4.1. *Let $G = B \wr A$ where A and B are finite p-groups. Then*

$$\mathcal{P}_G = \langle \underline{\underline{y}}^{(a)}\ (a \in A),\ \underline{\underline{x}}; \underline{\underline{s}}^{(a)}\ (a \in A),\ \underline{\underline{r}},\ y^{(a)} z^{(a')} = z^{(a')} y^{(a)}, \\ x^{-1} y^{(a)} x = y^{(aa_x)} \\ (a, a' \in A,\ a \neq a',\ x \in \underline{\underline{x}},\ y, z \in \underline{\underline{y}}) \rangle$$

is a standard presentation for G.

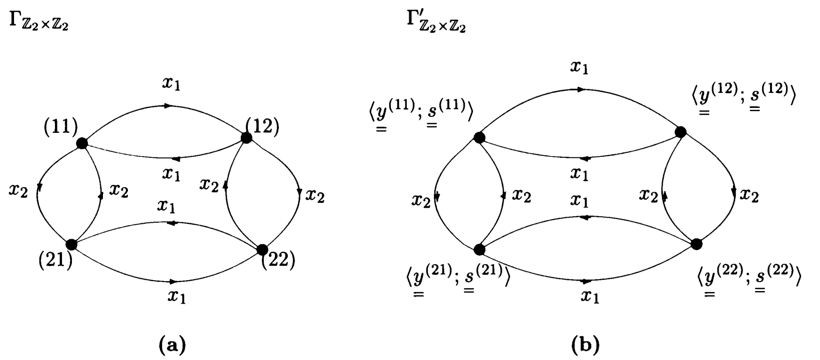


FIGURE 1.

• Finally, on $\mathcal{P}_G$, link the Tietze transformations to some geometric ideas related to the Cayley graph.

Remark 4.2. 1) In fact the construction above, as well as Lemma 4.1 cannot only be applied for p-groups but also for any finite groups A and B. Nevertheless, to prove our first result (Theorem 2.4), we will consider A and B as p-groups.

2) The reason for keeping track of the use of the Cayley graph in our construction is to obtain the set of relators $y^{(a)}z^{(a')} = z^{(a')}y^{(a)}$ and $x^{-1}y^{(a)}x = y^{(aa_x)}$, $a, a' \in A$, $a \neq a'$, $y, z \in \underline{\underline{y}}$, $x \in \underline{\underline{x}}$.

Remark 4.3. The construction above which is about obtaining a presentation for the standard wreath product has not been within our reach in literature. Besides that a presentation, say $\overline{\mathcal{P}_G}$, (similarly as $\mathcal{P}_G$ in Lemma 4.1), for $G = B \wr A$ where A and B are any finite groups, can be obtained by different methods, see, for example, [**6**]. In fact $\overline{\mathcal{P}_G}$ can be thought as a generalization of $\mathcal{P}_G$, and we will use $\overline{\mathcal{P}_G}$ for the proof of our second result, see subsection 4.2 below.

Example 4.4. Let B be a finite group. Now we will obtain a presentation for $G = B \wr \mathbf{Z}_2 \times \mathbf{Z}_2$ by using the Cayley graph based on the above construction.

Let $\mathcal{P}_{\mathbf{Z}_2\times\mathbf{Z}_2} = \langle x_1,\ x_2\ ;\ x_1^2,\ x_2^2,\ x_1x_2 = x_2x_1\rangle$ be a presentation for the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ on the generators $\{x_1,\ x_2\}$. For simplicity, let us label the elements 1, x_1, x_2 and x_1x_2 by the numbers (11), (12), (21)

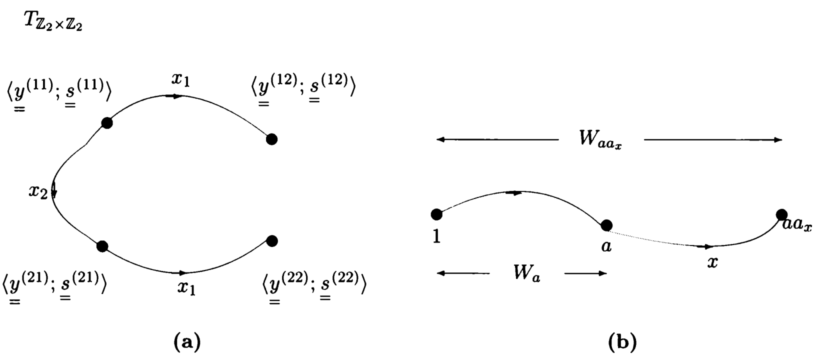


FIGURE 2.

and (22), respectively. For each vertex (ij), where $i, j \in \{1,2\}$, we take a copy $\langle \underline{\underline{y}}^{(ij)}\,;\ \underline{\underline{s}}^{(ij)}\rangle$ of $\mathcal{P}_B$. In fact, by fixing the four copies of $\mathcal{P}_B$ into each vertex in the Cayley graph $\Gamma_{\mathbf{Z}_2\times\mathbf{Z}_2}$, given in Figure 1 (a), we get the Cayley graph $\Gamma'_{\mathbf{Z}_2\times\mathbf{Z}_2}$, depicted in Figure 1 (b). Thus, we obtain relators

$$\left.\begin{array}{ll} x_1^{-1}y^{(11)}x_1 = y^{(12)}, & x_2^{-1}y^{(11)}x_2 = y^{(21)}, \\ x_1^{-1}y^{(12)}x_1 = y^{(11)}, & x_2^{-1}y^{(21)}x_2 = y^{(11)}, \\ x_1^{-1}y^{(21)}x_1 = y^{(22)}, & x_2^{-1}y^{(12)}x_2 = y^{(22)}, \\ x_1^{-1}y^{(22)}x_1 = y^{(21)}, & x_2^{-1}y^{(22)}x_2 = y^{(12)}, \end{array}\right\} \tag{4}$$

by using $\Gamma'_{\mathbf{Z}_2\times\mathbf{Z}_2}$. In other words, for $i, j \in \{1,2\}$,

$$x_1^{-1}y^{(ij)}x_1 = y^{(i\ j+1)}, \quad j \equiv 0 \pmod 2,$$
$$x_2^{-1}y^{(ij)}x_2 = y^{(i+1\ j)}, \quad i \equiv 0 \pmod 2.$$

For each of the vertices (ij) and (lm), in Figure 1 (b), where $(ij) < (lm)$, we also have relators

$$y^{(ij)}z^{(lm)} = z^{(lm)}y^{(ij)}.$$

Therefore, by Lemma 4.1, $G = B \wr \mathbf{Z}_2 \times \mathbf{Z}_2$ has a presentation

$$\begin{array}{rl} \mathcal{P}_G = \langle \underline{\underline{y}}^{(ij)}, x_1, x_2; \underline{\underline{s}}^{(ij)}, & x_1^2,\ x_2^2,\ x_1x_2 = x_2x_1,\ [y^{(ij)}, z^{(lm)}], \\ & x_1^{-1}y^{(ij)}x_1 = y^{(i\ j+1)}, \quad j \equiv 0 \pmod 2, \\ & x_2^{-1}y^{(ij)}x_2 = y^{(i+1\ j)}, \quad i \equiv 0 \pmod 2\rangle, \end{array} \tag{5}$$

where $i, j \in \{1,2\}$ and $ij < lm$. □

Suppose that we have a Cayley graph Γ_A. Now let us pick a maximal tree T in Γ_A and then, for each $a \in A$, let γ_a be the geodesic in T from 1 to a as in Figure 2 (b). Also, let W_a be the label on γ_a.

We can apply some Tietze transformations on the presentation $\mathcal{P}_G$, given in Lemma 4.1, as follows:

(T1) Add the relators $y^{(a)} = W_a^{-1} y^{(1)} W_a$, where $y^{(a)} \in \underline{\underline{y}}^{(a)}$, since these are consequences of the relators $x^{-1} y^{(a)} x = y^{(aa_x)}$ and $\underline{\underline{r}}$.

(T2) Delete the relators $x^{-1} y^{(a)} x = y^{(aa_x)}$ since these are consequences of the relators $y^{(a)} = W_a^{-1} y^{(1)} W_a$ and $\underline{\underline{r}}$. Let us show it:

For any $a \in A$, take $y^{(a)} = W_a^{-1} y^{(1)} W_a$ and conjugate it by x. Then we get $x^{-1} y^{(a)} x = x^{-1} W_a^{-1} y^{(1)} W_a x$, so $x^{-1} y^{(a)} x = W_{aa_x}^{-1} y^{(1)} W_{aa_x}$, see Figure 2 (b). Since $W_a^{-1} y^{(1)} W_a = y^{(a)}$, we write $W_{aa_x}^{-1} y^{(1)} W_{aa_x} = y^{(aa_x)}$. Thus, we have $y^{(aa_x)} = x^{-1} y^{(a)} x$.

(T3) Delete $\underline{\underline{s}}^{(a)}$, where $a \neq 1$, since these are consequences of the relators $\underline{\underline{s}}^{(1)}$ and $y^{(a)} = W_a^{-1} y^{(1)} W_a$. Thus, after deletion we will just have $\underline{\underline{s}}^{(1)}$ in the presentation. Let us show it:

We have

$$y^{(a)} = W_a^{-1} y^{(1)} W_a, \quad a \neq 1. \tag{6}$$

Let us take $S^{(1)} \in \underline{\underline{s}}^{(1)}$ and $S^{(a)} \in \underline{\underline{s}}^{(a)}$. That means the letters $S^{(1)}$ and $S^{(a)}$ belong to $\underline{\underline{y}}^{(1)}$ and $\underline{\underline{y}}^{(a)}$, $a \neq 1$, respectively. It follows from (6) that we have $S^{(a)} = W_a^{-1} S^{(1)} W_a$, $a \neq 1$. Thus, since $S^{(1)} \in \underline{\underline{s}}^{(1)}$ and $\underline{\underline{s}}^{(1)}$ is a relator in the presentation $\mathcal{P}_G$ (given in Lemma 4.1), we get $S^{(1)} \sim 1$ and so $S^{(a)} \sim 1$. That is, the relators $\underline{\underline{s}}^{(a)}$ are derivable from $\underline{\underline{s}}^{(1)}$. Thus we can delete $\underline{\underline{s}}^{(a)}$, $a \neq 1$, and we then have just $\underline{\underline{s}}^{(1)}$ in the presentation, as required.

(T4) Delete the generators $\underline{\underline{y}}^{(a)}$, where $a \neq 1$, and replace $\underline{\underline{y}}^{(a)}$ by $W_a^{-1} y^{(1)} W_a$, where $a \neq 1$, in $y^{(a)} z^{(a')} = z^{(a')} y^{(a)}$, $a, a' \in A$, $y, z \in \underline{\underline{y}}$. After deletion, we have only the generator $\underline{\underline{y}}^{(1)}$ in the set of generators $\underline{\underline{y}}^{(a)}$.

At this stage we have a commutator relator set with the form of $[y^{(1)}, W_a^{-1}z^{(1)}W_a]$, for each $a \in A$, which has total $|A|-1$ elements, and also have another commutator relator set with the form of

$$[W_{a_1}^{-1}y^{(1)}W_{a_1}, W_{a_2}^{-1}z^{(1)}W_{a_2}], \tag{7}$$

where $a_1, a_2 \in A$. Let us take relators (7) and conjugate them by $W_{a_2}^{-1}$. So we have

$$[W_{a_2}W_{a_1}^{-1}y^{(1)}W_{a_1}W_{a_2}^{-1}, z^{(1)}].$$

Then the inverses of these are $[z^{(1)}, W_{a_2}W_{a_1}^{-1}y^{(1)}W_{a_1}W_{a_2}^{-1}]$ and, actually, these relators are the form of

$$[y^{(1)}, W_{a_2}W_{a_1}^{-1}z^{(1)}W_{a_1}W_{a_2}^{-1}]. \tag{8}$$

But the relators in (8) are equal to some relators which are the form of $[y^{(1)}, W_a^{-1}z^{(1)}W_a]$ since $W_{a_1}W_{a_2}^{-1} \sim W_{a_1a_2^{-1}} \sim W_a$. Therefore we delete the relators in (7) since these are a consequence of the relators $[y^{(1)}, W_a^{-1}z^{(1)}W_a]$. Thus, we have just $[y^{(1)}, W_a^{-1}z^{(1)}W_a]$, for each $a \in A$, in commutator relator set in the presentation $\mathcal{P}_G$.

By omitting the superscript $^{(1)}$, we then obtain the presentation

$$\mathcal{P}_{1,G} = \langle \underline{\underline{y}},\ \underline{\underline{x}}\ ;\ \underline{\underline{s}},\ \underline{\underline{r}},\ [y,\ W_a^{-1}zW_a],\ (a \in A,\ a \neq 1,\ y, z \in \underline{\underline{y}})\rangle, \tag{9}$$

for the group $G = B \wr A$.

As in the homological part, we have two cases.

Case 1. *p is odd.* Let us take the presentation $\mathcal{P}_{1,G}$, given in (9). We recall that there are not any involutions in A since the order is odd. Hence, we will omit "m" in our calculations.

For any $a \in A$, let us take $[y,\ W_a^{-1}zW_a]$ and then conjugate it by $(W_a)^l$, where $l \equiv 1 \pmod p$ and $W_a(W_a)^l \sim 1$. Then we get $[(W_a)^{-l}y(W_a)^l,\ z]$, and the inverse of it is $[z, (W_a)^{-l}y(W_a)^l]$. In fact, it is the form of $[y, (W_a)^{-l}z(W_a)^l]$, for any $a \in A$. Since $W_a(W_a)^l \sim 1$, $a \in A$, (or, equivalently, $(W_a)^l \sim W_a^{-1}$), we have $[y, (W_a)^{-l}z(W_a)^l]$. But this is one of the relators in the relator set $[y,\ W_a^{-1}zW_a]$, $a \in A$, $a \neq 1$. In other words, for any $a \in A$, $[y, (W_a)^{-l}z(W_a)^l]$ is a consequence of $[y,\ W_a^{-1}zW_a]$. So we delete $[y,\ W_a^{-1}zW_a]$. Now if we apply the same process to each $a \in A$ then we delete half of the relators of the form $[y,\ W_a^{-1}zW_a]$. In fact, we delete $(|A|-1)/2$ elements from this set. Therefore, we have the presentation

(10)
$$\mathcal{P}_{2,G} = \langle \underline{\underline{y}},\ \underline{\underline{x}}\ ;\ \underline{\underline{s}},\ \underline{\underline{r}},\ [y,\ (W_a)^{-l}z(W_a)^{l}],\ (a \in A,\ a \neq 1,\ y,z \in \underline{\underline{y}})\rangle,$$

for G. Before we calculate the *Euler characteristic* of $\mathcal{P}_{2,G}$, we should remark that the number of elements in $[y,\ (W_a)^{-l}z(W_a)^{l}]$ is $(|A|-1)/2|\underline{\underline{y}}|^2$. Hence,

$$\begin{aligned}\chi(\mathcal{P}_{2,G}) &= 1-(|\underline{\underline{x}}|+|\underline{\underline{y}}|)+|\underline{\underline{r}}|+|\underline{\underline{s}}|+(|A|-1)/2|\underline{\underline{y}}|^2\\ &= 1-(|\underline{\underline{x}}|+|\underline{\underline{y}}|)+1-1+|\underline{\underline{r}}|+|\underline{\underline{s}}|+(|A|-1)/2|\underline{\underline{y}}|^2\\ &= (1-|\underline{\underline{x}}|+|\underline{\underline{r}}|)+(1-|\underline{\underline{y}}|+|\underline{\underline{s}}|)-1+(|A|-1)/2|\underline{\underline{y}}|^2\\ &= \chi(\mathcal{P}_A)+\chi(\mathcal{P}_B)-1+(|A|-1)/2|\underline{\underline{y}}|^2.\end{aligned}$$

By the assumption $|\underline{\underline{y}}| = d(B) = n$,

$$\chi(\mathcal{P}_{2,G}) = \chi(\mathcal{P}_A)+\chi(\mathcal{P}_B)-1+(|A|-1)/2n^2.$$

Therefore, $\chi(\mathcal{P}_{2,G})$ is equal to $\delta(G)$, given in equation (2). So $\mathcal{P}_{2,G}$ is an efficient presentation for G.

Case 2. *p is even.* Let us consider the set of relators $[y,\ W_a^{-1}zW_a]$, where $a \neq 1$, $a \in A$, $y,z \in \underline{\underline{y}}$, in the presentation $\mathcal{P}_{1,G}$ in (9). In fact, there are a total $|\underline{\underline{y}}|^2(|A|-1)$ elements in these relators and, clearly, we have $|\underline{\underline{y}}|^2$ choices of them. But we also have some mutually inverse terms in these $|\underline{\underline{y}}|^2$ choices, for instance, $[y_1,\ W_a^{-1}y_2W_a]$ and $[y_2,\ W_a^{-1}y_1W_a]$, $y_1,y_2 \in \underline{\underline{y}}$. The number of mutually inverse relators in these $|\underline{\underline{y}}|^2$ choices is $|\underline{\underline{y}}|(|\underline{\underline{y}}|-1)/2$. Moreover, since each relator in these $|\underline{\underline{y}}|^2$ choices has $|A|-1$ elements, the total number of such these mutually inverse relators is $(|A|-1)(|\underline{\underline{y}}|(|\underline{\underline{y}}|-1))/2$, say S_1. But, by

Tietze transformations, we can delete these S_1 elements since they are derivable from the others as follows:

For $y_1, y_2 \in \underline{\underline{y}}$ and $a \in A$, take $[y_1,\ W_a^{-1} y_2 W_a]$. Then, by conjugating W_a, we get

$$\begin{aligned}
[W_a y_1 W_a^{-1}, y_2] &\sim [y_2,\ W_a^{-1} y_1 W_a] \\
&\sim [y_2,\ (W_a^{-1})^{-1} y_1 W_a^{-1}] \\
&\sim [y_2,\ (W_{a^{-1}})^{-1} y_1 W_a^{-1}] \\
&\quad \text{since} \quad W_a^{-1} \sim_{\underline{\underline{r}}} W_{a^{-1}} =_A W_{a_1}, \quad \text{for any} \quad a_1 \in A \\
&\sim [y_2,\ W_{a_1}^{-1} y_1 W_{a_1}] \sim [y_2,\ W_a^{-1} y_1 W_a] \\
&\quad \text{since} \quad W_{a_1} \sim W_a.
\end{aligned}$$

After deletion we get the total

$$|\underline{\underline{y}}|^2(|A|-1) - S_1 = \frac{1}{2}(|A|-1)(|\underline{\underline{y}}|^2 + |\underline{\underline{y}}|)$$

relators, say S_2, in the set of relators $[y,\ W_a^{-1} z W_a]$, $a \in A$, $y, z \in \underline{\underline{y}}$.

We can still apply some deletions on these S_2 elements. Because we also have

$$\text{(11)} \qquad [y,\ W_a^{-1} y W_a], \quad a \in A, \quad y \in \underline{\underline{y}}$$

relators. In fact, the total number of these relators is $|\underline{\underline{y}}|$. Moreover, we can find some inverse elements in relators (11) and the number of these inverse elements, for each relator in (11), is $(|A|-1-m)/2$, where m is the number of involutions in A. Then, since we have total $|\underline{\underline{y}}|$ relators in (11), the total number of these inverse relators in (11) is $|\underline{\underline{y}}|(|A|-1-m)/2$, say S_3. Hence, by deleting these S_3 elements from the S_2 elements, we get $(1/2)|\underline{\underline{y}}|^2(|A|-1+(m/|\underline{\underline{y}}|))$, say S_4, in the set of relators $[y,\ W_a^{-1} z W_a]$.

Therefore, we have the presentation

$$\text{(12)} \quad \mathcal{P}_{3,G} = \langle \underline{\underline{y}},\ \underline{\underline{x}}\ ;\ \underline{\underline{s}},\ \underline{\underline{r}},\ [y,\ W_a^{-1} z W_a], (a \in A,\ a \neq 1,\ y, z \in \underline{\underline{y}})\rangle,$$

for $G = B \wr A$. Hence,

$$\begin{aligned}\chi(\mathcal{P}_{3,G}) &= 1 - (|\underline{\underline{x}}| + |\underline{\underline{y}}|) + |\underline{\underline{r}}| + |\underline{\underline{s}}| + \frac{1}{2}\left(|A| - 1 + \frac{m}{|\underline{\underline{y}}|}\right)|\underline{\underline{y}}|^2 \\ &= 1 - (|\underline{\underline{x}}| + |\underline{\underline{y}}|) + 1 - 1 + |\underline{\underline{r}}| + |\underline{\underline{s}}| \\ &\quad + \frac{1}{2}\left(|A| - 1 + \frac{m}{|\underline{\underline{y}}|}\right)|\underline{\underline{y}}|^2 \\ &= (1 - |\underline{\underline{x}}| + |\underline{\underline{r}}|) + (1 - |\underline{\underline{y}}| + |\underline{\underline{s}}|) - 1 \\ &\quad + \frac{1}{2}(|A| - 1 + \frac{m}{|\underline{\underline{y}}|})|\underline{\underline{y}}|^2 \\ &= \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + \frac{1}{2}\left(|A| - 1 + \frac{m}{|\underline{\underline{y}}|}\right)|\underline{\underline{y}}|^2.\end{aligned}$$

As p is the even case, by the assumption $|\underline{\underline{y}}| = d(B) = n$,

$$\chi(\mathcal{P}_{3,G}) = \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + \frac{1}{2}\left(|A| - 1 + \frac{m}{n}\right)n^2.$$

Thus $\chi(\mathcal{P}_{3,G})$ is equal to $\delta(G)$, given in (3). So $\mathcal{P}_{3,G}$ is an efficient presentation for G.

Example 4.4 (continued). Let us choose a maximal tree $T_{\mathbf{Z}_2 \times \mathbf{Z}_2}$, as depicted in Figure 2 (a), from the Cayley graph $\Gamma'_{\mathbf{Z}_2 \times \mathbf{Z}_2}$, given in Figure 1 (b). Then we can delete some relators in presentation (5) as follows:

Clearly $\underline{\underline{s}}^{(ij)}$, where $i, j \in \{1, 2\}$, $(ij) \neq (11)$, is a consequence of $\underline{\underline{s}}^{(11)}$. Let $S^{(ij)} \in \underline{\underline{s}}^{(ij)}$, and let $S^{(11)} \in \underline{\underline{s}}^{(11)}$. Suppose W_{ij} is the label on γ_{ij}. Then $W_{ij}^{-1} S^{(11)} W_{ij} = S^{(ij)}$. Since $S^{(11)} \in \underline{\underline{s}}^{(11)}$ and $\underline{\underline{s}}^{(11)}$ is a relator in the presentation and then since $S^{(11)} \sim 1$, we say that $\underline{\underline{s}}^{(ij)}$ is derivable from $\underline{\underline{s}}^{(11)}$, and so we delete $\underline{\underline{s}}^{(ij)}$ from the relator set in the presentation. Let γ_{11}, γ_{12}, γ_{21} and γ_{22} be geodesics in maximal tree $T_{\mathbf{Z}_2 \times \mathbf{Z}_2}$ from 1 to (ij), $(ij) \in \mathbf{Z}_2 \times \mathbf{Z}_2$, and let us suppose that

$$1 \text{ labels } \gamma_{11}, \quad x_1 \text{ labels } \gamma_{12}, \quad x_2 \text{ labels } \gamma_{21}, \quad x_2 x_1 \text{ labels } \gamma_{22}.$$

Hence, by using relations (4), we have

$$x_1^{-1}y^{(11)}x_1 = y^{(12)}, \quad x_2^{-1}y^{(11)}x_2 = y^{(21)} \text{ and } x_1^{-1}x_2^{-1}y^{(11)}x_2x_1 = y^{(22)}$$

or, equivalently,

$$\gamma_{12}^{-1}y^{(11)}\gamma_{12} = y^{(12)}, \quad \gamma_{21}^{-1}y^{(11)}\gamma_{21} = y^{(21)}, \quad \gamma_{22}^{-1}y^{(11)}\gamma_{22} = y^{(22)}. \tag{13}$$

From the Cayley graph in Figure 1 (b), we can easily see that

$$\left.\begin{array}{lll} \gamma_{12}\gamma_{12} = 1, & \gamma_{12}\gamma_{21} = \gamma_{22} = \gamma_{21}\gamma_{12}, & \gamma_{12}\gamma_{22} = \gamma_{21} = \gamma_{22}\gamma_{12}, \\ \gamma_{21}\gamma_{21} = 1, & \gamma_{21}\gamma_{22} = \gamma_{12} = \gamma_{22}\gamma_{21}. & \end{array}\right\} \tag{14}$$

Using (13), we get

$$\begin{aligned} [y^{(11)},\ z^{(12)}] &= [y^{(11)},\ \gamma_{12}^{-1}z^{(11)}\gamma_{12}], \\ [y^{(11)},\ z^{(21)}] &= [y^{(11)},\ \gamma_{21}^{-1}z^{(11)}\gamma_{21}], \\ [y^{(11)},\ z^{(22)}] &= [y^{(11)},\ \gamma_{22}^{-1}z^{(11)}\gamma_{22}], \\ [y^{(12)},\ z^{(21)}] &= [\gamma_{12}^{-1}y^{(11)}\gamma_{12}, \gamma_{21}^{-1}z^{(11)}\gamma_{21}], \\ [y^{(12)},\ z^{(22)}] &= [\gamma_{12}^{-1}y^{(11)}\gamma_{12}, \gamma_{22}^{-1}z^{(11)}\gamma_{22}] \end{aligned}$$

and

$$[y^{(21)},\ z^{(22)}] = [\gamma_{21}^{-1}y^{(11)}\gamma_{21}, \gamma_{22}^{-1}z^{(11)}\gamma_{22}].$$

After that, by applying (14) and using γ_{21}^{-1} and γ_{22}^{-1}, we can delete $[y^{(12)},\ z^{(21)}]$, $[y^{(12)},\ z^{(22)}]$ and $[y^{(21)},\ z^{(22)}]$.

Therefore we obtain the presentation

$$\begin{aligned} \mathcal{P}_G = \langle \underline{\underline{y}}^{(11)},\ x_1, x_2; \underline{\underline{s}}^{(11)},\ x_1^2,\ x_2^2,\ x_1x_2 = x_2x_1, [y^{(11)},\ x_1^{-1}z^{(11)}x_1], \\ [y^{(11)},\ x_2^{-1}z^{(11)}x_2],\ [y^{(11)},\ x_2^{-1}x_1^{-1}z^{(11)}x_1x_2]\rangle, \end{aligned} \tag{15}$$

for the group $G = B \wr \mathbf{Z}_2 \times \mathbf{Z}_2$.

Now assume that B is cyclic of order 4 with a presentation $\mathcal{P}_B = \langle y; y^4\rangle$. Then presentation (15) becomes

$$\begin{aligned} \mathcal{P}_G = \langle y, x_1, x_2; y^4, x_1^2, x_2^2, x_1x_2 \\ = x_2x_1, [y,\ x_1^{-1}yx_1], [y,\ x_2^{-1}yx_2], [y,\ x_2^{-1}x_1^{-1}yx_1x_2]\rangle. \end{aligned} \tag{16}$$

Since $\delta(\mathbf{Z}_4 \wr \mathbf{Z}_2 \times \mathbf{Z}_2) = \chi(\mathcal{P}_G) = 5$, presentation (16) is efficient for the group $G = \mathbf{Z}_4 \wr \mathbf{Z}_2 \times \mathbf{Z}_2$.

Until now we have proved group G has an efficient presentation $\mathcal{P}_{2,G}$, as in (10), or $\mathcal{P}_{3,G}$, as in (12). But, to complete the proof of Theorem 2.4, we must also show that these presentations are efficient on the minimal number of generators, that is, $d(G) = g+n$, by assuming $g = d(A)$, actually, $g = d(H_1(A))$ (by Proposition 3.5). This can be shown as follows:

Let us consider the presentation $\mathcal{P}_{3,G}$. Since $\mathcal{P}_{3,G}$ has $g+n$ generators, we certainly have $d(G) \leq g+n$. So we just need to show that $d(G) \geq g+n$. For this, we will use the fact that *the minimal number of generators of a group is greater than or equal to the minimal number of generators of a quotient group*, in particular, $d(G) \geq d(H_1(G))$. So we need to show that $d(H_1(G)) = g+n$.

Now let us choose an ordering $x_1 < x_2 < \cdots < x_g$ of the elements in the generating set $\underline{\underline{x}}$.

The first homology group of G can be presented by

$$\begin{aligned}\mathcal{P}_{H_1(G)} = \langle \underline{\underline{y}}, \underline{\underline{x}}; \underline{\underline{s}},\ \underline{\underline{r}},\ &[y, W_a^{-1} z W_a] \\ &(a \in A, a \neq 1,\ y, z \in \underline{\underline{y}}),\ [y,x]\ (y \in \underline{\underline{y}},\ x \in \underline{\underline{x}}), \\ &\quad [y,z]\ (y,z \in \underline{\underline{y}},\ y < z),\ [x,x']\ (x,x' \in \underline{\underline{x}},\ x < x')\rangle.\end{aligned}$$

By applying deletion operations on $\mathcal{P}_{H_1(G)}$, we have

$$\begin{aligned}\mathcal{P}_{H_1(G)} &= \langle \underline{\underline{y}},\ \underline{\underline{x}}; \underline{\underline{s}},\ \underline{\underline{r}},\ [y,x]\ (y \in \underline{\underline{y}},\ x \in \underline{\underline{x}}),\ [y,z]\ (y,z \in \underline{\underline{y}},\ y < z), \\ &\qquad [x,x']\ (x,x' \in \underline{\underline{x}},\ x < x')\rangle \\ &\cong H_1(A) \oplus H_1(B).\end{aligned}$$

So, by Lemma 3.4, $d(H_1(G)) = d(H_1(A)) + d(H_1(B))$. Since $d(H_1(A)) = d(A) = g$ and $d(H_1(B)) = d(B) = n$, by Proposition 3.5, we obtain $d(H_1(G)) = g+n$, as required. (We note that the ps being the odd case can be seen by using a similar way as above).

Example 4.4 (continued). The presentation $\mathcal{P}_G$, in (16), is efficient on 3 generators.

After all these processes of the proof of Corollary 2.3, we can prove Theorem 2.4, by induction on r, as follows:

a) Let $r = 1$. Then the result holds by Corollary 2.3.

b) Let $r > 1$; then $G_r = G_{r-1} \wr A_r$. By induction hypothesis, G_{r-1} has an efficient presentation on $d(G_{r-1})$ generators. Moreover, G_{r-1} is a p-group. Since A_r is an abelian p-group, again by Corollary 2.3, G_r has an efficient presentation on $d(G_r)$ generators.

This completes the proof of Theorem 2.4. □

Example 4.4 (continued). $G_r = (\cdots((\mathbf{Z}_4 \wr \mathbf{Z}_2 \times \mathbf{Z}_2) \wr \mathbf{Z}_2 \times \mathbf{Z}_2) \wr \cdots) \wr \mathbf{Z}_2 \times \mathbf{Z}_2$ has an efficient presentation on $2r + 1$ generators. □

4.2. Proof of Theorem 2.5. Suppose that G is the standard wreath product of any finite groups B by A satisfying conditions (i), (ii) and (iii). By the construction defined in the previous section (and so, by Lemma 4.1), we can get a presentation $\overline{\mathcal{P}_G}$, as depicted in Remark 4.3. After some deletion operations on $\overline{\mathcal{P}_G}$ considering the set $A - \{1\}$ can be divided into singletons $\{a\}$ ($a \in A$, a is an involution) and pairs $\{a, a^{-1}\}$ (a is not an involution) and choosing $y_1 < y_2 < \cdots < y_n$ for the elements of the generating set $\underline{\underline{y}}$, we get the following presentation $\mathcal{P}_{G_1}$ from $\overline{\mathcal{P}_G}$ for the group G. Let A^+ be a choice of one element from each pair $\{a, a^{-1}\}$, and let Inv be the set of involutions in group A. Thus,

$$\begin{aligned} \mathcal{P}_{G_1} = \langle \underline{\underline{y}}, \underline{\underline{x}}; \underline{\underline{s}}, \underline{\underline{r}}, \\ &[y, W_a^{-1} z W_a], (a \in A^+,\ y, z \in \underline{\underline{y}}), [y, W_a^{-1} z W_a] \\ &\qquad (a \in \mathrm{Inv},\ y, z \in \underline{\underline{y}},\ y \leq z)\rangle. \end{aligned}$$

We note that the above presentation $\mathcal{P}_{G_1}$ can be thought as a generalization of presentations $\mathcal{P}_{2,G}$ and $\mathcal{P}_{3,G}$ since they present only p-group G. In [**5**] has been proved that the presentation $\mathcal{P}_{G_1}$ is the simplest form for the group G, that is, it has the minimal number of generators and relators with $a \neq 1$. In that paper it has also been proved that this presentation is efficient with this minimal number of generators.

We also remark that there is no empty word in commutator relators in the presentation $\mathcal{P}_{G_1}$ since $a \neq 1$. In fact this is the main point of our proof.

By the definition of $G = B \wr A = B^{|A|} \rtimes_\theta A$, the group $B^{|A|}$ is normal in G. Let us denote by N the normal subgroup $B^{|A|}$. We have a set $G - B$ which has the elements obtained by the set $\underline{\underline{x}}$, and the elements satisfy the commutator form $[y, W_a^{-1} z W_a]$ $(a \in A^+,\ y, z \in \underline{\underline{y}}$ or $a \in \mathrm{Inv},\ y, z \in \underline{\underline{y}},\ y \leq z)$. Recall that γ_a is the geodesic in T from $\bar{1}$ to a (see Figure 2 (b)) and W_a is the label on γ_a. So W_a consists only the elements of the generating set $\underline{\underline{x}}$. Let U be the word obtained by the set $G - B$. Since $a \neq 1$ (and so $W_a \neq 1$), U cannot be the form of $[y, z]$ $(y, z \in \underline{\underline{y}},\ y \leq z)$. Also, $U \notin NB$ since NB does not have a word which contains the elements of $\underline{\underline{x}}$ and does not have elements satisfying the commutator form.

Due to $\mathcal{P}_{G_1}$ is the efficient presentation with the minimal number of generators that is our beginning point and, moreover, since the material in the above paragraph can be applied for this efficient group G, we can directly say that G is B-separable. In other words, by the meaning of minimal efficient presentation, we guaranteed that the set $G - B$ can also be obtained by this minimal number of elements, and so the set NB cannot contain any elements of the form $(yz)^k$ or $y^s z^i$, where $y, z \in \underline{\underline{y}},\ k, i, s \in \mathbf{Z}^+$.

This completes the proof. □

As a consequence of Corollary 2.3 and Theorem 2.5, we have the following results.

Corollary 4.5. *Suppose that A and B are finite 2-groups and, for the group $G = B \wr A$, $\mathcal{P}_{3,G}$, as in (12), is an efficient presentation on $g + n$ generators. Then G is B-separable.*

Proof. In the proof of Theorem 2.5, let us take the presentation $\mathcal{P}_{3,G}$ instead of $\mathcal{P}_{G_1}$. Then, by Corollary 2.3, we know that it is efficient on the minimal number of generators. So, by applying the same steps as in the proof of Theorem 2.5, we get the result. □

Similarly,

Corollary 4.6. *Suppose that A and B are finite p-groups, where p is odd, and $\mathcal{P}_{2,G}$, as in (10), is an efficient presentation on $g+n$ generators for $G = B \wr A$. Then G is B-separable.*

Furthermore, as an application of Theorems 2.4, 2.5 and Corollaries 4.5, 4.6, we have the following result.

Corollary 4.7. *Let $A_1, A_2, \dots, A_r$ and B be finite p-groups, and let*

$$G_0 = B, \quad G_1 = G_0 \wr A_1, \quad G_2 = G_1 \wr A_2, \quad \dots, \quad G_r = G_{r-1} \wr A_r.$$

Suppose that G_r has an efficient presentation on $d(G_r)$ generators. Then

$$\begin{aligned} G_1 &\text{ is } G_0\text{-separable}, \\ G_2 &\text{ is } G_1\text{-separable}, \dots, \\ G_r &\text{ is } G_{r-1}\text{-separable}. \end{aligned}$$

Example 4.4 (continued). $B^{|\mathbf{Z}_2\times\mathbf{Z}_2|}$ is the normal subgroup of $G = B^{|\mathbf{Z}_2\times\mathbf{Z}_2|} \rtimes_\theta (\mathbf{Z}_2 \times \mathbf{Z}_2)$. We have a set $G-B$ which has the elements obtained by the set $\{x_1, x_2\}$ and has the elements satisfying commutator forms $[x_1, x_2]$, $[y, x_1^{-1} y x_1]$, $[y, x_2^{-1} y x_2]$ and $[y, x_2^{-1} x_1^{-1} y x_1 x_2]$. We showed that presentation (16) is efficient on 3 generators. As we did in the proof of Theorem 2.5, the word U must contain x_1 or x_2. However $B^{|\mathbf{Z}_2\times\mathbf{Z}_2|}$ does not have a word that contains x_1 or x_2. Hence $U \notin B^{|\mathbf{Z}_2\times\mathbf{Z}_2|}B$. That means G is B-separable. □

Question. Is there a relationship between subgroup separability and efficiency for a standard (or general) wreath product of finite groups?

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BALIKESIR UNIVERSITESI, FEN-EDEBIYAT FAKULTESI, MATEMATIK BOLUMU, CAGIS KAMPUSU, 10145, BALIKESIR, TURKEY
Email address: firat@balikesir.edu.tr

BALIKESIR UNIVERSITESI, FEN-EDEBIYAT FAKULTESI, MATEMATIK BOLUMU, CAGIS KAMPUSU, 10145, BALIKESIR, TURKEY
Email address: scevik@balikesir.edu.tr